ON DERIVED EQUIVALENCES FOR SELFINJECTIVE ALGEBRAS

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Abstract. We show that if A is a representation-finite selfinjective artin algebra then every $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$ is a direct summand of a tilting complex, and that if A, B are derived equivalent representation-finite selfinjective artin algebras then there exists a sequence of selfinjective artin algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.

1. Introduction

Let A be an artin algebra. Rickard [7, Proposition 9.3] showed that for any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ the number of nonisomorphic indecomposabele direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$, the Grothendieck group of A, which generalizes earlier results [2, Proposition 3.2] and [6, Theorem 1.19]. He raised a question whether a complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ is a tilting complex or not if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of $K_0(A)$ (see also [6]). In case $P^\bullet$ is a projective resolution of a module $T \in \mathrm{mod}\text{-}A$ with proj dim $T_A \leq 1$, Bongartz [1, Lemma of 2.1] has settled the question affirmatively. More precisely, he showe d that every $T \in \mathrm{mod}\text{-}A$ with proj dim $T_A \leq 1$ and $\mathrm{Ext}^1_A(T, T) = 0$ is a direct summand of a classical tilting module, i.e., a tilting module of projective dimension ≤ 1. Unfortunetely, this is not true in general (see [7, Section 8]). Our first aim is to show that if A is a representation-finite selfinjective artin algebra then every $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ with $\mathrm{Hom}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$, where ν is the Nakayama functor, is a direct summand of a tilting complex (Theorem 4).

Rickard [8, Theorem 4.2] showed that the Brauer tree algebras over a field with the same numerical invariants are derived equivalent to each other. Subsequently, Okuyama pointed out that for any Brauer tree algebras A, B with the same numerical invariants there exists a sequence of Brauer tree algebras $A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1. These facts can be formulated as follows. For any tilting complex $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ associated with a certain sequence of idempotents in a ring A, there exists a sequence of rings $A = B_0, B_1, \cdots, B_m = \mathrm{End}_{\mathsf{K}(\mathrm{Mod}\text{-}A)}(P^\bullet)$ such that, for any $0 \leq i < m$, B_{i+1} is the endomorphism ring of a tilting complex for B_i of length ≤ 1 determined by an idempotent (see [4, Proposition 3.2]). We refer to [3], [5] for other examples of derived equivalences which are iterations of derived equivalences induced by tilting complexes of length ≤ 1. Our second aim is to show that for any derived equivalent representation-finite selfinjective artin algebras A, B there exists a sequence of selfinjective artin algebras

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$A = B_0, B_1, \cdots, B_m = B$ such that, for any $0 \le i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1 (Theorem 5).

2. Derived equivalences for selfinjective algebras

In the following, R is a commutative artinian ring with the Jacobson radical $\mathfrak{m}$ and A is an artin R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and is finitely generated as an R-module.

For any artin R-algebra A, we denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely generated modules. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. Also, we set $D = \mathrm{Hom}_R(-, E(R/\mathfrak{m}))$, where $E(R/\mathfrak{m})$ is an injective envelope of $R/\mathfrak{m}$ in Mod-R, and $\nu = D \circ \mathrm{Hom}_A(-, A)$, which is called the Nakayama functor.

Definition 1. Assume A is selfinjective and let $\{e_1, \cdots, e_n\}$ be a basic set of orthogonal local idempotents in A. Then there exists a permutation ρ of the set $I = \{1, \cdots, n\}$, called the Nakayama permutation, such that $\nu(e_i A) \simeq e_{\rho(i)} A$ for all $i \in I$.

Proposition 2. *Assume A is selfinjective and has a cyclic Nakayama permutation. Let B be a selfinjective artin R-algebra derived equivalent to A. Then B is Morita equivalent to A.*

For a cochain complex $X^\bullet$ over an abelian category $\mathcal{A}$, we denote by $\mathrm{H}^n(X^\bullet)$ the n-th cohomology of $X^\bullet$. For an additive category $\mathcal{B}$, we denote by $\mathsf{K}(\mathcal{B})$ (resp., $\mathsf{K}^+(\mathcal{B})$, $\mathsf{K}^-(\mathcal{B})$, $\mathsf{K}^\mathrm{b}(\mathcal{B})$) the homotopy category of complexes (resp., bounded below complexes, bounded above complexes, bounded complexes) over $\mathcal{B}$. As usual, we consider objects of $\mathcal{B}$ as complexes over $\mathcal{B}$ concentrated in degree zero.

Definition 3. For any nonzero $P^\bullet \in \mathsf{K}^-(\mathcal{P}_A)$ we set

$$a(P^\bullet) = \max\{i \in \mathbb{Z} \mid \mathrm{H}^i(P^\bullet) \neq 0\},$$

and for any nonzero $P^\bullet \in \mathsf{K}^+(\mathcal{P}_A)$ we set

$$b(P^\bullet) = \min\{i \in \mathbb{Z} \mid \mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet[i], A) \neq 0\}.$$

Then for any nonzero $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ we set $l(P^\bullet) = a(P^\bullet) - b(P^\bullet)$ and call it the length of $P^\bullet$. For the sake of convenience, we set $l(P^\bullet) = 0$ for $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ with $P^\bullet \simeq 0$.

For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X and by $X^{(n)}$ the direct sum of n copies of X.

Theorem 4. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ be a complex with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \neq 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists some $Q^\bullet \in \mathsf{K}^\mathrm{b}(\mathcal{P}_A)$ such that $Q^\bullet \oplus P^\bullet$ is a tilting complex. In particular, if the number of nonisomorphic indecomposable direct summands of $P^\bullet$ coincides with the rank of the Grothendieck group $K_0(A)$, then $P^\bullet$ is a tilting complex.*

Theorem 5. *Assume A is selfinjective and representation-finite. Then for any selfinjective artin R-algebra B derived equivalent to A the following hold.*

(1) *There exists a sequence of selfinjective artin R-algebras $A = B_0, B_1, \cdots$, $B_m = B$ such that for any $0 \le i < m$, B_{i+1} is the endomorphism algebra of a tilting complex for B_i of length ≤ 1.*
(2) *The Nakayama permutation of B coincides with that of A.*

The proofs of Theorems 4 and 5 follow by induction on the length of $P^\bullet$. But, in Theorem 5, we set $P^\bullet$ to be a tilting complex with $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet) \cong B$. The key of the induction is the following Lemma 6.

Lemma 6. *Assume A is selfinjective and representation-finite. Let $P^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a complex of length ≥ 1 with $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet, P^\bullet[i]) = 0$ for $i \ne 0$ and $\mathrm{add}(P^\bullet) = \mathrm{add}(\nu P^\bullet)$. Then there exists a tilting complex $T^\bullet \in \mathsf{K}^{\mathrm{b}}(\mathcal{P}_A)$ of length 1 such that*
(1) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet, P^\bullet[i]) = 0$ *for $i \ge l(P^\bullet)$,*
(2) $\mathrm{Hom}_{\mathsf{K}(\text{Mod-}A)}(P^\bullet[i], T^\bullet) = 0$ *for $i < 0$, and*
(3) $\mathrm{End}_{\mathsf{K}(\text{Mod-}A)}(T^\bullet)$ *is a selfinjective artin R-algebra whose Nakayama permutation coincides with that of A.*

References

[1] K. Bongartz, Tilted algebras, in: *Representations of algebras (Puebla, 1980)*, 26–38, Lecture Notes in Math., **903**, Springer, Berlin.
[2] D. Happel and C. M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443.
[3] D. Happel, J. Rickard and A. Schofield, Piecewise hereditary algebras, Bull. London Math. Soc. **20** (1988), no. 1, 23–28.
[4] M. Hoshino and Y. Kato, Tilting complexes associated with a sequence of idempotents, J. Pure Appl. Algebra **183** (2003), no. 1-3, 105–124.
[5] S. König and A. Zimmermann, Tilting selfinjective algebras and Gorenstein orders, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 191, 351–361.
[6] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), no. 1, 113–146.
[7] J. Rickard, Morita theory for derived categories, J. London Math. Soc. (2) **39** (1989), no. 3, 436–456.
[8] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra **61** (1989), 303–317.

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