SYMMETRY IN THE VANISHING OF EXT-GROUPS

IZURU MORI

ABSTRACT. In this note, we will find a class of rings R satisfying the following property: for every pair of finitely generated right R-modules M and N, $\operatorname{Ext}^i_R(M,N)=0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(N,M)=0$ for all $i \gg 0$. In particular, we will show that such a class of rings includes a group algebra of a finite group and the exterior algebra of odd degree.

1. Motivation

Throughout, we always assume that k is a field, R is a (right and left) noetherian ring, $\operatorname{mod} R$ is the category of finitely generated right R-modules, and $M, N \in \operatorname{mod} R$.

If R is a commutative local ring, then Serre [15] defined the intersection multiplicity of $M, N \in \operatorname{mod} R$ by

$$\chi(M,N) := \sum_{i=0}^{\infty} (-1)^i \operatorname{length} \operatorname{Tor}_i^R(M,N).$$

If R is not commutative, then $\operatorname{Tor}_i^R(M,N)$ do not make sense, but $\operatorname{Ext}^i_R(M,N)$ do, so Smith and I [14] defined a new intersection multiplicity of $M, N \in \operatorname{mod} R$ by

$$M \cdot N := (-1)^{\operatorname{codim} M} \sum_{i=0}^{\infty} (-1)^i \operatorname{length} \operatorname{Ext}^i_R(M,N)$$

in order to develop an intersection theory over a noncommutative ring. (Note that if R is not commutative, then $\operatorname{Ext}^i_R(M,N)$ are no longer R-modules, so we defined the above intersection multiplicity in [14] only over a k-algebra R, replacing $\operatorname{length} \operatorname{Ext}^i_R(M,N)$ by $\dim_k \operatorname{Ext}^i_R(M,N)$.) Fortunately, these two definitions of the intersection multiplicity agree over reasonably nice commutative rings.

Theorem 1. [5, Theorem 4, Theorem 5] *If R is a commutative local complete intersection ring, or a commutative local Gorenstein ring of $\operatorname{Kdim} R \le 5$, then*

$$M \cdot N = \chi(M,N)$$

for all $M, N \in \operatorname{mod} R$ such that

- $\operatorname{length}(M \otimes_R N) < \infty$,
- $\operatorname{pd}(M) < \infty$, $\operatorname{pd}(N) < \infty$, *and*
- $\operatorname{Kdim} M + \operatorname{Kdim} N \le \operatorname{Kdim} R$.

This note is basically a summary of [13] which has been accepted for publication in *J. Algebra*.

Three conditions on $M, N \in \operatorname{mod} R$ in the above theorem guarantee that both intersection multiplicities $\chi(M, N)$ and $M \cdot N$ are well-defined. In order to justify our new intersection theory, the following questions are natural over more general rings.

Question. Let R be an algebra or a commutative ring, and $M, N \in \operatorname{mod} R$.
(1) $M \cdot N = N \cdot M$ if both sides are well-defined?
(2) $M \cdot N$ is well-defined if and only if $N \cdot M$ is well-defined?

Over a commutative Gorenstein local ring, the first question above is equivalent to Serre's vanishing conjecture by [9]. In this note, we will focus on the second question above. Note that $M \cdot N$ is well-defined if and only if

- $\operatorname{length} \operatorname{Ext}_R^i(M, N) < \infty$ for all i, and
- $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$,

so we can split the second question above into the following two questions:

Question. Let R be an algebra or a commutative ring, and $M, N \in \operatorname{mod} R$.
(1) $\operatorname{length} \operatorname{Ext}_R^i(M, N) < \infty$ for all i if and only if $\operatorname{length} \operatorname{Ext}_R^i(N, M) < \infty$ for all i?
(2) $\operatorname{Ext}_R^i(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}_R^i(N, M) = 0$ for all $i \gg 0$?

The first question above was answered affirmatively over a commutative ring.

Theorem 2. [9, Corollary 3.2] *Let R be a commutative local ring. Then, for all $M, N \in \operatorname{mod} R$,*

$$\operatorname{length} \operatorname{Ext}_R^i(M, N) < \infty \textit{ for all } i \Leftrightarrow \operatorname{length} \operatorname{Ext}_R^i(N, M) < \infty \textit{ for all } i.$$

For the second question above, we will make the following definition.

Definition 3. We say that a ring R satisfies (**ee**) if, for all $M, N \in \operatorname{mod} R$,

$$\operatorname{Ext}_R^i(M, N) = 0 \text{ for all } i \gg 0 \Leftrightarrow \operatorname{Ext}_R^i(N, M) = 0 \text{ for all } i \gg 0.$$

First, we will make an easy observation.

Example 4. If R is regular, that is, $\operatorname{gldim} R < \infty$, then, for all $M, N \in \operatorname{mod} R$, $\operatorname{Ext}_R^i(M, N) = 0$ for all $i > \operatorname{gldim} R$, so R satisfies (**ee**).

Conversely, if R is a commutative local ring satisfying (**ee**), then $\operatorname{Ext}_R^i(R, k) = 0$ for all $i \geq 1$ where k is the residue field of R, so $\operatorname{Ext}_R^i(k, R) = 0$ for all $i \gg 0$, hence R is Gorenstein, that is, $\operatorname{id}(R) < \infty$.

It follows that the class of commutative local rings satisfying (**ee**) is somewhere between regular rings and Gorenstein rings. In commutative ring theory, there is a nice class of rings between them, namely complete intersection rings.

Theorem 5. [2] *Every commutative locally complete intersection ring satisfies (***ee***).*

It is not very difficult to find an example of non complete intersection ring which satisfies (**ee**). Very recently, Jorgensen and Sega [8] found an example of a commutative Gorenstein ring that does not satisfy (**ee**), so the class of commutative rings satisfying (**ee**) is strictly between complete intersection rings and Gorenstein rings.

2. Conjecture of Auslander

We will define another technical condition on a ring.

Definition 6. We say that a ring R satisfies (**ac**) if, for each $M \in \operatorname{mod} R$, there exists $n_M \in \mathbb{N}$ such that, for all $N \in \operatorname{mod} R$,

$$\operatorname{Ext}^i_R(M,N) = 0 \text{ for all } i \gg 0 \Rightarrow \operatorname{Ext}^i_R(M,N) = 0 \text{ for all } i > n_M.$$

There was a conjecture in representation theory of finite dimensional algebras.

Conjecture. (Auslander) Every artinian algebra satisfies (**ac**).

The above conjecture was important since it implies the famous conjecture below.

Conjecture. (Finitistic dimension conjecture) If R is an artinian algebra, then there exists $n_R \in \mathbb{N}$ such that, for all $M \in \operatorname{mod} R$,

$$\operatorname{pd}(M) < \infty \Rightarrow \operatorname{pd}(M) \leq n_R.$$

Although the above conjecture was raised in representation theory of finite dimensional algebras, it became also interested in commutative ring theory due to the following result.

Theorem 7. [6, Theorem 4.1], [13, Theorem 3.2] *Let R be a commutative local Gorenstein ring. If R satisfies (**ac**), then R satisfies (**ee**).*

Although the condition (**ac**) is interesting, it is not easy to find non-trivial examples of algebras satisfying (**ac**). In fact, there had been very few examples of algebras satisfying (**ac**) until recently.

Theorem 8. [4, Theorem 2.4] *Every group algebra of a finite group satisfies (**ac**).*

Theorem 9. [2, Theorem 4.7, Proposition 6.2] *Every commutative locally complete intersection ring satisfies (**ac**).*

Due to the above theorem, the following is a natural question.

Question. If R is a noncommutative analogue of a commutative complete intersection ring, then does R satisfy (**ac**) and/or (**ee**)?

On the positive side, we have the following result.

Theorem 10. [13, Corollary 2.3] *If R is a regular ring and $\{x_1, \dots, x_n\}$ is a regular central sequence of R, then $R/(x_1, \dots, x_n)$ satisfies (**ac**).*

The above theorem produces a new example of an algebra satisfying (**ac**).

Example 11. Every exterior algebra can be written as

$$\Lambda(k^n) \cong R/(x_1^2, \dots, x_n^2),$$

where

$$R = k\langle x_1, \dots, x_n\rangle/(x_ix_j + x_jx_i)_{1\le i<j\le n}$$

is a regular ring (an anti-commutative polynomial ring), and $\{x_1^2, \dots, x_n^2\}$ is a regular central sequence of R, so $\Lambda(k^n)$ satisfies (**ac**).

Jorgensen and Sega [7] found an example of a commutative Frobenius algebra that does not satisfy (**ac**), so the Auslander conjecture is false. The following theorem also shows that the Auslander conjecture is false. In particular, we cannot replace "central" by "normalizing" in the above theorem.

Theorem 12. [12, Theorem 6.5] *Let* $\Lambda = k\langle x_1, \dots, x_n\rangle/(x_ix_j + \alpha_{ij}x_jx_i, x_i^2)$ *be a skew exterior algebra where* $0 \neq \alpha_{ij} \in k$ *for* $1 \le i < j \le n$*. Then* Λ *satisfies (***ac***) if and only if* α_{ij} *are roots of unity for all* $1 \le i < j \le n$*.*

3. Stably Symmetric Algebras

In this section, we will define a stably symmetric algebra, which is a generalization of a symmetric algebra.

Definition 13. Let $\mathcal{C}$ be a k-linear Hom-finite category, that is,

$$\dim_k \operatorname{Hom}_{\mathcal{C}}(M, N) < \infty$$

for all $M, N \in \mathcal{C}$. A Serre functor on $\mathcal{C}$ is an autoequivalence $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ such that

$$\operatorname{Hom}_{\mathcal{C}}(M, N) \cong D \operatorname{Hom}_{\mathcal{C}}(N, \mathcal{K}(M))$$

for all $M, N \in \mathcal{C}$ where $D(-)$ is the functor taking the k-vector space dual.

A Serre functor on $\mathcal{C}$ is unique if it exists. Moreover, if $\mathcal{C}$ is a triangulated category, then a Serre functor $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is exact, so the following lemma is immediate.

Lemma 14. *Let* $\mathcal{C}$ *be a* k*-linear* Hom*-finite triangulated category. Then an exact autoequivalence* $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ *is a Serre functor on* $\mathcal{C}$ *if and only if*

$$\operatorname{Ext}^i_{\mathcal{C}}(M, N) \cong D \operatorname{Ext}^{-i}_{\mathcal{C}}(N, \mathcal{K}(M))$$

for all i *and all* $M, N \in \mathcal{C}$*.*

The definition of a Serre functor was motivated by the Serre duality.

Example 15. If X is a smooth projective scheme of finite type over k, then the bounded derived category of coherent $\mathcal{O}_X$-modules $\mathcal{D}^b(X)$ has a Serre functor

$$- \otimes_X \omega_X[d] : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

where ω_X is the canonical sheaf on X and $d = \dim X$, so that

$$\operatorname{Ext}^i_X(\mathcal{F}, \mathcal{G}) \cong D \operatorname{Ext}^{-i}_X(\mathcal{G}, \mathcal{F} \otimes_X \omega_X[d]) \cong D \operatorname{Ext}^{d-i}_X(\mathcal{G}, \mathcal{F} \otimes_X \omega_X)$$

for all i and all $\mathcal{F}, \mathcal{G} \in \operatorname{coh} X$. In particular, the classical Serre duality

$$\operatorname{H}^i(X, \mathcal{G}) \cong \operatorname{Ext}^i_X(\mathcal{O}_X, \mathcal{G}) \cong D \operatorname{Ext}^{d-i}_X(\mathcal{G}, \omega_X)$$

holds for all i and all $\mathcal{G} \in \operatorname{coh} X$.

We will apply the theory of a Serre functor to the triangulated category defined as follows. Let $\underline{\mathrm{mod}}\, R$ be the stable category of $\mathrm{mod}\, R$ by projective modules. In general, $\underline{\mathrm{mod}}\, R$ is not a triangulated category, but there is a natural way of making it a triangulated category. We define the category $\mathcal{S}(\underline{\mathrm{mod}}\, R)$, called the stabilization of $\underline{\mathrm{mod}}\, R$, whose objects are of the form $\Omega^i M$ where $M \in \underline{\mathrm{mod}}\, R$ and $i \in \mathbb{Z}$ modulo $M \cong N$ in $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ if $\Omega^i M \cong \Omega^i N$ in $\underline{\mathrm{mod}}\, R$ for all $i \gg 0$. It turns out that $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is a triangulated category with the translation functor

$$\Omega^{-1} : \mathcal{S}(\underline{\mathrm{mod}}\, R) \to \mathcal{S}(\underline{\mathrm{mod}}\, R).$$

We refer to [3] for more details on this construction. If R is a regular algebra, then, for all $M \in \mathrm{mod}\, R$, $\Omega^i M \cong 0$ for all $i > \mathrm{gldim}\, R$, so $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is trivial. On the other hand, if R is a Frobenius algebra, then $\underline{\mathrm{mod}}\, R$ is already a triangulated category, so $\mathcal{S}(\underline{\mathrm{mod}}\, R) \cong \underline{\mathrm{mod}}\, R$.

Definition 16. Let R be an algebra. We say that R is stably symmetric if

$$\mathcal{K} = \Omega^{-d} : \mathcal{S}(\underline{\mathrm{mod}}\, R) \to \mathcal{S}(\underline{\mathrm{mod}}\, R)$$

is a Serre functor for some $d \in \mathbb{Z}$.

In other words, R is stably symmetric if and only if $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is Calabi-Yau. However, we will see later that the definition of stably symmetric does not coincide with that of Calabi-Yau in the graded case. Note that if R is a regular ring, then $\mathcal{S}(\underline{\mathrm{mod}}\, R)$ is trivial, so R is stably symmetric. The following result is well known.

Lemma 17. *If R is a Frobenius algebra, then $\mathcal{S}(\underline{\mathrm{mod}}\, R) \cong \underline{\mathrm{mod}}\, R$ has a Serre functor*

$$\mathcal{K} = \Omega\mathcal{N} : \underline{\mathrm{mod}}\, R \to \underline{\mathrm{mod}}\, R$$

where

$$\mathcal{N}(-) = D\,\mathrm{Hom}_R(-, R) : \mathrm{mod}\, R \to \mathrm{mod}\, R$$

is the Nakayama functor.

If R is a symmetric algebra, then R is Frobenius such that the Nakayama functor is the identity, so we have the following.

Corollary 18. *Every symmetric algebra is stably symmetric.*

Example 19. The algebras below are examples of symmetric algebras, so they are stably symmetric by the above corollary.

- A commutative local Frobenius algebra.
- A semi-simple algebra.
- The trivial extension of an artinian algebra.
- The group algebra of a finite group.
- The exterior algebra $\Lambda(k^n)$ when n is odd.

4. Vogel Cohomology

In this section, we will interpret the two conditions (**ac**) and (**ee**) in terms of Vogel cohomologies. For $M, N \in \operatorname{mod} R$, the i-th Vogel cohomology is defined by

$$\widehat{\operatorname{Ext}}^i_R(M,N) := \lim_{n\to\infty} \underline{\operatorname{Hom}}_R(\Omega^{n+i}M, \Omega^n N).$$

Note that $\widehat{\operatorname{Ext}}^i_R(M,N)$ are defined for all integers $i \in \mathbb{Z}$. The below are two main results of this note.

Theorem 20. [13, Theorem 3.2] *Let R be a Gorenstein ring. Then the following conditions are equivalent:*

1. *R satisfies (**ac**).*
2. *For all $M, N \in \operatorname{mod} R$,*

$$(*) \quad \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \textit{ for all } i \gg 0 \Rightarrow \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \textit{ for all } i.$$

Theorem 21. [13, Theorem 4.6] *Let R be a stably symmetric Gorenstein algebra. Then the following conditions are equivalent:*

1. *R satisfies (**ee**).*
2. *For all $M, N \in \operatorname{mod} R$,*

$$(**) \quad \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \textit{ for all } i \gg 0 \Rightarrow \widehat{\operatorname{Ext}}^i_R(M,N) = 0 \textit{ for all } i \ll 0.$$

Since the condition (*) above is stronger than the condition (**) above, the following is immediate.

Corollary 22. [13, Theorem 4.7] *Let R be a stably symmetric Gorenstein algebra. If R satisfies (**ac**), then R satisfies (**ee**).*

The above corollary produces a few more examples of algebras satisfying (**ee**).

Example 23. Every group algebra of a finite group is a symmetric algebra satisfying (**ac**), so it satisfies (**ee**).

Example 24. The exterior algebra $\Lambda(k^n)$ where n is odd is a symmetric algebra satisfying (**ac**), so it satisfies (**ee**).

5. AS-Gorenstein Koszul Algebras

In this last section, we will make similar analysis for AS-Gorenstein Koszul algebras. From now on, we will assume that A is a connected graded algebra over k, $\operatorname{grmod} A$ is the category of finitely generated graded right A-modules, and $M, N \in \operatorname{grmod} A$.

If A is a Koszul algebra, then A is a quadratic algebra, that is, $A = T(V)/(W)$ where $T(V)$ is the tensor algebra on the finite dimensional vector space V over k, $W \subset V \otimes_k V$ is a subspace, and (W) is the two-sided ideal of $T(V)$ generated by W. It is known that its quadratic (Koszul) dual $A^! = T(V^*)/(W^\perp)$ is also Koszul where

$$W^\perp = \{\lambda \in V^* \otimes_k V^* \mid \lambda(w) = 0 \text{ for all } w \in W \subset V \otimes_k V\}.$$

Clearly, $(A^!)^! \cong A$ as graded algebras.

Example 25. An exterior algebra $\Lambda(k^n)$ is a Koszul algebra whose Koszul dual is a polynomial algebra $\Lambda(k^n)^! \cong S(k^n)$.

The class of algebras defined below plays an important role in noncommutative algebraic geometry.

Definition 26. A connected graded algebra A is called AS-Gorenstein if

- $\mathrm{id}(A) = d < \infty$, and
- $\mathrm{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

The following are versions of the Koszul duality.

Theorem 27. [10, Proposition 4.5], [11, Theorem 5.3] *If A is a noetherian AS-Gorenstein Koszul algebra such that $A^!$ is noetherian, then there is a duality*

$$E : \mathcal{D}^b(\mathrm{grmod}\, A) \to \mathcal{D}^b(\mathrm{grmod}\, A^!),$$

which induces a duality

$$E : \mathcal{S}(\underline{\mathrm{grmod}}\, A) \to \mathcal{D}^b(\mathrm{Proj}\, A^!)$$

as triangulated categories.

We refer to [1] for the definition of $\mathrm{Proj}\, A^!$ when $A^!$ is not commutative. We modify the definition of a stably symmetric algebra in the graded case.

Definition 28. Let A be a connected graded algebra. We say that A is stably symmetric in the graded sense if

$$\mathcal{K} = \Omega^{-d}(-)(\ell) : \mathcal{S}(\underline{\mathrm{grmod}}\, A) \to \mathcal{S}(\underline{\mathrm{grmod}}\, A)$$

is a Serre functor for some $d \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$ where $(\ell) : \mathrm{grmod}\, A \to \mathrm{grmod}\, A$ is the functor shifting degree by ℓ.

The theorem below produces many examples of stably symmetric graded algebras.

Theorem 29. [13, Corollary 5.7] *Let A be a noetherian AS-Gorenstein Koszul algebra such that $A^!$ is commutative. Then A is stably symmetric in the graded sense if and only if* $\mathrm{Proj}\, A^!$ *is smooth.*

Example 30. If $\Lambda(k^n)$ is an exterior algebra, then $\Lambda(k^n)$ is a noetherian AS-Gorenstein Koszul algebra such that $\Lambda(k^n)^! \cong S(k^n)$ is a commutative polynomial algebra. Since $\mathrm{Proj}\, \Lambda(k^n)^! \cong \mathbb{P}^{n-1}$ is a projective space, $\Lambda(k^n)$ is stably symmetric in the graded sense whether n is odd or even.

It follows that $\Lambda(k^n)$ satisfies (**ee**) in the graded sense, that is, the symmetry in the vanishing of Ext-groups holds for any pair of graded right modules over every exterior algebra.

We can construct many stably symmetric graded algebras which are not even artinian.

Example 31. If

$$A = k\langle x, y, z\rangle/(xz + zx, yz + zy, xy + yx + z^2, x^2, y^2),$$

then A is a noetherian AS-Gorenstein Koszul algebra such that

$$A^! \cong k[x, y, z]/(xy - z^2)$$

is commutative. Since $\operatorname{Proj} A^! \cong \mathbb{P}^1$ is smooth, A is stably symmetric in the graded sense. It is easy to see that A is not artinian.

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Department of Mathematics
Faculty of Science
Shizuoka University
Shizuoka 422-8529 JAPAN

E-mail address: simouri@ipc.shizuoka.ac.jp