BGP REFLECTION, TILTING MODULES AND TILTING COMPLEXES

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In this note, we review the transition from the notion of Bernstein-Gelfand-Ponomarev reflection functors to the notion of tilting complexes and triangulated equivalences.

1. Quivers and Path Algebras

Throughout this note, k is a field.

Definition 1.1. A quiver $\Delta = (\Delta_0, \Delta_1, h, t)$ is an oriented graph, where Δ_0 is a set of vertices and Δ_1 is a set of arrows between vertices. We use $h : \Delta_1 \to \Delta_0$, $t : \Delta_1 \to \Delta_0$ the maps defined by $h(\alpha) = j$, $t(\alpha) = i$ when $\alpha : i \to j$ is arrow from the vertex i to the vertex j. We denote by $\bar{\Delta}$ the underlying graph, that is obtained from Δ by forgetting the orientation of the arrows. Moreover, we often write $\Delta = (\bar{\Delta}, \Omega)$when we give an orientation Ω to $\bar{\Delta}$. For $x \in \Delta_0$, let

$$x^{\geq} = \{\alpha \in \Delta_1 | h(\alpha) = x\} \quad x^{\leq} = \{\alpha \in \Delta_1 | t(\alpha) = x\}$$

A vertex x in Δ is called a sink (resp., a source) if $x^{\leq} = \phi$ (resp., $x^{\geq} = \phi$). A quiver $\Delta = (\Delta_0, \Delta_1, h, t)$ is called a locally finite quiver if $\#x^{\geq}, \#x^{\leq} < \infty$ for any $x \in \Delta_0$, and it is called a finite quiver if $\#\Delta_0, \#\Delta_1 < \infty$. A path $w = (b|\alpha_r, \ldots, \alpha_1|a) : a \rightsquigarrow b$ from the vertex a to the vertex b in the quiver Δ is a sequence of ordered arrows $\alpha_1, \ldots, \alpha_r$ such that $a = t(\alpha_1), h(\alpha_i) = t(\alpha_{i+1})$ $(1 \leq i \leq r-1), h(\alpha_r) = b$. In this case, a (resp., b) is called the tail $t(w)$ (resp., the head $h(w)$) of w, and r is called the length of a path w. For every vertex i, the path $e_a = (a||a)$ of length 0 is called the empty path. A non-empty path w is called an oriented cycle if $h(w) = t(w)$.

Definition 1.2. Let $\Delta = (\Delta_0, \Delta_1, h, t)$ be a finite quiver with $\Delta_0 = \{1, \cdots, n\}$. For $\mathbf{x} = {}^t(x_1, \cdots, x_n), \mathbf{y} = {}^t(y_1, \cdots, y_n) \in \mathbb{Z}^n_{\geq 0}$, we define a bilinear form, a quadratic form and a symmetric bilinear form:

$$< \mathbf{x}, \mathbf{y} >_\Delta = \sum_{i \in \Delta_0} x_i y_i - \sum_{\alpha \in \Delta_1} x_{t(\alpha)} y_{h(\alpha)}$$

$$\chi_{\bar{\Delta}}(\mathbf{x}) = \sum_{i \in \Delta_0} x_i^2 - \sum_{\alpha \in \Delta_1} x_{t(\alpha)} x_{h(\alpha)}$$

$$(\mathbf{x}, \mathbf{y})_{\bar{\Delta}} = \frac{1}{2}(\chi_{\bar{\Delta}}(\mathbf{x} + \mathbf{y}) - \chi_{\bar{\Delta}}(\mathbf{x}) - \chi_{\bar{\Delta}}(\mathbf{y}))$$

Definition 1.3. Let $\Delta = (\Delta_0, \Delta_1, h, t)$ be a quiver. The k-linear path category $k\Delta$ of Δ is an additive category consisting of finite direct sums $\oplus_{a \in \Delta_0} a^{\oplus n_a}$ of vertices $a \in \Delta$ as objects, matrices of which entries are k-vectors spanned by all paths in Δ as morphisms, and compositions of morphisms are defined by compositions of paths

$$(c|\alpha_s, \ldots, \alpha_{r+1}|b) \circ (b|\alpha_r, \ldots, \alpha_1|a) = (c|\alpha_s, \ldots, \alpha_1|a).$$

For example, the Hom-set $k\Delta(a,b)$ for vertices a, b is the k-vector space spanned by all paths $a \rightsquigarrow b$ from a to b:

$$k\Delta(a,b) = < w|\ w : a \rightsquigarrow b >_k$$

Similarly, the path k-algebra $k\Delta$ is the k-vector space spanned by the set of all paths in Δ together with the multiplication induced by compositions of paths.

We often simply write $\alpha_r \ldots \alpha_1$ for $(b|\alpha_r, \ldots, \alpha_1|a)$.

Remark 1.4. If $\#\Delta_0 < \infty$, then $\sum_{x\in\Delta_0} e_x = 1$ in the k-algebra $k\Delta$.

Example 1.5. For a quiver

$$\Delta :\ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

we have

$$\begin{array}{lll} e_1k\Delta e_1 = < e_1 >_k & e_2k\Delta e_1 = < \alpha >_k & e_3k\Delta e_1 = < \beta\alpha >_k \\ e_1k\Delta e_2 = O & e_2k\Delta e_2 = < e_2 >_k & e_3k\Delta e_2 = < \beta >_k \\ e_1k\Delta e_3 = O & e_2k\Delta e_3 = O & e_3k\Delta e_3 = < e_3 >_k \end{array}$$

$$\chi_{\bar{\Delta}}(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3$$

Then we have

$$k\Delta \cong \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$$

Example 1.6. For a quiver

$$\Delta :\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$$

we have

$$\begin{array}{ll} e_1k\Delta e_1 = < e_1 >_k & e_2k\Delta e_1 = < \alpha, \beta >_k \\ e_1k\Delta e_2 = 0 & e_2k\Delta e_2 = < e_2 >_k \end{array}$$

$$\chi_{\bar{\Delta}}(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1x_2$$

Then we have

$$k\Delta \cong \begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix}$$

Example 1.7. For a quiver

$$\Delta :\ 1 \xrightarrow{\alpha} 2 \circlearrowright \beta$$

we have

$$\begin{array}{ll} e_1k\Delta e_1 = < e_1 >_k & e_2k\Delta e_1 = < \alpha, \beta^n\alpha|\ n \in \mathbb{N} >_k \\ e_1k\Delta e_2 = 0 & e_2k\Delta e_2 = < e_2, \beta^n|\ n \in \mathbb{N} >_k \end{array}$$

$$\chi_{\bar{\Delta}}(\mathbf{x}) = x_1^2 - x_1x_2$$

Then we have

$$k\Delta \cong \begin{bmatrix} k & 0 \\ k[x] & k[x] \end{bmatrix}$$

2. Representations and BGP reflection

Definition 2.1. Given a quiver $\Delta = (\Delta_0, \Delta_1, h, t)$ $(= (\bar{\Delta}, \Omega))$, a representation $M = (M(i); M(\alpha))$ of Δ over a field k is a family $(M(i))_{i\in\Delta_0}$ of k-vector spaces together with a family $(M(\alpha) : M(i) \to M(j))_{i\xrightarrow{\alpha} j\in\Delta_1}$ of k-linear maps. A representation $M = (M(i); M(\alpha))$ is called a (locally) finite dimensional representation if $M(i)$ is a finite dimensional k-vector space for every $i \in \Delta_0$. For a finite dimensional representation M, the dimension vector of M is $\underline{\mathbf{dim}}M = (\dim_k M(i))_{i\in\Delta_0}$.

For $(M(i); M(\alpha)), (N(i); N(\alpha))$, a morphism $f : (M(i); M(\alpha)) \to (N(i); N(\alpha))$ is a family $(f_i : M(i) \to N(i))_{i\in\Delta_0}$ of k-linear maps satisfying that we have a commutative diagram

$$\begin{tikzcd} M(i) \arrow[r, "M(\alpha)"] \arrow[d, "f_i"'] & M(j) \arrow[d, "f_j"] \\ N(i) \arrow[r, "N(\alpha)"] & N(j) \end{tikzcd}$$

for any $i \xrightarrow{\alpha} j \in \Delta_1$.

We denote by $\mathsf{Rep}_k\,\Delta$ or $\mathsf{Rep}_k(\bar{\Delta}, \Omega)$ (resp., $\mathsf{rep}_k\,\Delta$ or $\mathsf{rep}_k(\bar{\Delta}, \Omega)$) the category of representations (resp., finite dimensional representations) of Δ over k.

Remark 2.2. It is easy to see that $\mathsf{Rep}_k\,\Delta$ (resp., $\mathsf{rep}_k\,\Delta$) is equivalent to the category $\mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$ (resp., $\mathsf{Func}_k(k\Delta, \mathsf{mod}\,k)$) of k-linear additive functors from $k\Delta$ to the category of k-vector spaces (resp., finite dimensional k-vector spaces). Therefore, $\mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$ is an abelian category with direct sums and products. Let $h^a : k\Delta \to \mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$ (resp., $h_a : k\Delta \to \mathsf{Func}_k(k\Delta^{op}, \mathsf{Mod}\,k)$) be the functor defined by $h^a(x) = k\Delta(a, x)$ (resp., $h_a(x) = k\Delta(x, a)$) for any $x \in \Delta_0$. We often identify $\mathsf{Rep}_k\,\Delta$ with $\mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$. We often write $\mathsf{Mod}\,k\Delta = \mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$.

Definition 2.3. Let $\Delta = (\Delta_0, \Delta_1, h, t)$ $(= (\bar{\Delta}, \Omega))$ be a quiver a a vertex. We define the representation $(S_a, S_a(\alpha))$ by

$$S_a(x) = \begin{cases} k \text{ if } x = a \\ 0 \text{ if } x \neq a \end{cases} \qquad S_a(\alpha) = 0$$

We define the representation $(P_a(i), P_a(\alpha))$ by $k\Delta(a, -) \in \mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$. In other words, for any vertex x $P_a(x)$ is the k-vector space spanned by paths from a to x: $P_a(x) = k\Delta(a, x) =< w|\ w : a \rightsquigarrow x >_k$ and $P_a(\alpha)$ is the k-linear map defined by $P_a(\alpha)(w) = \alpha w$ for any arrow $\alpha : x \to y$ and any path $w : z \rightsquigarrow x$. Moreover, we define the representation $(Q_a(i), Q_a(\alpha))$ by $\mathrm{Hom}_k(k\Delta(-, a), k) : k\Delta \to \mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$.

Lemma 2.4 (Yoneda's Lemma)**.** *Let $a, b \in k\Delta$ and $M \in \mathsf{Func}_k(k\Delta, \mathsf{Mod}\,k)$. then the following hold.*

1. *We have the bijection $\theta_- : M(a) \to \mathrm{Hom}_{k\Delta}(k\Delta(a, -), M)$, where θ_- is defined by $(\theta_\lambda)(b)(f) = M(f)(x)$ for $\lambda \in M(a)$, $f \in k\Delta(a, b)$.*
2. *We have the bijection $\theta_- : k\Delta(b, a) \to \mathrm{Hom}_{k\Delta}(k\Delta(a, -), k\Delta(b, -))$.*

Example 2.5. For a quiver

$$\Delta :\ 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

$k\Delta =< e_1, e_2, e_3, \alpha, \beta >_k$. A representation M of Δ over k is the following

$$M(1) \xrightarrow{M(\alpha)} M(2) \xleftarrow{M(\beta)} M(3)$$

By the standard technique of linear algebra, all indecomposable representations are up to isomorphisms the following

$$\begin{array}{lll} M_1 = P_2 :\ 0 \to k \leftarrow 0 & M_2 = P_1 :\ k \to k \leftarrow 0 & M_3 = P_3 :\ 0 \to k \leftarrow k \\ M_4 = Q_2 :\ k \to k \leftarrow k & M_5 = Q_3 :\ 0 \to 0 \leftarrow k & M_6 = Q_1 :\ k \to 0 \leftarrow 0 \end{array}$$

$$\begin{tikzcd}
 & M_2 \arrow[dr] & & M_5 \\
M_1 \arrow[ur] \arrow[dr] & & M_4 \arrow[ur] \arrow[dr] & \\
 & M_3 \arrow[ur] & & M_6
\end{tikzcd}$$

Example 2.6. For a quiver

$$\Delta' :\ 1 \xleftarrow{\alpha} 2 \xrightarrow{\beta} 3$$

$k\Delta' =< e_1, e_2, e_3, \alpha, \beta >_k$. A representation M of Δ' over k is the following

$$N(1) \xleftarrow{N(\alpha)} N(2) \xrightarrow{N(\beta)} N(3)$$

By the standard technique of linear algebra, all indecomposable representations are up to isomorphisms the following

$$\begin{array}{lll} N_1 = Q_2 :\ 0 \leftarrow k \to 0 & N_2 = P_1 :\ k \leftarrow 0 \to 0 & N_3 = P_3 :\ 0 \leftarrow 0 \to k \\ N_4 = P_2 :\ k \leftarrow k \to k & N_5 = Q_3 :\ 0 \leftarrow k \to k & N_6 = Q_1 :\ k \leftarrow k \to 0 \end{array}$$

$$\begin{tikzcd}
N_2 \arrow[dr] & & N_5 \arrow[dr] & \\
 & N_4 \arrow[ur] \arrow[dr] & & N_1 \\
N_3 \arrow[ur] & & N_6 \arrow[ur] &
\end{tikzcd}$$

Definition 2.7. Let $\Delta = (\bar{\Delta}, \Omega)$ be a locally finite quiver, and a a sink (resp., a source) of $(\bar{\Delta}, \Omega)$. We define the new orientation $\sigma_a \Omega$ by reversing all arrows which are connected to the vertex a. We call σ_a the reflection. For a sink a in a quiver $(\bar{\Delta}, \Omega)$, we define the Bernstein-Gelfand-Ponomarev reflection functor (the BGP reflection functor) $\sigma_a^+ : \mathsf{Rep}_k(\bar{\Delta}, \Omega) \to \mathsf{Rep}_k(\bar{\Delta}, \sigma_a \Omega)$ as follows. For a k-representation $M = (M(i); M(\alpha))$ of $(\bar{\Delta}, \Omega)$, let

$$0 \to \sigma_a^+ M(a) \xrightarrow{(\beta_\alpha)} \oplus_{\alpha \in a^{\geq}} M(t(\alpha)) \xrightarrow{\sum_\alpha M(\alpha)} M(a)$$

be the canonical exact sequence and

$$\sigma_a^+ M(x) = \begin{cases} \sigma_a^+ M(a) & \text{if} \quad x = a \\ M(x) & \text{if} \quad x \neq a \end{cases} \qquad \sigma_a^+ M(\alpha) = \begin{cases} \beta_\alpha & \text{if} \quad \alpha \in a^{\geq} \\ \alpha & \text{if} \quad \alpha \notin a^{\geq} \end{cases}$$

Then $\sigma_a^+ M = (\sigma_a^+ M(i); \sigma_a^+ M(\alpha))$ is a representation of $(\bar{\Delta}, \sigma_a \Omega)$. Similarly, for a source b in a quiver $(\bar{\Delta}, \Omega)$, the BGP reflection functor $\sigma_b^- : \mathsf{Rep}_k(\bar{\Delta}, \Omega) \to \mathsf{Rep}_k(\bar{\Delta}, \sigma_b \Omega)$ is defined.

Theorem 2.8. *Let $\Delta = (\bar{\Delta}, \Omega)$ be a locally finite quiver, and a a sink of $(\bar{\Delta}, \Omega)$. Let $\mathcal{T}_a$ (resp., $\mathcal{Y}_a$) be the subcategory of $\mathsf{Rep}_k(\bar{\Delta}, \Omega)$ (resp., $\mathsf{Rep}_k(\bar{\Delta}, \sigma_a \Omega)$) consisting representations which don't have S_a as a direct summand. Then the BGP reflection functors $\sigma_a^+ : \mathsf{Rep}_k(\bar{\Delta}, \Omega) \to \mathsf{Rep}_k(\bar{\Delta}, \sigma_a \Omega)$ and $\sigma_a^- : \mathsf{Rep}_k(\bar{\Delta}, \sigma_a \Omega) \to \mathsf{Rep}_k(\bar{\Delta}, \Omega)$ induce the equivalence between $\mathcal{T}_a$ and $\mathcal{Y}_a$. A similar result holds for $\sigma_a^+ : \mathsf{rep}_k(\bar{\Delta}, \Omega) \to \mathsf{rep}_k(\bar{\Delta}, \sigma_a \Omega)$ and $\sigma_a^- : \mathsf{rep}_k(\bar{\Delta}, \sigma_a \Omega) \to \mathsf{rep}_k(\bar{\Delta}, \Omega)$.*

Proof. By the construction of BGP reflection, we have the canonical functorial morphisms $\sigma_a^- \circ \sigma_a^+ \to \mathbf{1}_{\mathsf{Rep}_k(\bar{\Delta}, \Omega)}$ and $\mathbf{1}_{\mathsf{Rep}_k(\bar{\Delta}, \sigma_a \Omega)} \to \sigma_a^+ \circ \sigma_a^-$. For a representation $M = (M(i); M(\alpha))$ of $(\bar{\Delta}, \Omega)$, it is easy to see that $M \in \mathcal{T}_a$ if and only if $\sum_{\alpha \in a^{\geq}} \alpha$ is an epimorphism. Similarly, $N \in \mathcal{Y}_a$ if and only if $(\alpha)_{\alpha \in a^{\leq}}$ is a monomorphism for a representation $N = (N(i); N(\alpha))$ of $(\bar{\Delta}, \sigma_a \Omega)$. For $M \in \mathcal{T}_a$, we have a short exact sequence

$$0 \to \sigma_a^+ M(a) \xrightarrow{(\sigma_a^+ M(\alpha))_\alpha} \oplus_{\alpha \in a^{\geq}} M(t(\alpha)) \xrightarrow{\sum_\alpha M(\alpha)} M(a) \to 0$$

Then we have $\mathrm{Im}(\sigma_a^+|_{\mathcal{T}_a}) \subset \mathcal{Y}_a$, and $\mathbf{1}_{\mathsf{Rep}_k(\bar{\Delta}, \Omega)} \to \sigma_a^- \circ \sigma_a^+|_{\mathcal{T}_a}$ is an isomorphism. Similarly, $\mathrm{Im}(\sigma_a^-|_{\mathcal{Y}_a}) \subset \mathcal{T}_a$, and $\sigma_a^+ \circ \sigma_a^-|_{\mathcal{Y}_a} \to \mathbf{1}_{\mathsf{Rep}_k(\bar{\Delta}, \sigma_a \Omega)}$ is an isomorphism. □

Definition 2.9. Let $\bar{\Delta}$ be underlying graph of a quiver $\Delta = (\Delta_0, \Delta_1, h, t)$ with $\Delta_0 = \{1, \cdots, n\}$, and $(-,-)_{\bar{\Delta}}$ the associated symmetric bilinear form. For a vertex a of $\bar{\Delta}$ and $\mathbf{x} \in \mathbb{Z}^n$, we define the following reflection of $\mathbb{Z}^n$

$$\sigma_a(\mathbf{x}) = \mathbf{x} - 2(\mathbf{x}, \mathbf{e}_a)_{\bar{\Delta}} \mathbf{e}_a$$

Here $\mathbf{e}_a$ is the a-th fundamental vector. For $\{a_1, \cdots, a_n\} = \{1, \cdots, n\}$, $c = \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_n}$ is called a Coxeter transformation. Moreover, we define the group generated by reflections

$$\mathrm{W}_{\bar{\Delta}} = \{\sigma_{a_1} \cdots \sigma_{a_r} | r \geq 0, \ \sigma_{a_1}, \cdots, \sigma_{a_r} \text{ are reflections}\}$$

For $\mathbf{x} \in \mathbb{Z}^n$, $\mathbf{x}$ is called positive $\mathbf{x} > \mathbf{0}$ if $\mathbf{x} \neq \mathbf{0}$ and $x_i \geq 0$ $(1 \leq i \leq n)$, $\mathbf{x}$ is called a root if $\chi_{\bar{\Delta}}(\mathbf{x}) = 1$, and $\mathbf{x}$ is called a radical vector if $\chi_{\bar{\Delta}}(\mathbf{x}) = 0$.
In the case that $\bar{\Delta}$ is Dynkin, $\mathrm{W}_{\bar{\Delta}}$ is called a Weyl group.

Definition 2.10. Let $\Delta = (\bar{\Delta}, \Omega)$ be a quiver with. A sequence of vertices $\{a_1, \cdots, a_n\}$ is called an absorbing sequence (resp., diverging sequence) for $(\bar{\Delta}, \Omega)$ if a_{i+1} is a sink (resp., source) of $(\bar{\Delta}, \sigma_{a_i} \cdots \sigma_{a_1} \Omega)$ for any $0 \leq i < n$. For a finite quiver Δ which does not contain oriented cycles, we have both an absorbing sequence and a diverging sequence which is coincides with the set of vertices.

Corollary 2.11. *Let $\Delta = (\bar{\Delta}, \Omega)$ be a finite quiver, a a sink and b a source of $(\bar{\Delta}, \Omega)$, and M an indecomposable representation in $\mathsf{rep}_k(\bar{\Delta}, \Omega)$.*

(1) *If $\sigma_a^+ M = 0$, then $M \cong S_a$.*

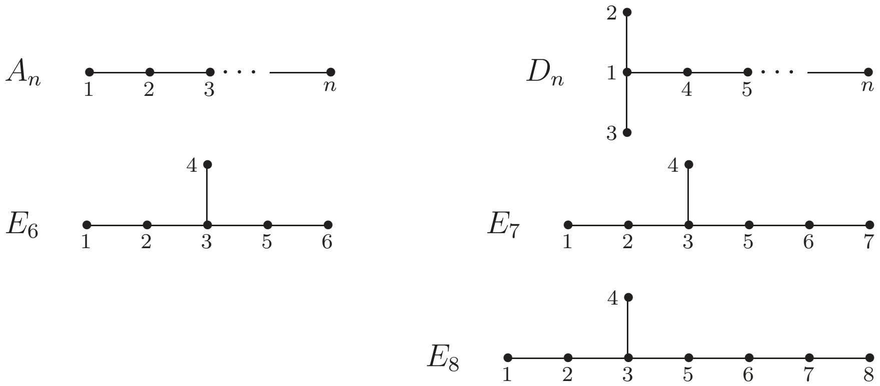

FIGURE 1. Dynkin graphs

(2) *If* $\sigma_a^+ M \neq 0$, *then* $\underline{\mathbf{dim}}\sigma_a^+ M = \sigma_a(\underline{\mathbf{dim}} M)$ *and* $\sigma_a^- \sigma_a^+ M \cong M$.
(3) *If* $\sigma_b^- M = 0$, *then* $M \cong S_b$.
(4) *If* $\sigma_b^- M \neq 0$, *then* $\underline{\mathbf{dim}}\sigma_b^- M = \sigma_b(\underline{\mathbf{dim}} M)$ *and* $\sigma_b^+ \sigma_b^- M \cong M$.

Example 2.12. In Examples 2.5 and 2.6, $\Delta' = (\bar{\Delta}, \sigma_2 \Omega)$, and we have the BGP reflections $\sigma_2^+ : \mathsf{rep}_k \Delta \to \mathsf{rep}_k \Delta'$ and $\sigma_2^- : \mathsf{rep}_k \Delta' \to \mathsf{rep}_k \Delta$ such that $\sigma_2^+ M_1 = 0$, $\sigma_2^- N_1 = 0$, $\sigma_2^+ M_i \cong N_i$ and $\sigma_2^- N_i \cong M_i$ $(2 \leq i \leq 6)$.

Theorem 2.13 (Root System and Weyl Group)**.** *Let* $\Delta = (\bar{\Delta}, \Omega)$ *be a quiver such that* $\bar{\Delta}$ *is a Dynkin diagram. Then the following hold.*

(1) *The Weyl group* $W_{\bar{\Delta}}$ *is a finite group.*
(2) *There is no radical vector except the zero vector* $\mathbf{0}$.
(3) *For any Coxeter transformation* c, $c\mathbf{v} = \mathbf{v}$ *implies* $\mathbf{v} = \mathbf{0}$.

Corollary 2.14. *Let* $\Delta = (\bar{\Delta}, \Omega)$ *be a quiver such that* $\bar{\Delta}$ *is a Dynkin diagram. For any indecomposable representation* $M \, \mathsf{rep}_k(\bar{\Delta}, \Omega)$, *there is a absorbing (resp,m diverging) sequence* $\{a_1, \cdots, a_s\}$ *and some vertex* a *such that* $M \cong \sigma_{a_s}^+ \cdots \sigma_{a_1}^+ S_a$ *(resp.,* $M \cong \sigma_{a_s}^- \cdots \sigma_{a_1}^- S_a$*).*

Proof. Let $\Delta_0 = \{1, \cdots, n\}$, $\{a_1, \cdots, a_n\}$ an absorbing (resp., diverging) sequence with $\{a_1, \cdots, a_n\} = \{1, \cdots, n\}$, and $c = \sigma_{a_n} \cdots \sigma_{a_1}$. Since $W_{\bar{\Delta}}$ is a finite group, there is an integer r such that $c^r = 1$. Let $\mathbf{v} = \sum_{i=1}^r c^i \underline{\mathbf{dim}} M$, then $c\mathbf{v} = \mathbf{v}$. By Theorem 2.13 (3) $\mathbf{v} = \mathbf{0}$, and therefore $c^i \underline{\mathbf{dim}} M \not> \mathbf{0}$ for some i. According to Corollary 2.11, we have the statement. □

Definition 2.15. Let $\Delta = (\bar{\Delta}, \Omega)$ be a finite connected quiver, and a a sink of $(\bar{\Delta}, \Omega)$. Then we have the canonical exact sequence in $\mathsf{Rep}_k(\bar{\Delta}, \Omega)$:

$$0 \to P_a \xrightarrow{(h^\alpha)_\alpha} \bigoplus_{\alpha \in a^{\geq}} P_{t(\alpha)} \xrightarrow{\sum_\alpha \sigma(\alpha)} T_a \to 0$$

We define the representation

$$T = T_a \oplus \bigoplus_{b \neq a} P_b$$

Proposition 2.16. *Let $\Delta = (\bar{\Delta}, \Omega)$ be a finite quiver, and a a sink of $(\bar{\Delta}, \Omega)$. By identifying $\mathsf{Rep}_k(\bar{\Delta}, \Omega)$ with $\mathsf{Mod}\, k\Delta$, then the following hold.*

(1) *The functor σ_a^+ is isomorphic to the functor $\mathrm{Hom}_{k\Delta}(T, -) : \mathsf{Mod}\, k(\bar{\Delta}, \Omega) \to \mathsf{Mod}\, k(\bar{\Delta}, \sigma_a\Omega)$.*

(2) $\mathcal{T}_a = \{M \in \mathsf{Rep}_k(\bar{\Delta}, \Omega) | \operatorname{Ext}^1_{k\Delta}(T, M) = 0\}$

Proof. (1) By Yoneda's lemma 2.4 we have the following isomorphism between exact sequences

$$\begin{tikzcd} \sigma_a^+ M(a) \arrow[r, "(\sigma_a^+ M(\alpha))_\alpha"] \arrow[d, "\wr"] & \oplus_{\alpha \in a^{\geq}} M(t(\alpha)) \arrow[r, "\sum_\alpha M(\alpha)"] \arrow[d, "\wr"] & M(a) \arrow[d, "\wr"] \\ \mathrm{Hom}(T_a, M) \arrow[r, "\mathrm{Hom}(\sum_\alpha \sigma(\alpha), M)"] & \mathrm{Hom}(\oplus_{\alpha \in a^{\geq}} P_{t(\alpha)}, M) \arrow[r, "\mathrm{Hom}((h^\alpha)_\alpha, M)"] & \mathrm{Hom}(P_a, M) \end{tikzcd}$$

(2) Since $\mathcal{T}_a$ is the subcategory consisting representations which don't have S_a as a direct summand, $M \in \mathcal{T}_a$ if and only if $\sum_\alpha M(\alpha) : \oplus_{\alpha \in a^{\geq}} M(t(\alpha)) \to M(a)$ is an epimorphism if and only if $\mathrm{Hom}((h^\alpha)_\alpha, M)$ is an epimorphism if and only if $\operatorname{Ext}^1(T, M) \cong \operatorname{Ext}^1(T_a, M) = 0$. □

Definition 2.17. Let $\mathcal{C}$ be an additive category. For $M \in \mathcal{C}$, We define $\mathsf{Add}\, M$ (resp., $\mathsf{add}\, M$) the full subcategory of $\mathcal{C}$ consisting of objects which are direct summands of coproducts (resp., finite coproducts) of copies of M.

Proposition 2.18. *Let $\Delta = (\bar{\Delta}, \Omega)$ be a finite quiver, and a a sink of $(\bar{\Delta}, \Omega)$, T a representation of Definition 2.15. Then the following hold.*

(1) $\mathrm{pdim}_{k\Delta} T \leq 1$.

(2) $\operatorname{Ext}^1_{k\Delta}(T, T) = 0$.

(3) *We have an exact sequence $0 \to \bigoplus_{x \in \Delta_0} P_x \to T^0 \to T^1 \to 0$ with $T^0, T^1 \in \mathsf{add}\, T$.*

Proof. By the definition of T, we have the exact sequence

$$0 \to \bigoplus_{x \in \Delta_0} P_x \to (\bigoplus_{x \in \Delta_0 \setminus \{a\}} P_x) \oplus (\bigoplus_{\alpha \in a^{\geq}} P_{t(\alpha)}) \to T_a \to 0$$

Then the statements (1) and (3) hold. If T_a has P_a as a direct summand, then so has $\bigoplus_{\alpha \in a^{\geq}} P_{t(\alpha)}$. This contradicts in a being a sink. Therefore T_a does not have P_a as a direct summand. By Proposition 2.16, we have $\operatorname{Ext}^1(T_a, T_a) = 0$. Similarly, $\operatorname{Ext}^1(T_a, P_b) = 0$ for $b \neq a$ because P_b does not have P_a as a direct summand. □

3. Tilting Modules

For a ring R, we denote by $\mathsf{Mod}\, R^{\mathrm{op}}$ (resp., $\mathsf{mod}\, R^{\mathrm{op}}$) the category of right (resp., finitely presented right) R-modules, and denote by $\mathsf{Proj}\, R^{\mathrm{op}}$ (resp., $\mathsf{proj}\, R^{\mathrm{op}}$, $\mathsf{Inj}\, R^{\mathrm{op}}$) the category of projective (resp., finitely projective, injective) R-modules.

Definition 3.1. Let R be a ring. A right R-module T is called a (classical) tilting module provided that the following hold.

(1) There is an exact sequence $0 \to P_1 \to P_0 \to T \to 0$ with $P_1, P_0 \in \mathsf{proj}\, R^{\mathrm{op}}$.
(2) $\mathrm{Ext}^1_R(T,T) = 0$.
(3) There is an exact sequence $0 \to R \to T^0 \to T^1 \to 0$ with $T^0, T^1 \in \mathsf{add}\, T$.

Lemma 3.2. *Let R be a ring, X a right R-module with $S = \mathrm{End}_R(X)$, and $X' \in \mathsf{add}\, X$.*

(1) $\mathrm{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\sim} \mathrm{Hom}_S(\mathrm{Hom}_R(X, X'), \mathrm{Hom}_R(X, Y))$
$(f \mapsto \mathrm{Hom}_R(X, f))$ *for all* $Y \in \mathsf{Mod}\, R^{\mathrm{op}}$.
(2) $\mathrm{Hom}_R(X, X') \otimes_S X \xrightarrow{\sim} X' \quad (f \otimes x \mapsto f(x))$
(3) $X' \xrightarrow{\sim} \mathrm{Hom}_S(\mathrm{Hom}_R(X', X), X) \quad (x' \mapsto (f \mapsto f(x')))$

Proof. Let $q_1, \cdots q_n : X' \to X$ and $p_1, \cdots, p_n : X \to X'$ be morphisms such that $\sum_{i=1}^n p_i q_i = 1$. Then the following are the inverse of the above:
$\mathrm{Hom}_S(\mathrm{Hom}_R(X, X'), \mathrm{Hom}_R(X, Y)) \to \mathrm{Hom}_{\mathcal{C}}(X', Y)(\phi \mapsto \sum_{i=1}^n \phi(p_i) q_i)$
$X' \to \mathrm{Hom}_R(X, X') \otimes_S X \quad (x' \mapsto \sum_{i=1}^n p_i \otimes q_i(x'))$
$\mathrm{Hom}_S(\mathrm{Hom}_R(X', X), X) \to X' \quad (\psi \mapsto \sum_{i=1}^n p_i \psi(q_i))$ □

Lemma 3.3. *Let A be a finite dimensional k-algebra, $M, N \in \mathsf{mod}\, A^{\mathrm{op}}$. Then there exists a morphism $f : M^{\oplus n} \to N$ such that* $\mathrm{Hom}(M, f) : \mathrm{Hom}_A(M, M^{\oplus n}) \to \mathrm{Hom}_A(M, N)$ *is surjective.*

Proof. Since $\mathrm{Hom}_A(M, N)$ is a finite dimensional k-vector space, we can take a k-basis $f_1, \cdots, f_n$ of $\mathrm{Hom}_A(M, N)$, and then $f = (f_1, \cdots, f_n) : M^{\oplus n} \to N$. □

Definition 3.4. Let A be a finite dimensional k-algebra, T_A a tilting right A-module. We define a pair of full subcategories of $\mathsf{mod}\, A^{\mathrm{op}}$

$$\mathcal{T}(T) = \{X \in \mathsf{mod}\, A^{\mathrm{op}} : \mathrm{Ext}^1_A(T, X) = 0\},$$
$$\mathcal{F}(T) = \{X \in \mathsf{mod}\, A^{\mathrm{op}} : \mathrm{Hom}_A(T, X) = 0\}.$$

For any $X \in \mathsf{mod}\, A^{\mathrm{op}}$, we define a subobject of X

$$t_T(X) = \sum\nolimits_{f \in \mathrm{Hom}_A(T, X)} \mathrm{Im}\, f$$

and an exact sequence in $\mathsf{mod}\, A^{\mathrm{op}}$

$$(e_X) : 0 \to t_T(X) \xrightarrow{j_X} X \to f_T(X) \to 0.$$

Definition 3.5. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories in an abelian category $\mathcal{A}$ is called a torsion pair of $\mathcal{A}$ provided that the following conditions are satisfied:

(i) $\mathcal{T} \cap \mathcal{F} = \{0\}$;
(ii) $\mathcal{T}$ is closed under factor objects;
(iii) $\mathcal{F}$ is closed under subobjects;
(iv) for any object X of $\mathcal{A}$, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

Proposition 3.6. *Let A be a finite dimensional k-algebra, T_A a tilting right A-module. Then $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair of $\mathsf{mod}\, A^{\mathrm{op}}$ such that $\mathcal{T}(T)$ is the category of finitely generated right A-modules which are generated by T.*

Proof. It is clear that $\mathcal{F}(T)$ is closed under submodules. Since $\mathrm{Ext}^2_A(T,-)=0$, $\mathcal{T}(T)$ is closed under factor modules. For any $X \in \mathsf{mod}\, A^{\mathrm{op}}$, we have an exact sequence

$$0 \to \mathrm{Hom}(T, t_T(X)) \xrightarrow{\sim} \mathrm{Hom}(T, X) \to \mathrm{Hom}(T, f_T(X)) \to \mathrm{Ext}^1(T, t_T X)$$

Since $\mathrm{Ext}^1(T, tX) = 0$, we have $\mathrm{Hom}(T, f_T(X)) = 0$, and hence $t_T(X) \in \mathcal{T}(T), f_T(X) \in \mathcal{F}(T)$. For any $Y \in \mathsf{mod}\, A^{\mathrm{op}}$, we have an exact sequence

$$0 \to \mathrm{Hom}(T^1, Y) \to \mathrm{Hom}(T^0, Y) \to \mathrm{Hom}(A, Y) \to \mathrm{Ext}^1(T^1, Y)$$

If $Y \in \mathcal{T}(T) \cap \mathcal{F}(T)$, then $\mathrm{Hom}(T^0, Y) = \mathrm{Ext}^1(T^1, Y) = 0$. Therefore $Y \cong \mathrm{Hom}(A, Y) = 0$. □

Proposition 3.7. *Let A be a finite dimensional k-algebra, T_A a tilting right A-module with $B = \mathrm{End}_A(T)$. Then the following hold for $M, N \in \mathcal{T}(T)$.*

1. *We have an exact sequence $\cdots \to T_1 \to T_0 \to M \to 0$ ($T_i \in \mathsf{add}\, T$) such that $\cdots \to \mathrm{Hom}_A(T, T_1) \to \mathrm{Hom}_A(T, T_0)$ is a projective resolution of* $\mathrm{Hom}_A(T, M)$.
2. $\mathrm{Tor}^B_1(\mathrm{Hom}_A(T, M), T) = 0$.
3. $\mathrm{Hom}_A(T, M) \otimes_B T \cong M$.
4. $\mathrm{Ext}^i_A(M, N) \xrightarrow{\sim} \mathrm{Ext}^i_B(\mathrm{Hom}_A(T, M), \mathrm{Hom}_A(T, N))$ *for any i.*

Proof. By Lemma 3.3, We have exact sequences $0 \to M_{i+1} \to T_i \to M_i \to 0$ such that $M_0 = M$, $T_i \in \mathsf{add}\, T$ and $M_i \in \mathcal{T}(T)$ for any i. Then we have exact sequences

$$0 \to \mathrm{Hom}_A(T, M_{i+1}) \to \mathrm{Hom}_A(T, T_i) \to \mathrm{Hom}_A(T, M_i) \to 0$$

Therefore, the resolution $T_\bullet \to M$ satisfies that $\mathrm{Hom}_A(T, T_\bullet) \to \mathrm{Hom}_A(T, M)$ is a projective resolution. By Lemma 3.2 we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(T, T_\bullet) \otimes_B T & \longrightarrow & \mathrm{Hom}_A(T, M) \otimes_B T \\ \downarrow \wr & & \downarrow \wr \\ T_\bullet & \longrightarrow & M \end{array}$$

For $N \in \mathcal{T}(T)$, we have an exact sequence and an isomorphism

$$0 \to \mathrm{Hom}_A(M_i, N) \to \mathrm{Hom}_A(T_i, N) \to \mathrm{Hom}_A(M_{i+1}, N) \to \mathrm{Ext}^1_A(M_i, N) \to 0$$
$$\mathrm{Ext}^{j+1}_A(M_{i+1}, N) \cong \mathrm{Ext}^{j+2}_A(M_i, N)$$

for any $i, j \geq 0$. By Lemma 3.2 we have

$$\begin{aligned} \mathrm{Ext}^i_B(\mathrm{Hom}_A(T, M), \mathrm{Hom}_A(T, N)) &\cong \mathrm{H}^i(\mathrm{Hom}_B(\mathrm{Hom}_A(T, T_\bullet), \mathrm{Hom}_A(T, N)) \\ &\cong \mathrm{H}^i(\mathrm{Hom}_A(T_\bullet, N)) \\ &\cong \begin{cases} \mathrm{Hom}_A(M, N) \quad (i = 0) \\ \mathrm{Ext}^1_A(M_i, N) \cong \mathrm{Ext}^i_A(M, N) \quad (i > 0) \end{cases} \end{aligned}$$

□

Proposition 3.8. *Let A be a finite dimensional k-algebra, T_A a tilting right A-module with $B = \mathrm{End}_A(T)$. Then the following hold for $M \in \mathsf{mod}\, A^{\mathrm{op}}$ and $N \in \mathsf{mod}\, B^{\mathrm{op}}$.*

(1) $0 \to \mathrm{Hom}_A(T, M) \otimes_B T \xrightarrow{\epsilon_M} M \to \mathrm{Tor}_1^B(\mathrm{Ext}_A^1(T, M), T) \to 0$
$\mathrm{Ext}_A^1(T, M) \otimes_B T = 0$
(2) $0 \to \mathrm{Ext}_A^1(T, \mathrm{Tor}_1^B(N, T)) \to N \to \mathrm{Hom}_A(T, N \otimes_B T) \to 0$
$\mathrm{Hom}_A(T, \mathrm{Tor}_1^B(N, T)) = 0$

Proof. By applying $\mathrm{Hom}_A(-, T)$ to the exact sequence $0 \to A \to T^0 \to T^1 \to 0$, we have a projective resolution of ${}_BT$: $0 \to Q_1 \to Q_0 \to T \to 0$.
(1) Let $M \to I^\bullet$ be an injective resolution and $F := \mathrm{Hom}_A(T, -)$, then by Proposition 3.7 we have the exact sequence

$$0 \to F(I^\bullet) \otimes_B Q_1 \to F(I^\bullet) \otimes_B Q_0 \to F(I^\bullet) \otimes_B T \to 0$$

By Proposition 3.7 we have $F(I^\bullet) \otimes_B T \cong I^\bullet$. Therefore we have the exact sequence

$$0 \to F(M) \otimes_B Q_1 \to F(M) \otimes_B Q_0 \to M \to \mathrm{Ext}_A^1(T, M) \otimes_B Q_1 \to \mathrm{Ext}_A^1(T, M) \otimes_B Q_0 \to 0$$

(2) Let $L_\bullet \to N$ be a projective resolution. Applying $\mathrm{Hom}(-, L_\bullet \otimes_B T)$ to the projective resolution $0 \to P_1 \to P_0 \to T \to$, we have the exact sequence

$$0 \to \mathrm{Hom}_A(T, L_\bullet \otimes_B T) \to \mathrm{Hom}_A(P_0, L_\bullet \otimes_B T) \to \mathrm{Hom}_A(P_1, L_\bullet \otimes_B T) \to 0$$

Since $\mathrm{Hom}_A(T, L_\bullet \otimes_B T) \cong L_\bullet$, we have the exact sequence

$$0 \to \mathrm{Hom}_A(P_0, \mathrm{Tor}_1^B(N, T)) \to \mathrm{Hom}_A(P_1, \mathrm{Tor}_1^B(N, T)) \to N \to \mathrm{Hom}_A(P_0, L_\bullet \otimes_B T) \to \mathrm{Hom}_A(P_1, L_\bullet \otimes_B T) \to 0$$

$\square$

For a finite dimensional k-algebra A, let $S_1, \cdots S_n$ be a complete set of simple right A-modules. Let $\mathrm{F}(A)$ be the free abelian group generated by isomorphism classes $[X]$ of right A-modules $X \in \mathsf{mod}\, A^{\mathrm{op}}$, $\mathrm{R}(A)$ the subgroup of $\mathrm{F}(A)$ generated by $[Y] - [X] - [Z]$ for all exact sequence $0 \to X \to Y \to Z \to 0$, and the Grothendieck group of A is $\mathrm{K}_0(A) = \mathrm{F}(A)/\mathrm{R}(A)$. Then $\mathrm{K}_0(A)$ is generated by $S_1, \cdots S_n$, and hence $\mathrm{K}_0(A) \cong \mathbb{Z}^n$. For $M \in \mathsf{mod}\, A^{\mathrm{op}}$, we define $\underline{\mathbf{dim}}M := (\#S_i\text{-composition factor of M})_i$.

Theorem 3.9. *Let A be a finite dimensional k-algebra, T_A a tilting right A-module with $B = \mathrm{End}_A(T)$. Let $F = \mathrm{Hom}_A(T, -), F' = \mathrm{Ext}_A^1(T, -), G = - \otimes_B T, G' = \mathrm{Tor}_1^B(-, T)$, and $\mathcal{X}(T) = \mathrm{Ker}\, G$, $\mathcal{Y}(T) = \mathrm{Ker}\, G'$ Then the following hold.*
(1) *$(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair of $\mathsf{mod}\, B^{\mathrm{op}}$.*
(2) *F and G induce the equivalence between $\mathcal{T}(T)$ and $\mathcal{Y}(T)$.*
(3) *F' and G' induce the equivalence between $\mathcal{F}(T)$ and $\mathcal{X}(T)$.*
(4) *$FG' = F'G = 0$ and $GF' = G'F = 0$.*
(5) *${}_BT$ is a tilting left B-module with $A^{\mathrm{op}} \cong \mathrm{End}_B(T)$.*
(6) *Let $f : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ be a function defined by $f(\underline{\mathbf{dim}}M) = \underline{\mathbf{dim}}F(M) - \underline{\mathbf{dim}}F'(M)$ for $M \in \mathsf{mod}\, A^{\mathrm{op}}$, then f is a group isomorphism.*

Proof. (4) $G(N) \in \mathcal{T}$ implies $F'G = 0$. By Proposition 3.7 $G'F = 0$. By Proposition 3.8 $FG' = 0$ and $GF' = 0$.
(1) By Proposition 3.8, $N \in \mathcal{X}(T) \cap \mathcal{Y}(T)$ implies $N = 0$. By (4) and Proposition 3.8 $(\mathcal{X}(T), \mathcal{Y}(T))$ is a torsion pair of $\mathsf{mod}\, B^{\mathrm{op}}$

(2), (3) By Proposition 3.8.
(5) Applying $(-)^{**} = \mathrm{Hom}_B(\mathrm{Hom}_A(-,T),T)$ to $0 \to A \to T^0 \to T^1 \to 0$, by Lemma 3.2 we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\wr"] & T^0 \arrow[r] \arrow[d, "\wr"] & T^1 \arrow[r] \arrow[d, "\wr"] & 0 \arrow[d, "\wr"] & \\
0 \arrow[r] & A^{**} \arrow[r] & T^{0**} \arrow[r] & T^{1**} \arrow[r] & \mathrm{Ext}^1_B(T,T) \arrow[r] & 0
\end{tikzcd}$$

where all vertical arrows are isomorphisms. It is easy to see that the composition $A \xrightarrow{\sim} A^{**} \xrightarrow{\sim} \mathrm{End}_B(T)$ is an anti-ring isomorphism.
(6) For an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathsf{mod}\, A^{\mathrm{op}}$, we have an exact sequence

$$0 \to F(X) \to F(Y) \to F(Z) \to F'(X) \to F'(Y) \to F'(Z) \to 0$$

Then $f(\underline{\mathbf{dim}}Y) = f(\underline{\mathbf{dim}}X) + f(\underline{\mathbf{dim}}Z)$ and f is a group morphism. By Lemma 3.8, f is an epimorphism, and $\mathrm{rank}\,\mathrm{K}_0(A) \geq \mathrm{rank}\,\mathrm{K}_0(B)$. By (5) we have $\mathrm{rank}\,\mathrm{K}_0(A^{\mathrm{op}}) \leq \mathrm{rank}\,\mathrm{K}_0(B^{\mathrm{op}})$. Therefore f is an isomorphism. $\square$

Theorem 3.10. *Let A be a finite dimensional k-algebra, T_A a tilting right A-module with $B = \mathrm{End}_A(T)$. Then the following hold.*

1. *For $M \in \mathcal{T}(T)$, $\mathrm{idim}\, F(M) \leq \mathrm{idim}\, M + 1$.*
2. *For $N \in \mathcal{F}(T)$, $\mathrm{idim}\, F'(N) \leq \mathrm{idim}\, N$ and $\mathrm{Ext}^n_B(F(-), F'(N)) = 0$ if $\mathrm{idim}\, N = n$.*
3. *For a right injective A-module I, we have a functorial isomorphism $\mathrm{Hom}_A -, I)|_{\mathcal{F}(T)} \cong \mathrm{Ext}^1_A(F'(-), F(I))|_{\mathcal{F}(T)}$.*

Proof. Let $0 \to M \to I^0 \to \cdots \to I^n \to 0$ be an injective resolution.
(1) Since any injective right A-module belong to $\mathcal{T}(T)$ and $\mathcal{T}(T)$ is closed under factor modules, we have an exact sequence $0 \to F(M) \to F(I^0) \to \cdots \to F(I^n) \to 0$. $F(A^\vee) \cong T^\vee$ implies $\mathrm{idim}\, F(A^\vee) \leq 1$. By $F(I) \in \mathsf{add}\, F(A^\vee)$, we have $\mathrm{idm}\, F(M) \leq n+1$.
(2) Assume that $\mathrm{idim}\, N \leq n$. Let $0 \to N \to Q \to K \to 0$ be an exact sequence with Q being injective, then we have an exact sequence $0 \to F(Q) \to F(K) \to F'(N) \to 0$. Since $\mathrm{idim}\, F(Q) \leq 1$ and $\mathrm{idim}\, F(K) \leq n-1+1$, $\mathrm{idim}\, F'(N) \leq n$. For $M \in \mathcal{T}(T)$, we have an exact sequence

$$\mathrm{Ext}^n_B(F(M), F(K)) \to \mathrm{Ext}^n_B(F(M), F'(N)) \to \mathrm{Ext}^{n+1}_B(F(M), F(Q))$$

By (1) and $\mathrm{Ext}^n_B(F(M), F(K)) \cong \mathrm{Ext}^n_A(M, K) = 0$, we have $\mathrm{Ext}^n_B(F(M), F'(N)) = 0$.
(3) By Proposition 3.7 we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(K,I) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(Q,I) \arrow[r] \arrow[d, "\wr"] & \mathrm{Hom}_A(N,I) \arrow[r] \arrow[d, "\alpha_N"] & 0 \\
\mathrm{Hom}_B(F(K),F(I)) \arrow[r] & \mathrm{Hom}_B(F(Q),F(I)) \arrow[r] & \mathrm{Ext}^1_B(F'(N),F(I)) \arrow[r] & \mathrm{Ext}^1_B(F(K),F(I))
\end{tikzcd}$$

Since $\mathrm{Ext}^1_B(F(K), F(I)) \cong \mathrm{Ext}^1_A(K, I) = 0$, α_N is an isomorphism. $\square$

Corollary 3.11. $\mathrm{gldim}\, B \leq \mathrm{gldim}\, A + 1$.

Corollary 3.12. *If* $\operatorname{gldim} A \leq 1$, *then* $(\mathcal{X}(T), \mathcal{Y}(T))$ *splits, that is*

$$\operatorname{Ext}^1_B(\mathcal{X}(T), \mathcal{Y}(T)) = 0$$

Lemma 3.13 (Bongartz's lemma)**.** *Let A be a finite dimensional k-algebra, T_A a finitely generated right A-module such that* $\operatorname{pdim} T \leq 1$ *and* $\operatorname{Ext}^1_A(T, T) = 0$. *Then there exists a finitely generated right A-module T' such that $T \oplus T'$ is a tilting module.*

Proof. Let $e_1, \cdots, e_n$ be a k-basis of $\operatorname{Ext}^1_A(T, A)$. Consider the push-out diagram

$$\begin{array}{rcccccccc}
\oplus_{i=1}^n e_i : 0 & \longrightarrow & A^{\oplus n} & \longrightarrow & \oplus X_i & \longrightarrow & T^{\oplus n} & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\downarrow & & \Big\| & & \\
e : 0 & \longrightarrow & A & \longrightarrow & T' & \longrightarrow & T^{\oplus n} & \longrightarrow & 0
\end{array}$$

then we have an exact sequence

$$\operatorname{Hom}_A(T, T^{\oplus n}) \xrightarrow{\delta} \operatorname{Ext}^1_A(T, A) \to \operatorname{Ext}^1_A(T, T') \to 0$$

By the construction of e, δ is an epimorphism, and hence $\operatorname{Ext}^1_A(T, T') = 0$. Moreover we have exact sequences

$$0 = \operatorname{Ext}^1_A(T^{\oplus n}, T) \to \operatorname{Ext}^1_A(T', T) \to \operatorname{Ext}^1_A(A, T) = 0$$
$$0 = \operatorname{Ext}^1_A(T^{\oplus n}, T') \to \operatorname{Ext}^1_A(T', T') \to \operatorname{Ext}^1_A(A, T') = 0$$

Therefore we have $\operatorname{Ext}^1_A(T \oplus T', T \oplus T') = 0$. It is clear that $\operatorname{pdim} T' \leq 1$. Hence $T \oplus T'$ is a tilting module. $\square$

Theorem 3.14. *Let A be a finite dimensional k-algebra, T_A a finitely generated right A-module such that* $\operatorname{pdim} T \leq 1$ *and* $\operatorname{Ext}^1_A(T, T) = 0$. *Then the following are equivalent.*

(1) *T is a tilting A-module.*
(2) *The number of non-isomorphic indecomposable modules which are direct summand of T is the number of non-isomorphic simple A-modules.*

Proof. (1) $\Rightarrow$ (2) By Theorem 3.10. (2) $\Rightarrow$ (1) By Lemma 3.13. $\square$

4. Triangulated Categories

Definition 4.1. A *triangulated category* $\mathcal{C}$ is an additive category together with
(1) an autofunctor $\Sigma : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ (i.e. there is Σ^{-1} such that $\Sigma \circ \Sigma^{-1} = \Sigma^{-1} \circ \Sigma = \mathbf{1}_{\mathcal{C}}$) called the *translation* (or suspension), and
(2) a collection $\mathcal{T}$ of sextuples (X, Y, Z, u, v, w):

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

called (*distinguished*) *triangles.* These data are subject to the following four axioms:
(TR1) (1) For a commutative diagram of which all vertical arrows are isomorphisms

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma(X')
\end{tikzcd}$$

if (X, Y, Z, u, v, w) is a (distinguished) triangle, then (X', Y', Z', u', v', w') is a (distinguished) triangle.
(2) Every morphism $u : X \to Y$ is embedded in a (distinguished) triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \qquad \begin{tikzcd} & Z \arrow[dl, "w"', "(1)" near end] & \\ X \arrow[rr, "u"] & & Y \arrow[ul, "v"'] \end{tikzcd}$$

(3) For any $X \in \mathcal{C}$,

$$X \xrightarrow{1} X \to 0 \to \Sigma(X)$$

is a (distinguished) triangle

(TR2) A sextuple

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$$

is a (distinguished) triangle if and only if

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X) \xrightarrow{-\Sigma(u)} \Sigma(Y)$$

is a (distinguished) triangle.

(TR3) For any (distinguished) triangles (X, Y, Z, u, v, w), (X', Y', Z', u', v', w') and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma(X) \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma(X') \end{tikzcd}$$

there exists $h : Z \to Z'$ which makes a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & \Sigma(X) \arrow[d, "\Sigma(f)"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma(X') \end{tikzcd}$$

(TR4) (Octahedral axiom) For any two consecutive morphisms $u : X \to Y$ and $v : Y \to Z$, if we embed u, vu and v in (distinguished) triangles (X, Y, Z', u, i, i'), (X, Z, Y', vu, k, k') and (Y, Z, X', v, j, j'), respectively, then there exist morphisms

$f : Z' \to Y'$, $g : Y' \to X'$ such that the following diagram commute

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, "v"] & Z' \arrow[r, "i'"] \arrow[d, dashed, "f"] & \Sigma(X) \arrow[d, equal] \\
X \arrow[r, "vu"] & Z \arrow[r, "k"] \arrow[d, "j"] & Y' \arrow[r, "k'"] \arrow[d, dashed, "g"] & \Sigma(X) \arrow[d, "\Sigma(u)"] \\
 & X' \arrow[r, equal] \arrow[d, "j'"] & X' \arrow[r, "j'"] \arrow[d, "\Sigma(i)j'"] & \Sigma(Y) \\
 & \Sigma(Y) \arrow[r, "\Sigma(i)"] & \Sigma(Z') &
\end{tikzcd}$$

and the third column is a triangle.
Sometimes, we write $X[i]$ for $\Sigma^i(X)$.

Definition 4.2 (∂-functor)**.** Let $\mathcal{C}$, $\mathcal{C}'$ be triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called a *∂-functor* (sometimes *exact functor*) provided that there is a functorial isomorphism $\alpha : F\Sigma_{\mathcal{C}} \xrightarrow{\sim} \Sigma_{\mathcal{C}'}F$ such that

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\alpha_X F(w)} \Sigma_{\mathcal{C}'}(F(X))$$

is a triangle in $\mathcal{C}'$ whenever $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma_{\mathcal{C}}(X)$ is a triangle in $\mathcal{C}$. Moreover, if a ∂-functor F is an equivalence, then F is called a *triangulated equivalence* . In this case, we denote by $\mathcal{C} \overset{\Delta}{\cong} \mathcal{C}'$.

For $(F, \alpha), (G, \beta) : \mathcal{C} \to \mathcal{C}'$ ∂-functors, a functorial morphism $\phi : F \to G$ is called a *∂-functorial morphism* if

$$(\Sigma_{\mathcal{C}'}\phi) \circ \alpha = \beta \circ \phi\Sigma_{\mathcal{C}}$$

We denote by $\partial(\mathcal{C}, \mathcal{C}')$ the collection of all ∂-functors from $\mathcal{C}$ to $\mathcal{C}'$, and denote by $\partial \operatorname{Mor}(F, G)$ the collection of ∂-functorial morphisms from F to G.

Proposition 4.3. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a ∂-functor between triangulated categories. If $G : \mathcal{C}' \to \mathcal{C}$ is a right (or left) adjoint of F, then G is also a ∂-functor.*

Definition 4.4. A contravariant (resp., covariant) additive functor $H : \mathcal{C} \to \mathcal{A}$ from a triangulated category $\mathcal{C}$ to an abelian category $\mathcal{A}$ is called a *homological functor* (resp., *a cohomological functor*), if for any triangle (X, Y, Z, u, v, w) in $\mathcal{C}$ the sequence

$$H(\Sigma(X)) \to H(Z) \to H(Y) \to H(X)$$
$$(\text{resp., } H(X) \to H(Y) \to H(Z) \to H(\Sigma(X))\ )$$

is exact. Taking $H(\Sigma^i(X)) = H^i(X)$, we have the long exact sequence:

$$\cdots \to H^{i+1}(X) \to H^i(Z) \to H^i(Y) \to H^i(X) \to \cdots$$
$$(\text{resp.,}\quad \cdots \to H^i(X) \to H^i(Y) \to H^i(Z) \to H^{i+1}(X) \to \cdots)$$

Proposition 4.5. *The following hold.*

(1) *If (X, Y, Z, u, v, w) is a triangle, then $vu = 0$, $wv = 0$ and $\Sigma(u)w = 0$.*

(2) *For any* $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(-,X) : \mathcal{C} \to \mathfrak{Ab}$ *(resp.,* $\mathrm{Hom}_{\mathcal{C}}(X,-) : \mathcal{C} \to \mathfrak{Ab}$*) is a homological functor (resp., a cohomological functor).*

(3) *For any homomorphism of triangles*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma(f)} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma(X') \end{array}$$

if two of f, g *and* h *are isomorphisms, then the rest is also an isomorphism.*

Proof. First, consider the following morphism between triangles

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma(X) \\ \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ 0 & \longrightarrow & M & = & M & \xrightarrow{w} & 0 \end{array}$$

(1) Taking $M = Z, \beta = v, \gamma = 1_Z$, we get the statement by (TR1) (3), (TR2), (TR3).
(2) Take β with $\beta \circ u = 0$, then there is γ by (TR2), (TR3) .
(3) By (2), we have a morphism between long exact sequences

$$\begin{array}{ccccccccc} \mathrm{h}_X & \xrightarrow{\mathrm{h}_u} & \mathrm{h}_Y & \xrightarrow{\mathrm{h}_v} & \mathrm{h}_Z & \xrightarrow{\mathrm{h}_w} & \mathrm{h}_{\Sigma(X)} & \xrightarrow{\mathrm{h}_{\Sigma(u)}} & \mathrm{h}_{\Sigma(Y)} \\ \downarrow{\scriptstyle \mathrm{h}_f} & & \downarrow{\scriptstyle \mathrm{h}_g} & & \downarrow{\scriptstyle \mathrm{h}_h} & & \downarrow{\scriptstyle \mathrm{h}_{\Sigma(f)}} & & \downarrow{\scriptstyle \mathrm{h}_{\Sigma(g)}} \\ \mathrm{h}_{X'} & \xrightarrow{\mathrm{h}_{u'}} & \mathrm{h}_{Y'} & \xrightarrow{\mathrm{h}_{v'}} & \mathrm{h}_{Z'} & \xrightarrow{\mathrm{h}_{w'}} & \mathrm{h}_{\Sigma(X')} & \xrightarrow{\mathrm{h}_{\Sigma(u')}} & \mathrm{h}_{\Sigma(Y')} \end{array}$$

Here $\mathrm{h}_M = \mathrm{Hom}_{\mathcal{C}}(-,M)$ for any object M. □

Proposition 4.6. *A triangle* $(X, Y, Z, u, v, 0)$ *is isomorphic to* $(X, Z \oplus X, Z, \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right], 0)$.

Proof. Since $\mathrm{Hom}_{\mathcal{C}}(Z,Z) \xrightarrow{0} \mathrm{Hom}_{\mathcal{C}}(Z, \Sigma(X))$, by Proposition 4.5, there is $s : Z \to Y$ such that $vs = 1_Z$. Then we have a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\mu} & Z \oplus X & \xrightarrow{\pi} & Z & \xrightarrow{0} & \Sigma(X) \\ \| & & {\scriptstyle \alpha}\downarrow & & \| & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{0} & \Sigma(X) \end{array}$$

where $\mu = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $\pi = \left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$, $\alpha = \left[\begin{smallmatrix} s & u \end{smallmatrix}\right]$. □

Definition 4.7 (Compact Object)**.** Let $\mathcal{C}$ be a triangulated category. An object $C \in \mathcal{C}$ is called a *compact* object in $\mathcal{C}$ if the canonical morphism

$$\coprod_{i \in I} \mathrm{Hom}_{\mathcal{C}}(C, X_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(C, \coprod_{i \in I} X_i)$$

is an isomorphism for any set $\{X_i\}_{i \in I}$ of objects (if $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$).

For a triangulated category $\mathcal{C}$, a set $\mathcal{S}$ of compact objects is called *a generating set* if $\mathrm{Hom}_{\mathcal{C}}(\mathcal{S}, X) = 0 \Rightarrow X = 0$, and if $\Sigma(\mathcal{S}) = \mathcal{S}$. A triangulated category $\mathcal{C}$ is *compactly generated* if $\mathcal{C}$ contains arbitrary coproducts, and if it has a generating set.

5. Derived Categories

Throughout this section, $\mathcal{A}$ is an abelian category and $\mathcal{B}$, $\mathcal{C}$ are additive subcategories of $\mathcal{A}$.

Definition 5.1 (Complex)**.** A (cochain) complex is a collection $X^{\cdot} = (X^n, d_X^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ of objects and morphisms of $\mathcal{B}$ such that $d_X^{n+1} d_X^n = 0$. A complex $X^{\cdot} = (X^n, d_X^n : X^n \to X^{n+1})_{n\in\mathbb{Z}}$ is called bounded below (resp., bounded above, bounded) if $X^n = 0$ for $n \ll 0$ (resp., $n \gg 0$, $n \ll 0$ and $n \gg 0$).

we define an objects of $\mathcal{A}$ for all $n \in \mathbb{Z}$

$$\begin{aligned} &\mathrm{Z}^n(X^{\cdot}) = \mathrm{Ker}\, d_X^n && \mathrm{B}^n(X^{\cdot}) = \mathrm{Im}\, d_X^{n-1} \\ &\mathrm{C}^n(X^{\cdot}) = \mathrm{Cok}\, d_X^{n-1} && \mathrm{H}^n(X^{\cdot}) = \mathrm{Z}^n(X^{\cdot}) / \mathrm{B}^n(X^{\cdot}) \\ & && \text{the } n\text{th cohomology,} \end{aligned}$$

A complex $X^{\cdot} = (X^n, d_X^n)$ is called a null complex if $\mathrm{H}^n(X^{\cdot}) = 0$ for all $n \in \mathbb{Z}$.

A morphism $f : X^{\cdot} \to Y^{\cdot}$ of complexes is a collection of morphisms $f^n : X^n \to Y^n$ satisfying $d_Y^n f^n = f^{n+1} d_X^n$ for any $n \in \mathbb{Z}$.

We denote by $\mathsf{C}(\mathcal{B})$ (resp., $\mathsf{C}^+(\mathcal{B})$, $\mathsf{C}^-(\mathcal{B})$, $\mathsf{C}^{\mathrm{b}}(\mathcal{B})$) the category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) of $\mathcal{B}$. An autofunctor $\Sigma : \mathsf{C}(\mathcal{B}) \to \mathsf{C}(\mathcal{B})$ is called translation if $(\Sigma(X^{\cdot}))^n = X^{n+1}$ and $(\Sigma(d_X))^n = -d_X^{n+1}$ for any complex $X^{\cdot} = (X^n, d_X^n)$.

In $\mathsf{C}(\mathcal{A})$, a morphism $u : X^{\cdot} \to Y^{\cdot}$ is called a quasi-isomorphism if $\mathrm{H}^n(u)$ is an isomorphism for any n.

In this section, "$*$" means "nothing", "$+$", "$-$" or "b".

Definition 5.2 (Truncations)**.** For a complex $X^{\cdot} = (X^i, d^i)$, we define the following truncations:

$$\begin{aligned} &\tau_{\geq n} X^{\cdot} : \ldots \to 0 \to X^n \to X^{n+1} \to X^{n+2} \to \ldots, \\ &\tau_{\leq n} X^{\cdot} : \ldots \to X^{n-2} \to X^{n-1} \to X^n \to 0 \to \ldots. \end{aligned}$$

Then we have exact sequences in $\mathsf{C}(\mathcal{A})$

$$O \to \tau_{\geq n}(X^{\cdot}) \to X^{\cdot} \to \tau_{\leq n+1}(X^{\cdot}) \to O$$

Definition 5.3 (Mapping Cone)**.** For $u \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, Y^{\cdot})$, the mapping cone of u is a complex $\mathrm{M}^{\cdot}(u)$ with

$$\begin{aligned} &\mathrm{M}^n(u) = X^{n+1} \oplus Y^n, \\ &d^n_{\mathrm{M}^{\cdot}(u)} = \begin{bmatrix} -d_X^{n+1} & 0 \\ u^{n+1} & d_X^n \end{bmatrix} : X^{n+1} \oplus Y^n \to X^{n+2} \oplus Y^{n+1}. \end{aligned}$$

$$\begin{tikzcd}
X^{\cdot} \arrow[d, "u"] & \cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "u^n"] & X^{n+1} \arrow[r] \arrow[d, "u^{n+1}"] & \cdots \\
Y^{\cdot} \arrow[d, "v"] & \cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] \arrow[d, "\binom{0}{1}"] & Y^{n+1} \arrow[r] \arrow[d, "\binom{0}{1}"] & \cdots \\
\mathrm{M}^{\cdot}(u) \arrow[d, "w"] & \cdots \arrow[r] & X^{n+1} \oplus Y^n \arrow[r, "d^n_{\mathrm{M}^{\cdot}(u)}"] \arrow[d, "(1\ 0)"] & X^{n+2} \oplus Y^{n+1} \arrow[r] \arrow[d, "(1\ 0)"] & \cdots \\
\Sigma(X^{\cdot}) & \cdots \arrow[r] & X^{n+1} \arrow[r, "-d_X^{n+1}"] & X^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 5.4 (Homotopy Category)**.** Two morphisms $f, g \in \mathrm{Hom}_{\mathsf{C}(\mathcal{B})}(X^{\cdot}, Y^{\cdot})$ is said to be *homotopic* (denote by $f \underset{h}{\simeq} g$) if there is a collection of morphisms $h = (h^n)$, $h^n : X^n \to Y^{n-1}$ such that

$$f^n - g^n = d_Y^{n-1} h^n + h^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$. The homotopy category $\mathsf{K}^*(\mathcal{B})$ of $\mathcal{B}$ is defined by

(1) $\mathrm{Ob}(\mathsf{K}^*(\mathcal{B})) = \mathrm{Ob}(\mathsf{C}^*(\mathcal{B}))$,
(2) $\mathrm{Hom}_{\mathsf{K}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) = \mathrm{Hom}_{\mathsf{C}^*(\mathcal{B})}(X^{\cdot}, Y^{\cdot}) / \underset{h}{\simeq}$ for $X^{\cdot}, Y^{\cdot} \in \mathrm{Ob}(\mathsf{K}^*(\mathcal{B}))$.

Proposition 5.5. *A category $\mathsf{K}^*(\mathcal{B})$ is a triangulated category whose distinguished triangles are defined to be isomorphic to*

$$X^{\cdot} \xrightarrow{u} Y^{\cdot} \xrightarrow{v} \mathrm{M}^{\cdot}(u) \xrightarrow{w} \Sigma(X^{\cdot})$$

for any $u : X^{\cdot} \to Y^{\cdot}$ in $\mathsf{K}^*(\mathcal{B})$.

Definition 5.6 (Derived Category)**.** The derived category $\mathsf{D}^*(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the quotient category by quasi-isomorphisms, that is the category satisfying

(1) $\mathrm{Ob}(\mathsf{D}^*(\mathcal{A})) = \mathrm{Ob}(\mathsf{K}^*(\mathcal{A}))$.
(2) For $X, Y \in \mathrm{Ob}(\mathsf{D}^*(\mathcal{A}))$, let $V(X^{\cdot}, Y^{\cdot}) = \{(s, Y'^{\cdot}, f) | s : Y^{\cdot} \to Y'^{\cdot} \in \mathsf{Qis}, f : X^{\cdot} \to Y^{\cdot}\}$. In $V(X^{\cdot}, Y^{\cdot})$, we define $(s, Y'^{\cdot}, f) \sim (s', Y''^{\cdot}, f')$ if there is $(s'', Y'''^{\cdot}, f')$ such that all triangles are commutative in the following diagram:

$$\begin{tikzcd}
 & Y'^{\cdot} \arrow[d, dashed] & \\
X^{\cdot} \arrow[ur, "f"] \arrow[r, dashed, "f''"] \arrow[dr, "f'"'] & Y'''^{\cdot} & Y^{\cdot} \arrow[ul, "s"'] \arrow[l, dashed, "s''"'] \arrow[dl, "s'"] \\
 & Y''^{\cdot} \arrow[u, dashed] &
\end{tikzcd}$$

Then we define a morphism from $X^{\cdot}$ to $Y^{\cdot}$ by an equivalence class $s^{-1}f$ of (s, Y', f).

(3) For $s^{-1}f : X^{\cdot} \to Y^{\cdot}, t^{-1}g : Y^{\cdot} \to Z^{\cdot}$, there are $s' : Z'^{\cdot} \to Z''^{\cdot} \in \mathsf{Qis}$ and $g' : Y'^{\cdot} \to Z''^{\cdot}$ such that $s' \circ g = g' \circ s$. Then we define $(t^{-1}g) \circ (s^{-1}f) = (s' \circ t)^{-1} g \circ f$.

$$\begin{tikzcd}
X^{\cdot} \arrow[dr, "f"] & Y^{\cdot} \arrow[d, "s"] \arrow[dr, "g"] & Z^{\cdot} \arrow[d, "t"] \\
 & Y'^{\cdot} \arrow[dr, dashed, "g'"] & Z'^{\cdot} \arrow[d, dashed, "s'"] \\
 & & Z''^{\cdot}
\end{tikzcd}$$

Moreover, we define the quotient functor $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ by

(Q1) $Q(X^{\cdot}) = X^{\cdot}$ for $X^{\cdot} \in \mathsf{K}^*(\mathcal{A})$.

(Q2) $Q(f) = 1_{Y^{\cdot}}^{-1} f$ for a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $\mathsf{D}^*(\mathcal{A})$.

Proposition 5.7. *The following hold.*

(1) $\mathsf{D}^*(\mathcal{A})$ *is a triangulated category, and the canonical functor* $Q : \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ *is a* ∂*-functor.*

(2) *The* i*-th cohomology of complexes is a cohomological functor in the sense of Definition 4.4.*

Lemma 5.8. *Let* A *be a ring. For* $X^\bullet \in \mathsf{K}(\mathsf{Mod}\, A)$ *and* $I^\bullet \in \mathsf{K}^+(\mathsf{Inj}\, A)$ *(resp.,* $P^\bullet \in \mathsf{K}^-(\mathsf{Proj}\, A)$*), if* $X^\bullet$ *is null, then we have*

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^\bullet, I^\bullet) = 0.$$
$$(\textit{resp.},\ \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(P^\bullet, X^\bullet) = 0)$$

Proposition 5.9. *The following hold for a ring* A*.*

(1) $\mathsf{K}^-(\mathsf{Proj}\, A) \overset{\triangle}{\cong} \mathsf{D}^-(\mathsf{Mod}\, A)$.

(2) $\mathsf{K}^+(\mathsf{Inj}\, A) \overset{\triangle}{\cong} \mathsf{D}^+(\mathsf{Mod}\, A)$.

6. Tilting Complexes

Definition 6.1. Let $\mathcal{C}$ be a triangulated category. A subcategory $\mathcal{B}$ of $\mathcal{C}$ is said to *generates* $\mathcal{C}$ as a triangulated category if $\mathcal{C}$ is the smallest triangulated full subcategory which is closed under isomorphisms and contains $\mathcal{B}$.

Theorem 6.2. *Let* A*,* B *be rings. The following are equivalent.*

(1) $\mathsf{D}^-(\mathsf{Mod}\, A) \overset{\triangle}{\cong} \mathsf{D}^-(\mathsf{Mod}\, B)$.

(2) $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \overset{\triangle}{\cong} \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$.

(3) $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A) \overset{\triangle}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, B)$.

(4) $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A) \overset{\triangle}{\cong} \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, B)$.

(5) *There exists* $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ *with* $B \cong \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(T^\bullet)$ *such that*
(a) $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^\bullet, T^\bullet[i]) = 0$ *for* $i \neq 0$*,*
(b) $\mathsf{add}\, T^\bullet_A$ *generates* $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

(6) *There exists* $T^{\cdot} \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ *with* $B \cong \mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(T^{\cdot})$ *such that*
   (a) $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^{\cdot}, T^{\cdot}[i]) = 0$ *for* $i \neq 0$,
   (b) For $X^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\, A)$, $X^{\cdot} = O$ *whenever* $\mathrm{Hom}_{\mathsf{K}^{-}(\mathsf{Proj}\, A)}(T^{\cdot}, X^{\cdot}[i])$ $= 0$ *for all* i.

Definition 6.3. A complex $T^{\cdot}_A \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ is called a tilting complex for A provided that

(1) $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^{\cdot}, T^{\cdot}[i]) = 0$ for $i \neq 0$.
(2) $\mathsf{add}\, T^{\cdot}_A$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

We say that B is *derived equivalent* to A if there is a tilting complex $T^{\cdot}_A$ such that $B \cong \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^{\cdot})$.

Remark 6.4. Miyashita defined a tilting module of finite projective dimension as follows. Let R be a ring. A right R-module T is called a tilting module of projective dimension n provided that the following hold.

(1) There is an exact sequence $0 \to P_n \to \cdots \to\to P_0 \to T \to 0$ with $P_0, \cdots, P_n \in \mathsf{proj}\, R^{\mathrm{op}}$.
(2) $\mathrm{Ext}^i_R(T, T) = 0$ $(i > 0)$.
(3) There is an exact sequence $0 \to R \to T^0 \to \cdots \to T^n \to 0$ with $T^0, \cdots, T^n \in \mathsf{add}\, T$.

Then the projective resolution of T is a tilting complex. Happel and Cline-Parshall-Scott showed that the derived functor $\boldsymbol{R}^{\mathrm{b}}\, \mathrm{Hom}_R(T, -) : \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, R) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, S)$ is an equivalence.

Lemma 6.5. *For* $X^{\cdot} \in \mathsf{D}^{-}(\mathsf{Mod}\, A)$, *the following are equivalent.*

(1) $X^{\cdot} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$.
(2) *For any* $Y^{\cdot} \in \mathsf{D}^{-}(\mathsf{Mod}\, A)$, *there is* n *such that* $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A}(Y^{\cdot}, X^{\cdot}[i])$ $= 0$ *for all* $i < n$.

Proof. $1 \Rightarrow 2$. We may assume $X^{\cdot} \in \mathsf{C}^{\mathrm{b}}(\mathsf{Mod}\, A)$, $Y^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\, A)$. Then $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(Y^{\cdot}, X^{\cdot}[i]) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(Y^{\cdot}, X^{\cdot}[i])$.

$2 \Rightarrow 1$. Since $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{b}})}(A, X^{\cdot}[i]) \cong \mathrm{H}^i(X^{\cdot})$, it is easy. □

For an additive category $\mathcal{B}$ and $m \leq n$, we write $\mathsf{K}^{[m,n]}(\mathcal{B})$ for the full subcategory of $\mathsf{K}(\mathcal{B})$ consisting of complexes $X^{\cdot}$ with $X^i = O$ for $i < m$, $n < i$.

Lemma 6.6. *For* $X^{\cdot} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, *the following are equivalent.*

(1) $X^{\cdot}$ *is isomorphic to an object of* $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\, A)$.
(2) *For any* $Y^{\cdot} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, *there is* n *such that* $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A}(X^{\cdot}, Y^{\cdot}[i])$ $= 0$ *for all* $i > n$.

Proof. $1 \Rightarrow 2$. It is trivial.

$2 \Rightarrow 1$. We may assume $X^{\cdot} \in \mathsf{K}^{-}(\mathsf{Proj}\, A)$. Let $M = \prod_{i \in \mathbb{Z}} \mathrm{C}^i(X^{\cdot})$. If $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(X^{\cdot}, \mathrm{C}^i(X^{\cdot})[-i]) = 0$, then we have exact sequences

$$\mathrm{Hom}_A(X^{i+1}, C^i(X^{\cdot})) \to \mathrm{Hom}_A(C^i(X^{\cdot}), C^i(X^{\cdot})) \to O.$$

This means that the canonical morphisms $C^i(X^\bullet) \to X^{i+1}$ are split monomorphisms. $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Mod}\,A)}(X^\bullet, M[i]) = 0$ for all $i > n$ if and only if $X^\bullet$ is isomorphic to an object in $\mathsf{K}^{[-n,\infty)}(\mathsf{Proj}\,A)$. $\square$

Definition 6.7 (Perfect Complex)**.** A complex $X^\bullet \in \mathsf{D}(\mathsf{Mod}\,A)$ is called *a perfect complex* if $X^\bullet$ is isomorphic to a complex of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ in $\mathsf{D}(\mathsf{Mod}\,A)$. We denote by $\mathsf{D}(\mathsf{Mod}\,A)_{\mathrm{perf}}$ the triangulated full subcategory of $\mathsf{D}(\mathsf{Mod}\,A)$ consisting of perfect complexes.

Lemma 6.8. *For $X^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\,A)$, the following are equivalent.*

(1) *$X^\bullet$ is a compact object in $\mathsf{K}^{\mathrm{b}}(\mathsf{Proj}\,A)$.*
(2) *$X^\bullet$ is isomorphic to an object of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$.*

Proof. $2 \Rightarrow 1$. It is easy.

$1 \Rightarrow 2$. Let $X^\bullet = X^0 \xrightarrow{d^0} X^1 \to \dots \to X^n$, with $X^i \in \mathsf{Proj}\,A$. By adding $P \xrightarrow{1} P$ to $X^\bullet$, we may assume that X^0 is a free A-module $A^{(I)}$. If I is a finite set, then by $2 \Rightarrow 1$ X^0 is also compact, and hence $\tau_{\geq 1}X^\bullet$ is compact. by induction on n, we get the assertion. Otherwise, since we have $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(X^\bullet, A^{(I)}) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(X^\bullet, A)^{(I)}$, the canonical morphism $X^\bullet \to A^{(I)}$ factors through a direct summand $\mu : A^m \hookrightarrow A^{(I)}$ for some $m \in \mathbb{N}$. Then there is a homotopy morphism $h : X^1 \to A^{(I)}$ such that $1_{A^{(I)}} - \mu g = hd^0$ with some $g : A^{(I)} \to A^m$. Let $A^{(I)} = A^m \oplus A^{(J)}$ be the canonical decomposition, then $A^{(J)} \xrightarrow{d^0|_{A^{(J)}}} X^1 \xrightarrow{ph} A^{(J)} = 1_{A^{(J)}}$, where $p : A^{(I)} \to A^{(J)}$ is the canonical projection. Therefore $X^\bullet \cong \mathrm{M}^\bullet(1_{A^{(J)}})[-1] \oplus X'^\bullet$, where $X'^\bullet : A^m \to X'^1 \to \dots \to X^n$ with X'^1 being a direct summand of X^1. Then we reduce the case of X^0 being a finitely generated free A-module. $\square$

Lemma 6.9. *Let $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathsf{Mod}\,A)}(T^\bullet, T^\bullet[i]) = 0$ for $i \neq 0$, and $B = \mathrm{End}_{\mathsf{K}(\mathsf{Mod}\,A)}(T)$. Then there exists a fully faithful ∂-functor $F : \mathsf{K}^-(\mathsf{Proj}\,B) \to \mathsf{K}^-(\mathsf{Proj}\,A)$ such that*

(1) *$FB \cong T^\bullet$.*
(2) *F preserves coproducts.*
(3) *F has a right adjoint $G : \mathsf{K}^-(\mathsf{Proj}\,A) \to \mathsf{K}^-(\mathsf{Proj}\,B)$.*

Proof. [Skip] This lemma is important. But the proof is out of the methods of derived categories. $\square$

Lemma 6.10. *If $T^\bullet$ satisfies the condition (G), then $F : \mathsf{K}^-(\mathsf{Proj}\,B) \to \mathsf{K}^-(\mathsf{Proj}\,A)$ is an equivalence.*

(G) *For $X^\bullet \in \mathsf{K}^-(\mathsf{Proj}\,A)$, $X^\bullet = O$ whenever $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\,A)}(T^\bullet, X^\bullet[i]) = 0$ for all i.*

Proof. Let $X^\bullet \in \mathsf{K}^-(\mathsf{Proj}\,A)$ such that $GX^\bullet = O$. Then $\mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\,A)}(T^\bullet, X^\bullet[i]) \cong \mathrm{Hom}_{\mathsf{K}^-(\mathsf{Proj}\,B)}(B, GX^\bullet[i]) = 0$ for all i. Therefore $\mathrm{Ker}\,G = \{O\}$. By the left version of Proposition 6.11, G and F are equivalences. $\square$

Proposition 6.11. *Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories, $F : \mathcal{C} \to \mathcal{C}'$ a ∂-functor which has a fully faithful left adjoint $S : \mathcal{C}' \to \mathcal{C}$. Then F induces an equivalence between $\mathcal{C}/\operatorname{Ker} F$ and $\mathcal{C}'$.*

Proof. By the universal property of $Q : \mathcal{C} \to \mathcal{C}/\operatorname{Ker} F$, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d, "Q"'] \arrow[dr, "F"] & \\ \mathcal{C}/\operatorname{Ker} F \arrow[r, "F'"'] & \mathcal{C}' \end{tikzcd}$$

If $f : X \to Y$ is a morphism in $\mathcal{C}$, then Ff is an isomorphism if and only if Qf is an isomorphism. For every object $M \in \mathcal{C}$, $FSFM \to FM$ is an isomorphism, and then $QSFM \to QM$ is an isomorphism. Therefore $QSF \to Q$ is an isomorphism. By the universal property of Q and $QSF = QSF'Q$, we have $\mathbf{1}_{\mathcal{C}/\operatorname{Ker} F} \cong QSF'$. Since, $F'QS = FS \cong \mathbf{1}_{\mathcal{C}'}$, F' is an equivalence. □

Remark 6.12. Let $\mathcal{C}$ be a triangulated category. For an additive subcategory $\mathcal{B}$ of $\mathcal{C}$, we can construct the smallest triangulated full subcategory $\mathcal{EB}$ which is closed under isomorphisms and contains $\mathcal{B}$ as follows.

Let $\mathcal{E}^0\mathcal{B} = \mathcal{B}$. For $n > 0$, let $\mathcal{E}^n\mathcal{B}$ be the full subcategory of $\mathcal{C}$ consisting of objects X there exist $U, V \in \mathcal{E}^{n-1}\mathcal{B}$ satisfying that either of $(X, U, V, *, *, *)$ or $(U, V, X, *, *, *)$ is a triangle in $\mathcal{C}$. Then it is easy to see that $\mathcal{EB} = \bigcup_{n\geq 0}\mathcal{E}^n\mathcal{B}$ is the smallest triangulated full subcategory which is closed under isomorphisms and contains $\mathcal{B}$

Theorem 6.13. *Let $T^{\cdot}$ be a complex of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ such that*

(a) $\operatorname{Hom}_{\mathsf{K}(\mathsf{Mod}\, A)}(T^{\cdot}, T^{\cdot}[i]) = 0$ *for* $i \neq 0$,

(b) $\mathsf{add}\, T^{\cdot}_A$ *generates* $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$.

Then $F : \mathsf{K}^{-}(\mathsf{Proj}\, B) \to \mathsf{K}^{-}(\mathsf{Proj}\, A)$ *is an equivalence.*

Proof. It suffices to show that $T^{\cdot}$ satisfies the condition of Lemma 6.10. Since $\mathsf{add}\, T^{\cdot}_A$ generates $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$, if $\operatorname{Hom}_{\mathsf{K}^{-}(\mathsf{Proj}\, A)}(T^{\cdot}, X^{\cdot}[i]) = 0$ for all i, then $\operatorname{Hom}_{\mathsf{K}^{-}(\mathsf{Proj}\, A)}(A, X^{\cdot}[i]) = 0$ for all i. Thus $X^{\cdot} = O$. □

References of Section

§1. [ASS], [草], [Ri], [Po], [Br].
§2. [ASS], [草], [Ri], [Po], [Br].
§3. [ASS], [Bo], [Ri], [AHK].
§4. [We], [Ve], [RD], [KV], [Ne3], [BBD] ([Ne1], [Ne2]).
§5. [We], [Ve], [RD].
§6. [Rd1], [Rd2], [AHK] ([BBD], [CPS], [Ha], [Mi]).

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