AN INTRODUCTION TO NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Abstract. There are several research fields called noncommutative algebraic geometry. In this note, we will introduce the one founded by M. Artin. Roughly speaking, in this research field, we study noncommutative algebras using ideas and techniques of algebraic geometry. Since classification of low dimensional schemes is one of the most active projects in algebraic geometry, classification of low dimensional noncommutative schemes is the main project in noncommutative algebraic geometry. In fact, since noncommutative projective curves were classified by Artin and Stafford, one of the most active projects in this field is to classify noncommutative projective surfaces. In this note, we will focus on the classification of the simplest noncommutative projective surfaces, namely, quantum projective planes, due to Artin, Tate, and Van den Bergh. We will also relate this project to the study of Frobenius Koszul algebras.

1. Overview

1.1. **Motivations.** In this note, we fix a field k. An algebra always means an algebra finitely generated over k, and a scheme always means a scheme of finite type over k. That is, every algebra is of the form $R = T(V)/I$ where V is a finite dimensional vector space over k, $T(V)$ is the tensor algebra on V over k, and I is a two-sided ideal of $T(V)$. By choosing a basis $\{x_1, \dots, x_n\}$ for V over k, we may also write $R = k\langle x_1, \dots, x_n\rangle/I$ where $k\langle x_1, \dots, x_n\rangle$ is the free algebra on $\{x_1, \dots, x_n\}$ over k. In particular, if R is commutative, then we may write $R = S(V)/I = k[x_1, \dots, x_n]/I$ where $S(V)$ is the symmetric algebra on V over k and $k[x_1, \dots, x_n]$ is the polynomial algebra on $\{x_1, \dots, x_n\}$ over k. A scheme of finite type over k is a scheme which can be covered by a finite number of affine schemes of commutative algebras finitely generated over k. We denote by $\operatorname{Mod} R$ the category of right R-modules, and by $\operatorname{mod} R$ the full subcategory of $\operatorname{Mod} R$ consisting of finitely generated ones.

Our impossible dream is to classify all algebras. Following algebraic geometry or algebraic topology, it is natural to start classifying algebras of low dimensions. Depending on the research fields, there are several candidates for which dimension function of an algebra we should use for this purpose. Since we follow the ideas of algebraic geometry, we use Gelfand-Kirillov dimension (GKdimension) defined below.

Definition 1. Let $R = T(V)/I$ be an algebra and $R_n = (k+V)^n$ the standard filtration of R. We define the Gelfand-Kirillov dimension (GKdimension) of R by

$$\operatorname{GKdim} R = \limsup_{n\to\infty} \log(\dim_k R_n)/\log n.$$

If R is a commutative algebra, then $\operatorname{GKdim} R = \operatorname{Kdim} R$, the Krull dimension of R.

Let R be an algebra. Since $\operatorname{GKdim} R = 0$ if and only if R is finite dimensional over k, classifying all algebras of GKdimension 0 is the same as classifying all artinian algebras, which is already an impossible dream even in the commutative case. There are two natural directions to proceed:

(1) Classify only nice algebras. The concept "nice" largely depends on the research fields. For example, in representation theory of finite dimensional algebras, classifying all Frobenius (self-injective) algebras is active.
(2) Classify algebras up to something weaker than isomorphism. This also depends on the research fields. For example, in representation theory of finite dimensional algebras, classifying algebras up to Morita equivalence, derived equivalence, stable equivalence, etc. is active.

In noncommutative algebraic geometry, we follow ideas of algebraic geometry. So we will first review the classification problem in algebraic geometry.

1.2. **Commutative Algebras.** Recall that every commutative algebra is of the form $R = k[x_1, \dots, x_n]/I$. Roughly speaking, the affine scheme associated to R is defined as a set by

$$X = \operatorname{Spec} R = \mathcal{V}(I) := \{p = (a_1, \dots, a_n) \in k^n \mid f(p) = 0 \text{ for all } f \in I\}$$

endowed with some topology, called Zariski topology, together with the sheaf of algebras $\mathcal{O}_X$ on it, called the structure sheaf. Note that if $I = (f_1, \dots, f_m)$ where $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, then

$$\mathcal{V}(I) = \{p = (a_1, \dots, a_n) \in k^n \mid f_1(p) = \cdots = f_m(p) = 0\}.$$

There are close relationships between an algebra R and a topological space $\operatorname{Spec} R$. For example:

(1) $\operatorname{Kdim} R = \dim \operatorname{Spec} R$.
(2) R is an integral domain if and only if $\operatorname{Spec} R$ is (reduced and) irreducible, that is, it is not a union of a finite number of smaller schemes. Such a scheme is often called an affine variety.
(3) R is regular if and only if $\operatorname{Spec} R$ is smooth, that is, it has no singularities.

We have the following lemma.

Lemma 2. *Let R, R' be commutative algebras. Then*

$$\begin{aligned} & R \cong R' \\ \iff & \operatorname{Mod} R \cong \operatorname{Mod} R' \\ \iff & \operatorname{Spec} R \cong \operatorname{Spec} R'. \end{aligned}$$

It follows that:

$$\text{Classifying commutative} \left\{ \begin{array}{c} \text{algebras} \\ \text{integral domains} \\ \text{regular algebras} \end{array} \right\} \text{of Krull dimension } d$$

up to isomorphism

$$\Longleftrightarrow \text{Classifying commutative} \left\{ \begin{array}{c} \text{algebras} \\ \text{integral domains} \\ \text{regular algebras} \end{array} \right\} \text{of Krull dimension } d$$

up to Morita equivalence

$$\Longleftrightarrow \text{Classifying affine} \left\{ \begin{array}{c} \text{schemes} \\ \text{varieties} \\ \text{smooth schemes} \end{array} \right\} \text{of dimension } d$$

up to isomorphism.

The following example suggests that the third classification is easiest.

Example 3. Let

- $R = \mathbb{R}[x, y]/(x^3 - y^2)$.
- $R' = \mathbb{R}[x, y]/(x^3 + x^2 - y^2)$.
- $R'' = \mathbb{R}[x, y]/(x^2 - 2xy - y^3 + 3y^2 - y)$.

It is not easy to find an algebra homomorphism between the above algebras nor a functor between modules categories over the above algebras. However, it is easy to see that

- $\operatorname{Spec} R = \mathcal{V}(x^3 - y^2)$ is a cuspidal curve.
- $\operatorname{Spec} R' = \mathcal{V}(x^3 + x^2 - y^2)$ is a nodal curve.
- $\operatorname{Spec} R'' = \mathcal{V}(x^2 - 2xy - y^3 + 3y^2 - y)$ is a nodal curve.

In fact, using algebraic geometry, we can show that

$$\operatorname{Spec} R \not\cong \operatorname{Spec} R' \cong \operatorname{Spec} R''.$$

As we have already mentioned, it is too difficult to classify all commutative algebras even of dimension 0, so it is too difficult to classify all affine schemes of dimension 0. In the classification of schemes in algebraic geometry, it is acceptable to assume that k is algebraically closed because classification problems become far more difficult if we do not assume so. It is also reasonable to classify only irreducible schemes (varieties) because every scheme is a finite union of irreducible ones. In dimension 0, there is only one affine variety up to isomorphism, namely a single point, or there is only one integral domain up to isomorphism, namely k itself. To classify higher dimensional ones, there are again two directions to proceed:

(1) Add more conditions on a scheme, namely, classify only smooth projective schemes.
(2) Classify schemes up to something weaker than isomorphism, namely, classify schemes up to birational equivalence.

We will explain both methods as below.

(1) **Classification of smooth projective schemes.** Unfortunately, classifying all smooth affine schemes of low dimensions is not yet easy. In algebraic topology, it is far

easier to classify only compact surfaces than to classify arbitrary surfaces. In this principle, we want to classify only "compact" schemes. One way to proceed is to "compactify" affine schemes to make them projective schemes. A basic process is as follows.

Definition 4. An algebra A is graded if it is endowed with a k-vector space decomposition $A = \oplus_{i\in\mathbb{Z}} A_i$ such that $A_iA_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{Z}$. Elements of A_i are called homogeneous of degree i. A right A-module M is graded if it is endowed with a k-vector space decomposition $M = \oplus_{i\in\mathbb{Z}} M_i$ such that $M_iA_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules, and by $\operatorname{grmod} A$ the full subcategory of $\operatorname{GrMod} A$ consisting of finitely generated ones.

In this note, a graded algebra always means a graded algebra finitely generated in degree 1 over k, that is, $A_i = 0$ for all $i < 0$, $A_0 = k$, and $A_i = A_1 \cdots A_1$ (i-factors) for all $i \geq 1$. Typical examples are the tensor algebra $T(V)$ where $T(V)_i = V^{\otimes i}$, and a polynomial algebra $k[x_1, \cdots, x_n]$ where $k[x_1, \cdots, x_n]_i = \{$ homogeneous polynomials of degree $i\} \cup \{0\}$. In fact, every graded algebra is of the form $A = T(V)/I$ where I is a homogeneous two-sided ideal of $T(V)$, that is, I is generated by homogeneous elements of $T(V)$. Again, by choosing a basis $\{x_1, \dots, x_n\}$ for V over k, we may also write $A = k\langle x_1, \dots, x_n\rangle/I$. In particular, if A is commutative, then we may write $A = k[x_1, \dots, x_n]/I$ where I is a homogeneous ideal of $k[x_1, \dots, x_n]$, that is, I is generated by (finitely many) homogeneous polynomials.

Recall that the projective space is defined by $\mathbb{P}^{n-1} = (k^n \setminus \{(0, \dots, 0)\})/\sim$ where $(a_1, \dots, a_n) \sim (\lambda a_1, \dots, \lambda a_n)$ for all $\lambda \in k \setminus \{0\}$. For example, $(a, b) = (c, d)$ in $\mathbb{P}^1$ if and only if $ad = bc$. Let $A = k[x_1, \dots, x_n]/I$ be a commutative graded algebra where I is a homogeneous ideal of $k[x_1, \dots, x_n]$. Roughly speaking, the projective scheme associated to A is defined as a set by

$$X = \operatorname{Proj} A = \mathcal{V}(I) := \{p = (a_1, \dots, a_n) \in \mathbb{P}^{n-1} \mid f(p) = 0 \text{ for all homogeneous } f \in I\}$$

endowed with some topology, called Zariski topology, together with the sheaf of algebras $\mathcal{O}_X$ on it, called the structure sheaf. Note that if $I = (f_1, \dots, f_m)$ where $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ are homogeneous polynomials, then

$$\mathcal{V}(I) = \{p = (a_1, \dots, a_n) \in \mathbb{P}^{n-1} \mid f_1(p) = \cdots = f_m(p) = 0\}.$$

Given a commutative algebra R, we can always homogenize the relations of R by adding one more generator t to make it a graded algebra $\tilde{R}$ generated in degree 1, so we can take $\operatorname{Proj} \tilde{R}$ as examples below. Moreover, we can recover the original algebra from $\tilde{R}$ by $R \cong (\tilde{R}[t^{-1}])_0$. Although $\operatorname{Proj} \tilde{R}$ is hardly compact in Zariski topology, it somehow behaves like a compact manifold in algebraic topology. Since $\operatorname{Spec} R$ is open and dense in $\operatorname{Proj} \tilde{R}$, this process looks like a compactification of $\operatorname{Spec} R$.

Example 5. Let

- $R = \mathbb{R}[x, y]/(x^2 - y)$ so that $\operatorname{Spec} R = \mathcal{V}(x^2 - y)$ is a parabola.
- $R' = \mathbb{R}[x, y]/(xy - 1)$ so that $\operatorname{Spec} R' = \mathcal{V}(xy - 1)$ is a hyperbola.
- $R'' = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ so that $\operatorname{Spec} R'' = \mathcal{V}(x^2 + y^2 - 1)$ is a circle.

It follows that none of the above affine schemes are isomorphic to one another. However,

- $\tilde{R} = \mathbb{R}[x,y,t]/(x^2 - yt)$ so that $\operatorname{Proj}\tilde{R} = \mathcal{V}(x^2 - yt)$ becomes a circle by adding one point $(0,1,0) \in \mathbb{P}^2$.
- $\tilde{R}' = \mathbb{R}[x,y,t]/(xy - t^2)$ so that $\operatorname{Proj}\tilde{R}' = \mathcal{V}(xy - t^2)$ becomes a circle by adding two points $(0,1,0),(1,0,0) \in \mathbb{P}^2$.
- $\tilde{R}'' = \mathbb{R}[x,y,t]/(x^2 + y^2 - t^2) = \mathbb{R}[x,y,t]/(x^2 - (y+t)(-y+t))$ so that $\operatorname{Proj}\tilde{R}'' = \mathcal{V}(x^2 + y^2 - t^2)$ is a circle as before.

It follows that

$$\operatorname{Proj}\tilde{R} \cong \operatorname{Proj}\tilde{R}' \cong \operatorname{Proj}\tilde{R}''.$$

(2) **Classification of schemes up to birational equivalence.** The second method is as follows:

Definition 6. We say that two (affine or projective) varieties X and Y are birationally equivalent if there are open dense subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$.

If R is an integral domain, then we can construct the field of quotients $Q(R) = \{ab^{-1} \mid a,b \in R, b \neq 0\}$ of R, which is an extension field of k. We define the function field of $X = \operatorname{Spec} R$ by $k(X) = Q(R)$. If A is a graded integral domain, then we can construct $Q_{gr}(A)_0 = \{ab^{-1} \mid a,b \in A \text{ are homogeneous of the same degree}, b \neq 0\}$, which is an extension field of k. We define the function field of $X = \operatorname{Proj} A$ by $k(X) = Q_{gr}(A)_0$.

Theorem 7. *Two (affine or projective) varieties X and Y are birationally equivalent if and only if $k(X) \cong k(Y)$.*

That is, two affine varieties $\operatorname{Spec} R$ and $\operatorname{Spec} R'$ are birationally equivalent if and only if $Q(R) \cong Q(R')$. It follows that classifying all varieties of dimension d up to birational equivalence is the same as classifying all extension fields of transcendence degree d.

Example 8. If $R = k[x,y]/(x^3 - y^2)$, then the map

$$\mathcal{V}(x^3 - y^2) \setminus \{(0,0)\} \to k \setminus \{0\};\ (a,b) \mapsto b/a$$

is an isomorphism. If $R' = k[x,y]/(x^3 + x^2 - y^2)$, then the map

$$\mathcal{V}(x^3 + x^2 - y^2) \setminus \{(0,0)\} \to k \setminus \{0\};\ (a,b) \mapsto b/a$$

is an isomorphism. So $\operatorname{Spec} R, \operatorname{Spec} R', \operatorname{Spec} k[t]$ are all birationally equivalent. In fact, $Q(R) \cong Q(R') \cong k(t)$.

By resolution of singularities, every variety is birationally equivalent to a smooth projective scheme, so classifying all varieties up to birational equivalence is weaker than classifying all smooth projective schemes up to isomorphism. However, in dimension 1, they are the same in the sense below. From now on, we will call a variety of dimension 1 a curve, and a variety of dimension 2 a surface.

Theorem 9. *For each curve X, there exists a unique smooth projective curve up to isomorphism which is birationally equivalent to X.*

It is a common agreement that the classification of curves had been completed in a sense that, for a each curve, there is a birational invariant g, called a genus. For each genus g, there is a variety $\mathfrak{M}_g$, called the variety of moduli of curves of genus g such that

$\dim \mathfrak{M}_g = \begin{cases} 0 & \text{if } g = 0 \\ 1 & \text{if } g = 1 \\ 3g-3 & \text{if } g \geq 2, \end{cases}$ which parameterizes all smooth projective curves of genus g.

In dimension 2, these two classifications are not the same. In fact, classifying all smooth projective surfaces up to isomorphism is difficult. On the other hand, the classification of surfaces up to birational equivalence is considered to be successful in a sense that, for each surface, there is a unique special surface, called the minimal model, which is birationally equivalent to it, except for a rational surface (a surface birationally equivalent to $\mathbb{P}^2$), and a birationally ruled surface (a surface birationally equivalent to $\mathbb{P}^1 \times C$ where C is a curve). For these two exceptions, minimal models are not unique, but they are well-known. A minimal model is important since every surface can be obtained by blowing-up a minimal model several times.

1.3. **Noncommutative Projective Varieties.** Now we turn to noncommutative algebras. Since classification of low dimensional varieties has been successful in algebraic geometry, we would like to classify noncommutative algebras (varieties) of low dimensions, following ideas and techniques of algebraic geometry. As in the commutative case, we restrict ourselves to domains over an algebraically closed field k. It is easy to see that the only domain of GKdimension 0 is k. The following theorem due to Small and Warfield is rather surprising.

Theorem 10. [17] *Every finitely generated domain of GKdimension 1 is commutative.*

The above theorem says that every noncommutative affine curve is in fact commutative. It is already too difficult to classify all domains of GKdimension 2. Since every noncommutative algebra can be homogenized as in the commutative case (see examples below), we will focus on graded algebras, and classify their associated projective schemes.

In modern algebraic geometry, the category $\operatorname{Mod} \mathcal{O}_X$ of quasi-coherent $\mathcal{O}_X$-modules play an essential role to study scheme X. In fact, every scheme X determines and is determined by the category $\operatorname{Mod} \mathcal{O}_X$ by Rosenberg [15], so it is reasonable to identify a scheme X with the category $\operatorname{Mod} \mathcal{O}_X$. The following classical result is due to Serre.

Theorem 11. [16] *If A is a commutative graded algebra finitely generated in degree 1 over k and $X = \operatorname{Proj} A$, then*

$$\operatorname{Mod} \mathcal{O}_X \cong \operatorname{GrMod} A / \operatorname{Fdim} A$$

where $\operatorname{Fdim} A$ is the full subcategory of $\operatorname{GrMod} A$ consisting of direct limits of finite dimensional modules over k.

The following definition of a noncommutative projective scheme due to Artin and Zhang was motivated by the above result.

Definition 12. [5] Let A be a graded algebra. We define a noncommutative projective scheme associated to A by the quotient category

$$\operatorname{Proj} A := \operatorname{GrMod} A / \operatorname{Fdim} A.$$

If A is a noetherian graded domain, then we define the function field of $X = \operatorname{Proj} A$ by

$$k(X) := Q_{gr}(A)_0 = \{ab^{-1} \mid a, b \in A \text{ are homogeneous of the same degree}, b \neq 0\}.$$

As usual, we define $\operatorname{proj} A := \operatorname{grmod} A / \operatorname{fdim} A$ where $\operatorname{fdim} A$ is the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules over k. Roughly speaking, objects in $\operatorname{proj} A$ are the same as those in $\operatorname{grmod} A$, but two modules $M, N \in \operatorname{grmod} A$ are isomorphic in $\operatorname{proj} A$ if and only if $M_{\geq n} \cong N_{\geq n}$ in $\operatorname{grmod} A$ for some n. Note that $k(X)$ is a division algebra over k. We have the following lemma.

Lemma 13. *Let A, A' be noetherian graded domains, and $X = \operatorname{Proj} A, X' = \operatorname{Proj} A'$. Then*

$$\begin{aligned} & A \cong A' \\ \Longrightarrow\ & \operatorname{GrMod} A \cong \operatorname{GrMod} A' \ \textit{(graded Morita equivalent)} \\ \Longrightarrow\ & X \cong X' \\ \Longrightarrow\ & k(X) \cong k(X') \ \textit{(birationally equivalent).} \end{aligned}$$

This implies that there are four levels of classifying noetherian graded domains. If A is a noetherian graded domain of GKdimension 2, then it is reasonable to call $\operatorname{Proj} A$ a noncommutative projective curve. Artin and Stafford [3] classified noncommutative projective curves in this sense as follows.

Let A be a noetherian graded domain of GKdimension 2, $X = \operatorname{Proj} A$, and $K = Z(k(X))$, the center of $k(X)$. Potentially, there are two possibilities, either $\operatorname{tr.deg}_k K = 0$ or $\operatorname{tr.deg}_k K = 1$, however, since $\operatorname{GKdim} k(X) = 1$, it follows that $k(X)$ is finite dimensional over its center K by Small and Warfield [17], so $\operatorname{tr.deg}_k K = 1$ and $k(X) \in \operatorname{Br}(K)$, the Brauer group of K. Since k is algebraically closed, $k(X) = K$ by Tsen's theorem. By the classification of commutative curves, there exists a curve E such that $k(X) \cong K \cong k(E)$. It says that every noncommutative projective curve is birationally equivalent to a commutative curve. Surprisingly, more is true.

Theorem 14. [3] *If A is a graded domain of GKdimension 2 generated in degree 1, then A is noetherian. Moreover, there is a pair (E, σ) where E is a curve, and $\sigma \in \operatorname{Aut} E$ such that*

$$A_n \cong \mathrm{H}^0(E, \mathcal{L} \otimes_E \sigma^* \mathcal{L} \otimes_E \cdots \otimes_E (\sigma^{n-1})^* \mathcal{L})$$

for all $n \gg 0$ where $\mathcal{L} = \mathcal{O}_E(1)$ is an ample invertible sheaf on E. In particular, $\operatorname{Proj} A \cong \operatorname{Mod} \mathcal{O}_E$.

The above theorem says that every noncommutative projective curve is isomorphic to a commutative curve, so the classification of noncommutative projective curve can be regarded as settled. We will see later that a geometric pair (E, σ) above play an essential role in classifying higher dimensional noncommutative projective schemes.

If A is a noetherian graded domain of GKdimension 3, then it is reasonable to call $\operatorname{Proj} A$ a noncommutative projective surface. The classification of noncommutative projective surfaces is wide open even up to birational equivalence, which is the same as the classification of division algebras of transcendence degree 2. Here is a conjecture due to Artin (slightly modified by the author).

Conjecture 15. [1] Let A be a noetherian graded domain of GKdimension 3, $X = \operatorname{Proj} A$, and $K = Z(k(X))$.

(1) (quantum rational surface) If $\operatorname{tr.deg}_k K = 0$ so that $K = k$, then $k(X) \cong K(\text{q-}\mathbb{P}^2)$ where q-$\mathbb{P}^2$ is a quantum projective plane defined later.
(2) (birationally quantum ruled surface) If $\operatorname{tr.deg}_k K = 1$ so that there is a curve E such that $K \cong k(E)$, then $k(X) \cong K(t; \sigma)$ for some $\sigma \in \operatorname{Aut} K$ (or $\sigma \in \operatorname{Aut} E$).
(3) If $\operatorname{tr.deg}_k K = 2$ so that there is a surface S such that $K \cong k(S)$, then $k(X) \in \operatorname{Br}(K)$, that is, $k(X)$ is finite dimensional over K.

Although classification of noncommutative projective surfaces is nowhere in sight, many important techniques to classify commutative surfaces has been extended to noncommutative settings:

(1) Serre's duality [6], [23].
(2) Intersection theory [7], [14].
(3) Riemann-Roch theorem [7], [9].
(4) Blowing up [21].

In the next section, we will focus on the classification of the simplest noncommutative surfaces called quantum projective planes.

2. Quantum Projective Planes

2.1. **AS-regular Algebras.** The simplest surface in algebraic geometry is the affine plane, which is $\operatorname{Spec} k[x, y]$, so the simplest noncommutative surfaces must be a "quantum" affine plane, which should be "$\operatorname{Spec} R$", where R is a noncommutative analogue of $k[x, y]$. There are some noncommutative algebras analogous to $k[x, y]$ such as:

- $R = k\langle x, y\rangle/(xy - \alpha yx)$ where $\alpha \in k \setminus \{0\}$ (a skew polynomial algebra).
- $R' = k\langle x, y\rangle/(xy - yx - x)$ (the enveloping algebra of the 2-dimension non-abelian Lie algebra).
- $R'' = k\langle x, y\rangle/(xy - yx - 1)$ (the 1st Weyl algebra).

All of the above algebras are regular algebras of GKdimension 2. Although these algebras can be regarded as coordinate rings of a "quantum" affine plane, we do not have a precise definition of it yet. As in the commutative case, we can homogenize any algebras. For example:

- $\tilde{R} = k\langle x, y\rangle[z]/(xy - \alpha yx) = k\langle x, y, z\rangle/(xy - \alpha yx, yz - zy, zx - xz)$.
- $\tilde{R}' = k\langle x, y\rangle[z]/(xy - yx - xz) = k\langle x, y, z\rangle/(xy - yx - xz, yz - zy, zx - xz)$.
- $\tilde{R}'' = k\langle x, y\rangle[z]/(xy - yx - z^2) = k\langle x, y, z\rangle/(xy - yx - z^2, yz - zy, zx - xz)$.

All of the above algebras are regular graded algebras of GKdimension 3 and of global dimension 3. Since the only commutative regular graded algebra is the polynomial algebra, it may be reasonable to define a "quantum" projective space of dimension d as $\operatorname{Proj} A$ for some regular graded algebra of $\operatorname{gldim} A = d + 1$. It is easy to see that the only regular graded algebra of $\operatorname{gldim} A = 0$ is k. However, since the global dimension of a free algebra $k\langle x_1, \dots, x_n\rangle$ is 1, we must add some additional conditions on a regular graded algebra to make the definition more reasonable. Although the conditions such as A is noetherian and A is a domain are most reasonable, it turns out that these conditions are very difficult to check in practice. One of the ingenuous ideas of Artin and Schelter was to define AS-regular algebras as below and classify them up to dimension 3.

Definition 16. [2] A graded algebra A is called a d-dimensional AS-regular algebra if

(1) $\operatorname{gldim} A = d < \infty$.
(2) $\operatorname{GKdim} A < \infty$.
(3) A satisfies Gorenstein condition, that is, $\dim_k \operatorname{Ext}^i_A(k,k) < \infty$ for all i, and $\operatorname{Ext}^i_A(k,A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

If A is a $(d+1)$-dimensional quadratic AS-regular algebra, then we call $\operatorname{Proj} A$ a d-dimensional quantum projective space. In particular, if A is a 3-dimensional quadratic AS-regular algebra, then we call $\operatorname{Proj} A$ a quantum projective plane.

Let A be a graded algebra. The opposite graded algebra of A is denoted by A^o and the enveloping graded algebra of A is denoted by $A^e = A^o \otimes_k A$. A graded left A-module will be identified with the graded right A^o-module, and a graded A-A bimodule will be identified with a graded right A^e-module. Then Gorenstein condition above is equivalent to the following condition: if

$$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to k \to 0$$

is the minimal free resolution of $k \in \operatorname{GrMod} A$, then $F_i \in \operatorname{grmod} A$ and

$$0 \to F_0^\vee \to F_1^\vee \cdots \to F_{d-1}^\vee \to F_d^\vee \to k \to 0$$

is the minimal free resolution of $k \in \operatorname{GrMod} A^o$ where $F_i^\vee := \operatorname{Hom}_A(F_i, A) \in \operatorname{grmod} A^o$.

Classifying AS-regular algebras up to dimension 2 is easy.

Lemma 17. *Let A be an AS-regular algebra.*

(1) $\operatorname{gldim} A = 1$ *if and only if* $A \cong k[x]$.
(2) *If* $\operatorname{gldim} A = 2$, *then* $A \cong k\langle x, y\rangle/(\alpha x^2 + \beta xy + \gamma yx + \delta y^2)$ *where* $(\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3$.

Conversely, if $A \cong k\langle x, y\rangle/(\alpha x^2 + \beta xy + \gamma yx + \delta y^2)$, *then:*

	$\alpha\delta - \beta\gamma \neq 0$	$\alpha\delta - \beta\gamma = 0$ *but* $\beta \neq \gamma$	$\alpha\delta - \beta\gamma = 0$ *and* $\beta = \gamma$
$\operatorname{gldim} A$	*2*	*2*	∞
$\operatorname{GKdim} A$	*2*	*2*	∞
noetherian	*Yes*	*No*	*No*
domain	*Yes*	*No*	*No*
Gorenstein condition	*Yes*	*No*	*No*

Question 18. Let A be a graded algebra such that $\operatorname{gldim} A < \infty$ and $\operatorname{GKdim} A < \infty$. Then A is noetherian if and only if A is a domain if and only if A satisfies Gorenstein condition?

2.2. **Geometric Algebras.** Artin, Tate and Van den Bergh [4] classified all 3-dimensional AS-regular algebras using geometric techniques. Recall that every graded algebra generated in degree 1 is of the form $A = T(V)/I$. A homogeneous element $f \in I_i \subset V^{\otimes i}$ defines a linear map $f : (V^{\otimes i})^* \cong V^{*\otimes i} \to k$ where V^* is the k-vector space dual of V, or equivalently a multilinear form $f : V^{*\times i} \to k$, so we may define a sequence of schemes

$$\Gamma_i := \mathcal{V}(I_i) = \{(p_1, \dots, p_i) \in \mathbb{P}(V^*)^{\times i} \mid f(p_1, \dots, p_i) = 0 \text{ for all } f \in I_i\}.$$

If $\pi_i : \Gamma_{i+1} \to \Gamma_i$ is the projection map onto the first i coordinates, then (Γ_i, π_i) is an inverse system of schemes. We define the point scheme of A by taking the inverse limit $\Gamma := \varprojlim \Gamma_i$.

Example 19. If $A = T(V)$ is the tensor algebra, then $\Gamma_i = \mathcal{V}(0) = \mathbb{P}(V^*)^{\times i}$, so the point scheme of A is $\Gamma \cong \mathbb{P}(V^*)^{\times \infty}$.

Example 20. On the other hand, if $A = S(V) = k[x_1, \dots, x_n]$ is the polynomial algebra, then $\Gamma_i = \{(p, p, \dots, p) \in \mathbb{P}(V^*)^{\times i} \mid p \in \mathbb{P}(V^*)\}$, so the point scheme of A is $\Gamma \cong \mathbb{P}(V^*)$.

Example 21. In fact, if $A = S(V)/I$ is a commutative graded algebra generated in degree 1, then $\Gamma_i = \{(p, p, \dots, p) \in \mathbb{P}(V^*)^{\times i} \mid p \in \operatorname{Proj} A\}$ for all $i \gg 0$, so the point scheme of A is $\Gamma \cong \operatorname{Proj} A$.

Example 22. If A is artinian, then $\Gamma_i = \emptyset$ for all $i \gg 0$, so the point scheme of A is $\Gamma = \emptyset$.

Question 23. [4] If A is a noetherian finitely presented graded algebra, then does $\varprojlim \Gamma_i$ converge?

To simplify the story, we will focus on only quadratic algebras. A quadratic algebra is of the form $A = T(V)/(R)$ where $R \subset V \otimes_k V$ is a subspace and (R) is the two-sided ideal of $T(V)$ generated by R.

Definition 24. A quadratic algebra $A = T(V)/(R)$ is called geometric if there is a pair (E, σ) where $E \subseteq \mathbb{P}(V^*)$ is a scheme and $\sigma \in \operatorname{Aut} E$ is an automorphism such that
G1 $\Gamma_2 = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$.
G2 $R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \text{ for all } p \in E\}$.

If A satisfies the condition (**G1**), then A determines a geometric pair (E, σ), which is written as $\mathcal{P}(A) = (E, \sigma)$. In this case, $\Gamma_i = \{(p, \sigma(p), \dots, \sigma^{i-1}(p)) \in \mathbb{P}(V^*)^{\times i} \mid p \in E\}$ for all $i \geq 2$, so the point scheme of A is $\Gamma \cong E$. If A satisfies the condition (**G2**), then A is determined by a geometric pair (E, σ), which is written as $A = \mathcal{A}(E, \sigma)$.

Classifying geometric algebras is equivalent to classifying geometric pairs in the following sense.

Theorem 25. *Let $A = T(V)/(R) = \mathcal{A}(E, \sigma), A' = T(V')/(R') = \mathcal{A}(E', \sigma')$ be geometric algebras. Then $A \cong A'$ if and only if there is an isomorphism $\tau : E \to E'$ which extends to an isomorphism $\bar{\tau} : \mathbb{P}(V^*) \to \mathbb{P}(V'^*)$ such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow{\tau} & E' \end{array}$$

commutes.

Although the definition is technical, many noetherian quadratic algebras are geometric.

Example 26. If A is a commutative quadratic domain, then $A = \mathcal{A}(\operatorname{Proj} A, \operatorname{Id})$ is geometric.

Example 27. Let $A = k\langle x, y\rangle/(f)$ where

$$f = \alpha x^2 + \beta xy + \gamma yx + \delta y^2, \quad (\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3.$$

Then

$$\begin{aligned}
& (p, q) = ((a, b), (c, d)) \in \Gamma_2 = \mathcal{V}(f) \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \\
\Longleftrightarrow\ & f(p, q) = \alpha ac + \beta ad + \gamma bc + \delta bd = 0 \\
\Longleftrightarrow\ & (\beta a + \delta b)d = (-\alpha a - \gamma b)c \\
\Longleftrightarrow\ & \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \beta a + \delta b \\ -\alpha a - \gamma b \end{pmatrix} = \begin{pmatrix} \beta & \delta \\ -\alpha & -\gamma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{P}^1 \\
\Longleftrightarrow\ & q = \sigma(p) \text{ where } p \in \mathbb{P}^1 \text{ and } \sigma = \begin{pmatrix} \gamma & \delta \\ -\alpha & -\beta \end{pmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
& A \text{ satisfies } (\mathbf{G1}) \\
\Longleftrightarrow\ & \sigma \in \operatorname{Aut} \mathbb{P}^1 = \mathrm{PGL}(2, k) = \mathrm{GL}(2, k)/\sim \\
\Longleftrightarrow\ & \det \sigma = \alpha\delta - \beta\gamma \neq 0.
\end{aligned}$$

In fact, $A = \mathcal{A}(\mathbb{P}^1, \sigma)$ is geometric if and only if $\alpha\delta - \beta\gamma \neq 0$. It follows that A is a 2-dimensional AS-regular algebra if and only if $A = \mathcal{A}(\mathbb{P}^1, \sigma)$ is a geometric algebra for some $\sigma \in \operatorname{Aut} \mathbb{P}^1$.

Using geometric pairs, 3-dimensional AS-regular algebras were also classified by Artin, Tate, and Van den Bergh. We state their theorem only in the quadratic case.

Theorem 28. [4] *Let A be a quadratic algebra. Then A is a 3-dimensional AS-regular algebra if and only if $A \cong \mathcal{A}(E, \sigma)$ where*
 (1) *$E = \mathbb{P}^2$ and $\sigma \in \operatorname{Aut} \mathbb{P}^2$, or*
 (2) *$E \subseteq \mathbb{P}^2$ is a cubic divisor, that is, $E = \mathcal{V}(f)$ for some $f \in S(V)_3$, and $\sigma \in \operatorname{Aut} E$ such that $\sigma^*\mathcal{L} \not\cong \mathcal{L}$ but $(\sigma^2)^*\mathcal{L} \otimes_E \mathcal{L} \cong \sigma^*\mathcal{L} \otimes_E \sigma^*\mathcal{L}$ where $\mathcal{L} = \mathcal{O}_E(1)$ is the very ample invertible sheaf on E.*

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quadratic AS-regular algebra. The following is a list of possibilities for E: (1) $\mathbb{P}^2$. (2) triple lines. (2) union of a double line and a single line. (3) three lines meeting at one point. (4) a triangle. (5) a line and a conic meeting at one point. (6) a line and conic meeting at two points. (7) an elliptic curve.

Example 29. If $A = k\langle x, y, z\rangle/(zy - \alpha yz, xz - \beta zx, yx - \gamma xy)$ where $\alpha, \beta, \gamma \in k \setminus \{0\}$, then A is a 3-dimensional quadratic AS-regular algebra such that

$$E = \begin{cases} \mathbb{P}^2 & \text{if } \alpha\beta\gamma = 1, \\ \mathcal{V}(xyz) \subset \mathbb{P}^2 \text{ (a triangle)} & \text{if } \alpha\beta\gamma \neq 1. \end{cases}$$

Example 30. Let $A = k\langle x, y, z\rangle/(\alpha yz + \beta zy + \gamma x^2, \alpha zx + \beta xz + \gamma y^2, \alpha xy + \beta yx + \gamma z^2)$. For a generic choice of $\alpha, \beta, \gamma \in k \setminus \{0\}$, A is a 3-dimensional quadratic AS-regular algebra such that

$$E = \mathcal{V}(\alpha\beta\gamma(x^3 + y^3 + z^3) - (\alpha^3 + \beta^3 + \gamma^3)xyz) \subset \mathbb{P}^2$$

is an elliptic curve, and σ is given by the translation by the point $(\alpha, \beta, \gamma) \in E$ in the group law on E. In this case, A is called a 3-dimensional Sklyanin algebra.

Unfortunately, there exists a 4-dimensional quadratic AS-regular algebra which does not satisfy (**G2**) (see [20]), however, almost all known examples of 4-dimensional quadratic AS-regular algebras are geometric.

For geometric algebras, graded Morita equivalence can be characterized in terms of their geometric pairs.

Theorem 31. [10] *Let* $A = T(V)/(R) = \mathcal{A}(E, \sigma), A' = T(V')/(R') = \mathcal{A}(E', \sigma')$ *be geometric algebras. Then* $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ *if and only if there is a sequence of isomorphisms* $\tau_n : E \to E'$ *which extend to isomorphisms* $\bar{\tau}_n : \mathbb{P}(V^*) \to \mathbb{P}(V'^*)$ *such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau_n} & E' \\ \sigma \downarrow & & \downarrow \sigma' \\ E & \xrightarrow{\tau_{n+1}} & E' \end{array}$$

commute for all $n \in \mathbb{Z}$.

3. Frobenius Koszul Algebras

3.1. **Koszul Algebras.**

So far, it seems that there is no connection between two research fields, noncommutative algebraic geometry and representation theory of finite dimensional algebras because the projective schemes associated to any graded algebra finite dimensional over k is empty. However, we will see that some of the ideas and techniques of noncommutative algebraic geometry can be transferred to the study of Frobenius Koszul algebras via Koszul duality.

Definition 32. Let A be a graded algebra. A linear resolution of $M \in \operatorname{GrMod} A$ is a free resolution of the form

$$\cdots \xrightarrow{[v_{ij}^{(2)}]\cdot} \oplus A \xrightarrow{[v_{ij}^{(1)}]\cdot} \oplus A \xrightarrow{[v_{ij}^{(0)}]\cdot} \oplus A \to M \to 0$$

where $v_{ij}^{(k)} \in V = A_1$ for all i, j, k. We say that A is Koszul if $k := A/A_{\geq 1}$ has a linear resolution.

It is known that if A is Koszul, then $A = T(V)/(R)$ is quadratic, and its quadratic dual $A^! = T(V^*)/(R^\perp)$ where

$$R^\perp := \{\lambda \in V^* \otimes_k V^* \mid \lambda(r) = 0 \text{ for all } r \in R\}$$

is also Koszul, which is called the Koszul dual of A.

Example 33. The following algebras are Koszul.

- A free algebra. For example,
$$A = k\langle x, y\rangle \qquad \Longleftrightarrow \qquad A^! \cong k\langle x, y\rangle/(x^2, xy, yx, y^2).$$
- A skew polynomial algebra, and a skew exterior algebra. For example,
$$A = k\langle x, y\rangle/(xy - \alpha yx) \qquad \Longleftrightarrow \qquad A^! \cong k\langle x, y\rangle/(x^2, \alpha xy + yx, y^2).$$

- A monomial quadratic algebra. For example,

$$A = k\langle x, y\rangle/(x^2, xy) \qquad \Longleftrightarrow \qquad A^! \cong k\langle x, y\rangle/(yx, y^2).$$

Let A be a graded algebra. The complexity of A is defined by $c_A := \mathrm{GKdim}(\bigoplus \mathrm{Ext}^i_A(k, k))$. This section was motivated by the following result due to Smith.

Theorem 34. [18] *Let A be a graded algebra. Then A is Frobenius Koszul of $c_A < \infty$ if and only if $A^!$ is AS-regular Koszul.*

It follows that classifying all AS-regular Koszul algebras of GKdimension d is equivalent to classifying all Frobenius Koszul algebras of complexity d. We will see below that there are four levels of classification.

As in the commutative case, a dualizing complex defined below plays an essential role in noncommutative algebraic geometry. We define the i-th local cohomology of $M \in \mathrm{GrMod}\, A$ by

$$\mathrm{H}^i_{\mathfrak{m}}(M) = \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M) \in \mathrm{GrMod}\, A.$$

For an abelian category $\mathcal{C}$, we denote by $\mathcal{D}^b(\mathcal{C})$ the category of bounded complexes in $\mathcal{C}$.

Definition 35. [22] Let A be a graded algebra. A bounded complex $D \in \mathcal{D}^b(\mathrm{GrMod}\, A^e)$ of graded A-A bimodules is called dualizing if it satisfies the following conditions:

(1) D has finite injective dimension over A and over A^o.
(2) D is a complex of modules finitely generated over A and over A^o.
(3) There are isomorphisms of graded A-A bimodules

$$\mathrm{Ext}^i_A(D, D) \cong \mathrm{Ext}^i_{A^o}(D, D) \cong \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

A dualizing complex D over A is called balanced if there are isomorphisms of graded A-A bimodules

$$\mathrm{H}^i_{\mathfrak{m}}(D) \cong \mathrm{H}^i_{\mathfrak{m}^o}(D) \cong \begin{cases} A^* := \mathrm{Hom}_k(A, k) \in \mathrm{GrMod}\, A^e & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Almost all noetherian graded algebras we usually consider, typically, graded quotient algebras of a noetherian AS-regular algebra, have balanced dualizing complexes.

Let $\underline{\mathrm{grmod}} A$ be the stable category of $\mathrm{grmod}\, A$ modulo projectives. We can form a triangulated category $\mathcal{S}(\underline{\mathrm{grmod}} A)$, called the stabilization of $\underline{\mathrm{grmod}} A$, by formally inverting the syzygy functor $\Omega : \underline{\mathrm{grmod}} A \to \underline{\mathrm{grmod}} A$ so that $\Omega^{-1} : \mathcal{S}(\underline{\mathrm{grmod}} A) \to \mathcal{S}(\underline{\mathrm{grmod}} A)$ is the translation functor.

Theorem 36. *Let A, B be graded algebras.*

(1) *If A and B are quadratic, then*

$$A \cong B \qquad \Longleftrightarrow \qquad A^! \cong B^!.$$

(2) [13] *If A and B are Koszul, then*

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, B \qquad \Longleftrightarrow \qquad \mathrm{GrMod}\, A^! \cong \mathrm{GrMod}\, B^!.$$

(3) [8] *If A and B are Koszul and $A, B, A^!, B^!$ are noetherian and having balanced dualizing complexes, then*

$$\mathcal{D}^b(\operatorname{grmod} A) \cong \mathcal{D}^b(\operatorname{grmod} B) \qquad \Longleftrightarrow \qquad \mathcal{D}^b(\operatorname{grmod} A^!) \cong \mathcal{D}^b(\operatorname{grmod} B^!).$$

(4) [9] *If A and B are Koszul and $A, B, A^!, B^!$ are noetherian and having balanced dualizing complexes, then*

$$\mathcal{D}^b(\operatorname{proj} A) \cong \mathcal{D}^b(\operatorname{proj} B) \qquad \Longleftrightarrow \qquad \mathcal{S}(\underline{\operatorname{grmod}} A^!) \cong \mathcal{S}(\underline{\operatorname{grmod}} B^!).$$

Note that if A is a Frobenius algebra, then $\mathcal{S}(\underline{\operatorname{grmod}} A) \cong \underline{\operatorname{grmod}} A$ as triangulated categories. If we accept the conjecture that every AS-regular (Koszul) algebra is noetherian, then we have the following correspondences:

$$\begin{array}{ll} & \text{Classifying all Frobenius Koszul algebras of complexity } d \\ & \qquad \left\{ \begin{array}{l} \text{up to isormophism} \\ \text{up to graded Morita equivalence} \\ \text{up to graded derived equivalence} \\ \text{up to graded stable equivalence} \end{array} \right. \\ \Longleftrightarrow & \text{Classifying all (noetherian) AS-regular Koszul algebras of GKdimension } d \\ & \qquad \left\{ \begin{array}{l} \text{up to isormophism} \\ \text{up to graded Morita equivalence} \\ \text{up to graded derived equivalence} \\ \text{up to derived equivalence of the associated projective schemes.} \end{array} \right. \end{array}$$

Since classification of Frobenius algebras is an active project in representation theory of finite dimensional algebras, there will be deep interactions between these two research fields.

3.2. **Co-point Modules.** If A is a Koszul algebra, then we expect that some of the techniques to study A can be used to study $A^!$. In fact, by transferring techniques of noncommutative algebraic geometry described in the previous section, we have already obtained some results in representation theory of finite dimensional algebras (see [18], [11]). We would like to make more progress in this direction.

Definition 37. A quadratic algebra A is called co-geometric if its quadratic dual $A^! = \mathcal{A}(E, \sigma)$ is geometric.

If A is co-geometric, then A determines a geometric pair by $(E, \sigma) = \mathcal{P}(A^!)$, and A is determined by a geometric pair (E, σ) by $A \cong \mathcal{A}^!(E, \sigma) := \mathcal{A}(E, \sigma)^!$. The following results are simple interpretations of the ones in the previous section.

Theorem 38. *Let $A = T(V)/(R) = \mathcal{A}^!(E, \sigma), A' = T(V')/(R') = \mathcal{A}^!(E', \sigma')$ be co-geometric algebras.*

(1) *$A \cong A'$ if and only if there is an isomorphism $\tau : E \to E'$ which extends to an isomorphism $\bar{\tau} : \mathbb{P}(V) \to \mathbb{P}(V')$ such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow{\tau} & E' \end{array}$$

commutes.

(2) *Moreover, if A, A' are Koszul algebras, then* $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ *if and only if there is a sequence of isomorphisms $\tau_n : E \to E'$ which extend to isomorphisms $\bar{\tau_n} : \mathbb{P}(V) \to \mathbb{P}(V')$ such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau_n} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow{\tau_{n+1}} & E' \end{array}$$

commute for all $n \in \mathbb{Z}$.

Since A is a Frobenius Koszul algebra of complexity 3 if and only if $A^!$ is a 3-dimensional quadratic AS-regular algebra, Frobenius Koszul algebras of complexity 3 can be completely classified in term of geometric pairs up to isomorphism and up to graded Morita equivalence. In this section, we will find the geometric pair (E, σ) directly from A without passing through its Koszul dual.

Let $A = T(V)/I$ be a graded algebra. For $p \in \mathbb{P}(V)$, we define $N_p := A/vA \in \operatorname{GrMod} A$ where $v \in V = A_1$ such that $p = [v] \in \mathbb{P}(V)$. This notation is useful because, for $p, q \in \mathbb{P}(V)$, $N_p \cong N_q$ if and only if $p = q$.

Example 39. If $A = k\langle x, y, z\rangle/I$ and $p = (a, b, c) \in \mathbb{P}^2$, then

$$N_p = A/(ax + by + cz)A \in \operatorname{GrMod} A.$$

Definition 40. Let $A = T(V)/I$ be a graded algebra. We say that $N \in \operatorname{GrMod} A$ is a co-point module if N has a minimal free resolution of the form

$$\cdots \xrightarrow{v_2\cdot} A \xrightarrow{v_1\cdot} A \xrightarrow{v_0\cdot} A \to N \to 0$$

where $v_i \in V = A_1$ for all $i \in \mathbb{N}$.

We denote by $\operatorname{clin} A$ the full subcategory of $\operatorname{GrMod} A$ consisting of co-point modules. If $N \in \operatorname{clin} A$ is a co-point module having the resolution as above, then $\Omega^i N \cong N_{p_i} \in \operatorname{clin} A$ are co-point modules where $p_i = [v_i] \in \mathbb{P}(V)$ for all $i \in \mathbb{N}$. We define $\mathcal{P}^!(A) = (E, \sigma)$ where $E := \{p \in \mathbb{P}(V) \mid N_p \in \operatorname{clin} A\}$, and $\sigma : E \to E$ is a map defined by $\Omega N_p \cong N_{\sigma(p)}$, so that

$$\Omega^i N_p \cong N_{\sigma^i(p)}$$

for all $i \in \mathbb{N}$.

For the purpose below, we assume that the following technical condition. We say that a Koszul algebra A satisfies (*) if

(1) $\operatorname{GKdim} A^! < \infty$ (equivalently $c_A < \infty$),
(2) $A^!$ is noetherian, and

(3) $A^!$ satisfies Cohen-Macaulay property, that is, $\operatorname{grade} M + \operatorname{GKdim} M = \operatorname{GKdim} A^!$ for all $M \in \operatorname{grmod} A^!$ where $\operatorname{grade} M := \inf\{i \mid \operatorname{Ext}^i_{A^!}(M, A^!) \neq 0\}$.

We expect that every Frobenius Koszul algebra of finite complexity satisfies (*).

Theorem 41. [11] *If $A = \mathcal{A}^!(E,\sigma)$ is a co-geometric Frobenius Koszul algebra satisfying (*), then $\mathcal{P}^!(A) = (E,\sigma)$.*

Many important functors in representation theory of finite dimensional algebras preserve co-point modules.

Lemma 42. [12] *If $A = \mathcal{A}^!(E,\sigma)$ is a Frobenius Koszul algebra satisfying (*), then there are following functors:*

$$\begin{array}{ccl} \Omega & : & \operatorname{clin} A \to \operatorname{clin} A \\ \operatorname{Tr} & : & \operatorname{clin} A \to \operatorname{clin} A^o \\ (-)^\vee := \operatorname{Hom}_A(-,A) & : & \operatorname{clin} A \to \operatorname{clin} A^o \\ (-)^* := \operatorname{Hom}_k(-,k) & : & \operatorname{clin} A \to \operatorname{clin} A^o \\ \mathcal{N}(-) := ((-)^\vee)^* & : & \operatorname{clin} A \to \operatorname{clin} A. \end{array}$$

It follows that the Nakayama functor $\mathcal{N} : \operatorname{clin} A \to \operatorname{clin} A$ induces an automorphism $\nu \in \operatorname{Aut} E$, which is called the Nakayama automorphism.

Conjecture 43. Let $A = \mathcal{A}^!(E,\sigma), A' = \mathcal{A}^!(E',\sigma')$ be co-geometric Frobenius Koszul algebras of complexity d satisfying (*), and let $\nu \in \operatorname{Aut} E, \nu' \in \operatorname{Aut} E'$ be Nakayama automorphisms. Then $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ if and only if $\mathcal{A}^!(E,\nu\sigma^d) \cong \mathcal{A}^!(E',\nu'\sigma'^d)$.

It is easy to prove the above conjecture if $c_A \leq 2$. If $c_A = 3$ and E is an elliptic curve, then the above conjecture was proved in [10].

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