STABLE EQUIVALENCES INDUCED FROM GENERALIZED TILTING MODULES III

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Abstract. For a generalized tilting module ${}_BT_A$ and a nilpotent symmetric algebra $({}_AM_A, \varphi, \psi)$, under natural assumptions, the stable functors $\mathcal{K}er : \underline{\text{mod}}-\Lambda(\psi, \varphi) \to \underline{\text{mod}}-\Lambda(\varphi^T, \psi^T)$ and $\mathcal{C}oker : \underline{\text{mod}}-\Lambda(\varphi^T, \psi^T) \to \underline{\text{mod}}-\Lambda(\psi, \varphi)$ has been constructed and it was proved that they induce an equivalence $\underline{\text{mod}}-\Lambda(\psi, \varphi) \approx \underline{\text{mod}}-\Lambda(\psi^T, \varphi^T)$ in [2]. In this note, it is proved that those functors $\mathcal{K}er$ and $\mathcal{C}oker$ preserve the distinguished triangles and, therefore, the stable categories $\underline{\text{mod}}-\Lambda(\psi, \varphi)$ and $\underline{\text{mod}}-\Lambda(\psi^T, \varphi^T)$ are equivalent as triangulated categories.

1. Introduction

Let A and B be finite dimensional algebras over a field K. A bimodule ${}_BT_A$ is called a generalized tilting module if

(1) $B = \text{End}(T_A)$ and $\text{End}({}_BT) = A$, and
(2) $\text{Ext}^n_B(T, T) = 0 = \text{Ext}^n_A(T, T)$ for any $n > 0$.

A system $({}_AM_A, \psi, \varphi)$ consisting of a bimodule ${}_AM_A$ and two homomorphisms $\varphi : {}_AM \otimes_A M_A \to {}_AM_A$ and $\psi : {}_AM \otimes_A M_A \to {}_A\text{D}A_A$ is called a nilpotent symmetric algebra if

(1) the algebra (M, φ) is associative and nilpotent,
(2) the homomorphism ψ satisfies
   (i) $\psi(\varphi(m_1 \otimes m_2) \otimes m_3) = \psi(m_1 \otimes \varphi(m_2 \otimes m_3))$,
   (ii) $\psi(m_1 \otimes m_2)(1_A) = \psi(m_2 \otimes m_1)(1_A)$

   for all elements $m_1, m_2, m_3 \in M$, and
(3) the homomorphism ψ is non-degenerate in the sense that the condition $\psi(m \otimes M) = 0$ implies $m = 0$ for an element $m \in M$,

where D stands for the canonical duality functor $\text{Hom}_K(?, K)$. Let ${}_BT_A$ is a generalized tilting module and $({}_AM_A, \varphi, \psi)$ a nilpotent symmetric algebra. The induced system $({}_BM^T_B, \varphi^T, \psi^T)$ is defined as $M^T = T \otimes_A \text{Hom}_A(T, M)$ and

$$\varphi^T(t_1 \otimes f_1 \otimes t_2 \otimes f_2) = t_1 \otimes \varphi(f_1(t_2) \otimes f_2(?)) \in M^T,$$

$$\psi^T(t_1 \otimes f_1 \otimes t_2 \otimes f_2) = \psi(f_1(t_2) \otimes f_2(?t_1))(1_A) \in \text{D}B$$

for elements t_1, $t_2 \in T$ and $f_1, f_2 \in \text{Hom}_A(T, M)$. Then, the system (φ^T, ψ^T) is again a nilpotent symmetric algebra if the homomorphism

$$\theta_{T,M} : {}_BT \otimes_A \text{Hom}_A(T, M)_B \to {}_B\text{Hom}_A(T, T \otimes_A M)_B$$

defined by $\theta_{T,M}(t \otimes f)(t') = t \otimes f(t')$ for t, $t' \in T$ and $f \in \text{Hom}_A(T, M)$ is bijective. In this case, we have two symmetric algebras

$$\Lambda(\varphi, \psi) = A \oplus M \oplus \text{D}A$$

The detailed version of this paper will be submitted for publication elsewhere.

and

$$\Lambda(\varphi^T, \psi^T) = B \oplus M^T \oplus \mathrm{D}B.$$

The multiplication of the algebra $\Lambda(\varphi, \psi)$ is defined as

$$(a, m, s) \cdot (a', m', s') = (aa', am' + ma' + \varphi(m \otimes m'), as' + sa' + \psi(m \otimes m'))$$

for $a, a' \in A$, $m, m' \in M$ and $s, s' \in \mathrm{D}A$. In the same way, the multiplication of the algebra $\Lambda(\varphi^T, \psi^T)$ is defined by using homomorphisms φ^T and ψ^T. For such symmetric algebras $\Lambda(\varphi, \psi)$ and $\Lambda(\varphi^T, \psi^T)$, assuming several conditions, it is proved that the kernel functor $\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ and the cokernel functor $\mathcal{C}oker : \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi)$ are defined and that those functors induce a category equivalence $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \approx \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$.

It is known by D. Happel [1] that the stable module category $\underline{\mathrm{mod}}{-}\Lambda$ of any self-injective algebra Λ has a natural structure of triangulated category with Ω_Λ^{-1} as the translation functor. In this note, we prove that our functor $\mathcal{K}er$ preserves the distinguished triangles and, therefore, the stable module categories $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi)$ and $\underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ are equivalent as triangulated categories.

2. The stable functor $\mathcal{K}er$

In order to check that the functor $\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ preserves distinguished triangles in the next section, we recall here its definition.

Let $({}_AM_A, \varphi, \psi)$ be a nilpotent symmetric algebra and ${}_BT_A$ a generalized tilting module. We call an exact sequence

$$\cdots\cdots \to T_1 \to T_0 \to X \to 0$$

a dominant right T_A-resolusion of a module X_A if (1) $T_k \in \mathrm{add}(T_A)$ for all $k \geqslant 0$ and (2) the sequence

$$\cdots\cdots \to \mathrm{Hom}_A(T, T_1) \to \mathrm{Hom}_A(T, T_0) \to \mathrm{Hom}_A(T, X) \to 0$$

is exact again. We denote by $\mathrm{gen}^*(T_A)$ the class of all modules X_A for which there exist dominant right T_A-resolutions. The notion of dominant left $\mathrm{D}T_B$-resolutions of B-modules and the class $\mathrm{cog}^*(\mathrm{D}T_B)$ are defined in the dual manner. To define the stable functors

$$\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \leftrightarrows \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T) : \mathcal{C}oker$$

and to prove that those induce an equivalence $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \approx \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$, we suppose that the following four conditions

(A) the map $\theta_{T,M} : T \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, T \otimes_A M)$ is bijective,
(B) the modules M_A and $T \otimes_A M_A$ are in the class $\mathcal{C}(T_A)$,
(C) the class $\mathcal{C}(T_A)$ is contravariantly finite in the category $\mathrm{mod}{-}A$, and
(D) the class $\mathcal{D}(\mathrm{D}T_B)$ is covariantly finite in the category $\mathrm{mod}{-}B$

are satisfied, where $\mathcal{C}(T_A) = (T_A)^\perp \cap \mathrm{gen}^*(T_A)$ and $\mathcal{D}(\mathrm{D}T_B) = {}^\perp(\mathrm{D}T_B) \cap \mathrm{cog}^*(\mathrm{D}T_B)$.

Let $X_{\Lambda(\varphi,\psi)}$ be a module over the symmetric algebra $\Lambda(\varphi, \psi) = A \oplus M \oplus \mathrm{D}A$. Since A is a subalgebra of $\Lambda(\varphi, \psi)$, X can be seen as a module over A, which we call the underlying module of $X_{\Lambda(\varphi,\psi)}$ and denote by X_A. Then, the multiplication $X \times \Lambda(\varphi, \psi) \to X$ defines two homomorphisms $\alpha_X : X \otimes_A M_A \to X_A$ and $\beta_X : X \otimes_A \mathrm{D}A_A \to X_A$ and they satisfy the four conditions

(M-1) $\beta_X \cdot (\beta_X \otimes \mathrm{D}A) = 0$,
(M-2) $\alpha_X \cdot (\beta_X \otimes M) = 0$,
(M-3) $\beta_X \cdot (\alpha_X \otimes \mathrm{D}A) = 0$, and
(M-4) $\alpha_X \cdot (\alpha_X \otimes M) = \alpha_X \cdot (X \otimes \varphi) + \beta_X \cdot (X \otimes \psi)$.

Conversely, for a module X_A and two homomorphisms $\alpha_X : X \otimes_A M_A \to X_A$ and $\beta_X : X \otimes_A \mathrm{D}A_A \to X_A$ satisfying the four conditions above, we can define a $\Lambda(\varphi, \psi)$-module structure on X by $x \cdot (a, m, s) = xa + \varphi(x \otimes m) + \psi(x \otimes s)$ for elements $x \in X$ and $(a, m, s) \in \Lambda(\varphi, \psi)$. In this way, we may identify any module $X_{\Lambda(\varphi,\psi)}$ with the triple (X_A, α_X, β_X). Similarly, a homomorphism of $\Lambda(\varphi, \psi)$-modules $f : X_{\Lambda(\varphi,\psi)} \to Y_{\Lambda(\varphi,\psi)}$ is a homomorphism of underlying modules $X_A \to Y_A$ which satisfies the following two conditions

(H-1) $f \cdot \alpha_X = \alpha_Y \cdot (f \otimes M)$ and
(H-2) $f \cdot \beta_X = \beta_Y \cdot (f \otimes \mathrm{D}A)$.

Let (X_A, α_X, β_X), (Y_A, α_Y, β_Y) be $\Lambda(\varphi, \psi)$-modules and $f : X_{\Lambda(\varphi,\psi)} \to Y_{\Lambda(\varphi,\psi)}$ a homomorphism. By condition (C), there exist exact sequences of the form

$$0 \to V_X \to W_X \xrightarrow{\gamma_X} X \to 0 \quad \text{and} \quad 0 \to V_Y \to W_Y \xrightarrow{\gamma_Y} Y \to 0$$

such that $V_X, V_Y \in \mathcal{C}(T_A)$ and $W_X, W_Y \in {}^{\perp}\mathcal{C}(T_A)$. Since $\mathrm{Ext}^1_A(W_X, V_Y) = 0$, we get two homomorphisms $W_f : W_X \to W_Y$ and $V_f : V_X \to V_Y$ over A such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_X & \longrightarrow & W_X & \xrightarrow{\gamma_X} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle V_f} & & \downarrow{\scriptstyle W_f} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & V_Y & \longrightarrow & W_Y & \xrightarrow[\gamma_Y]{} & Y & \longrightarrow & 0 \end{array}$$

is commutative.

It is checked that there is an isomorphism $\Lambda(\varphi^T, \psi^T) \otimes_B T \cong T \otimes_A \Lambda(\varphi, \psi)$ of K-spaces and this defines a $(\Lambda(\varphi^T, \psi^T), \Lambda(\varphi, \psi))$-bimodule, which we denote by ${}_{\Lambda(\varphi^T,\psi^T)}\Theta_{\Lambda(\varphi,\psi)}$. Then, the $\Lambda(\varphi^T, \psi^T)$-modules $\mathcal{K}er(X)$, $\mathcal{K}er(Y)$ and a $\Lambda(\varphi^T, \psi^T)$-homomorphism $\mathcal{K}er(f) : \mathcal{K}er(X) \to \mathcal{K}er(Y)$ are defined by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}er(X) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_X \otimes_A \mathrm{D}T) & \xrightarrow{\lambda_X} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathcal{K}er(f)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda(\varphi^T,\psi^T), W_f \otimes \mathrm{D}T)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Theta, f)} & & \\ 0 & \longrightarrow & \mathcal{K}er(Y) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_Y \otimes_A \mathrm{D}T) & \xrightarrow[\lambda_Y]{} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Y) & \longrightarrow & 0 \end{array}$$

where the homomorphism λ_X is defined as follows: First the underlying module of $\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X)$ is $\mathrm{Hom}_A(T, X)$ since $\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) = \mathrm{Hom}_{\Lambda(\varphi,\psi)}(T \otimes_A \Lambda(\varphi, \psi), X) \cong \mathrm{Hom}_A(T, X)$. Second, the underlying module of the $\Lambda(\varphi^T, \psi^T)$-module

$$\mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_X \otimes_A \mathrm{D}T) = \mathrm{Hom}_B(B \oplus T \otimes_A \mathrm{Hom}_A(T, M) \oplus \mathrm{D}B, W_X \otimes_A \mathrm{D}T)$$

is isomorphic to a direct sum of three modules

$$\mathrm{Hom}_B(\mathrm{D}B, W_X \otimes_A \mathrm{D}T) \cong \mathrm{Hom}_B(T \otimes_A \mathrm{D}T, W_X \otimes_A \mathrm{D}T) \cong \mathrm{Hom}_B(T, W_X),$$

$$\mathrm{Hom}_B(T \otimes_A \mathrm{Hom}_A(T, M), W_X \otimes_A \mathrm{D}T) \cong \mathrm{D}(\mathrm{Hom}_A(T, \mathrm{D}\mathrm{Hom}_A(T, M)) \otimes_B \mathrm{Hom}_A(W_X, T))$$
$$\cong \mathrm{D}\mathrm{Hom}_A(W_X, \mathrm{D}\mathrm{Hom}_A(T, M)) \cong W_X \otimes_A \mathrm{Hom}_A(T, M)$$

and

$$\mathrm{Hom}_B(B, W_X \otimes_A \mathrm{D}T) \cong W_X \otimes_A \mathrm{D}T.$$

Using those modules, the map λ_X is defined by giving its three components

$$\lambda_{X,1} = \mathrm{Hom}(T, \gamma_X) : \mathrm{Hom}_A(T, W_X) \to \mathrm{Hom}_A(T, X),$$

$$\lambda_{X,2} = \alpha_X^* \cdot (\gamma_X \otimes \mathrm{Hom}_A(T, M)) : W_X \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, X)$$

and

$$\lambda_{X,3} = \beta_X^* \cdot (\gamma_X \otimes \mathrm{D}T) : W_X \otimes_A \mathrm{D}T \to \mathrm{Hom}_A(T, X),$$

where $\alpha_X^* : X \otimes_A \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T, X)$ and $\beta_X^* : X \otimes_A \mathrm{D}T \to \mathrm{Hom}_A(T, X)$ are the adjoint maps of the structure maps α_X and β_X, respectively.

This defines a K-linear functor $\mathcal{K}er : \mathrm{mod}{-}\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$ and it induces a stable functor $\mathcal{K}er : \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$. Similarly, by using the condition (D), the functor $\mathcal{C}oker : \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T) \to \underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi)$ is defined. Finally, by the condition (B), it is ckecked that those functors define the stable equivalence $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi) \approx \underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$.

3. Equivalences of triangulated categories

A distinguished triangle

$$X_1 \xrightarrow{f} X_2 \longrightarrow C_f \longrightarrow \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)$$

in the stable module category $\underline{\mathrm{mod}}{-}\Lambda(\varphi, \psi)$ is given by the push-out diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & E(X_1) & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X_2 & \longrightarrow & C_f & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0 \end{array}$$

in the module category $\mathrm{mod}{-}\Lambda(\varphi, \psi)$, where $X_1 \hookrightarrow E(X_1)$ is an injection into an injective module $E(X_1)$ and $X \xrightarrow{f} X_2$ an arbitrary homomorphism of $\Lambda(\varphi, \psi)$-modules. We have to prove that the sequence

$$\mathcal{K}er(X_1) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(X_2) \longrightarrow \mathcal{K}er(C_f) \longrightarrow \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1))$$

is again a distinguished triangle in the category $\underline{\mathrm{mod}}{-}\Lambda(\varphi^T, \psi^T)$.

We start with the following result:

Lemma 1. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence of* $\Lambda(\varphi, \psi)$*-modules. Then there exist right* ${}^{\perp}\mathcal{C}(T_A)$*-approximations* $W_X \xrightarrow{\gamma_X} X \to 0$, $W_Y \xrightarrow{\gamma_Y} Y \to 0$ *and* $W_Z \xrightarrow{\gamma_Z} Z \to 0$

such that all the rows and columns are exact in the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_X & \xrightarrow{V_f} & V_Y & \xrightarrow{V_g} & V_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & W_X & \xrightarrow{W_f} & W_Y & \xrightarrow{W_g} & W_Z & \longrightarrow & 0 \\
 & & {\scriptstyle \gamma_X}\downarrow & & {\scriptstyle \gamma_Y}\downarrow & & {\scriptstyle \gamma_Z}\downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. We choose first any right ${}^{\perp}\mathcal{C}(T_A)$-approximations $W_X \stackrel{\gamma_X}{\to} X$ and $W'_Y \stackrel{\gamma'_Y}{\to} Y$ and get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_X & \xrightarrow{s} & W_X & \xrightarrow{\gamma_X} & X & \longrightarrow & 0 \\
 & & {\scriptstyle V'_f}\downarrow & & {\scriptstyle W'_f}\downarrow & & {\scriptstyle f}\downarrow & & \\
0 & \longrightarrow & V'_Y & \xrightarrow[t]{} & W'_Y & \xrightarrow[\gamma'_Y]{} & Y & \longrightarrow & 0
\end{array}$$

In the diagram, W'_f may not be injective, but since $W_X \in {}^{\perp}\mathcal{C}(T_A) \subseteq \operatorname{cog}^*(T_A)$, we can take a left $\operatorname{add}(T_A)$-approximation $0 \to W_X \stackrel{u}{\to} T_0$ and, by setting

$$V_f = \begin{pmatrix} V'_f \\ u \cdot s \end{pmatrix}, \quad W_f = \begin{pmatrix} W'_f \\ u \end{pmatrix} \quad \text{and} \quad \gamma_Y = (\gamma'_Y, 0), \quad t' = \begin{pmatrix} t & 0 \\ 0 & \mathrm{id}_{T_0} \end{pmatrix},$$

we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_X & \xrightarrow{s} & W_X & \xrightarrow{\gamma_X} & X & \longrightarrow & 0 \\
 & & {\scriptstyle V_f}\downarrow & & {\scriptstyle W_f}\downarrow & & {\scriptstyle f}\downarrow & & \\
0 & \longrightarrow & V'_Y \oplus T_0 & \xrightarrow{t'} & W'_Y \oplus T_0 & \xrightarrow{\gamma_Y} & Y & \longrightarrow & 0
\end{array}$$

Here we put $W_Y = W'_Y \oplus T_0$, $V_Y = V'_Y \oplus T_0$, $W_Z = \operatorname{Coker}(W_f)$, $V_Z = \operatorname{Coker}(V_f)$ and denote the cokernels of the maps W_f and V_f by $W_Y \stackrel{W_g}{\to} W_Z \to 0$ and $V_Y \stackrel{V_g}{\to} V_Z \to 0$, respectively. Then, by the snake lemma, we get an exact sequence

$$0 \longrightarrow V_Z \longrightarrow W_Z \xrightarrow{\gamma_Z} Z \longrightarrow 0$$

in which $V_Z \in \mathcal{C}(T_A)$ and $W_Z \in {}^{\perp}\mathcal{C}(T_A)$ hold as easily seen. It is now obvious that those modules and homomorphisms make the diagram as stated in the lemma. **q.e.d**

For a short exact sequence of $\Lambda(\varphi,\psi)$-modules

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

we choose three ${}^{\perp}\mathcal{C}(T_A)$-approximations $\gamma_X : W_X \to X$, $\gamma_Y : W_Y \to Y$, $\gamma_Z : W_Z \to Z$ and two homomorphisms $W_f : W_X \to W_Y$, $W_g : W_Y \to W_Z$ as stated in the lemma. By making use of those modules and homomorphisms, the sequence

$$\mathcal{K}er(X) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(Y) \xrightarrow{\mathcal{K}er(g)} \mathcal{K}er(Z)$$

is defined in the module category $\text{mod}-\Lambda(\varphi^T,\psi^T)$ by the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}er(X) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_X \otimes_A \mathrm{D}T) & \xrightarrow{\lambda_X} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{K}er(f)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda(\varphi^T,\psi^T), W_f \otimes \mathrm{D}T)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Theta,f)} & & \\
0 & \longrightarrow & \mathcal{K}er(Y) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_Y \otimes_A \mathrm{D}T) & \xrightarrow{\lambda_Y} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Y) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathcal{K}er(g)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Lambda(\varphi^T,\psi^T), W_g \otimes \mathrm{D}T)} & & \downarrow{\scriptstyle \mathrm{Hom}(\Theta,g)} & & \\
0 & \longrightarrow & \mathcal{K}er(Z) & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), W_Z \otimes_A \mathrm{D}T) & \xrightarrow{\lambda_Z} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \longrightarrow & 0
\end{array}$$

and we get the following lemma.

Lemma 2. *When we choose right* ${}^{\perp}\mathcal{C}(T_A)$*-approximations* $W_X \overset{\gamma_X}{\to} X$, $W_Y \overset{\gamma_Y}{\to} Y$ *and* $W_Z \overset{\gamma_Z}{\to} Z$ *as in the previous lemma, the sequence*

$$0 \longrightarrow \mathcal{K}er(X) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(Y) \xrightarrow{\mathcal{K}er(g)} \mathcal{K}er(Z) \longrightarrow 0$$

is exact in the module category $\text{mod}-\Lambda(\varphi^T,\psi^T)$.

Proof. Applying the functor $\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, ?)$ to the exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, X) & \xrightarrow{\mathrm{Hom}(\Theta,f)} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Y) & \xrightarrow{\mathrm{Hom}(\Theta,g)} & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & & \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_A(T, X) & \xrightarrow[\mathrm{Hom}(T,f)]{} & \mathrm{Hom}_A(T, Y) & \xrightarrow[\mathrm{Hom}(T,g)]{} & \mathrm{Hom}_A(T, Z) & \longrightarrow & \mathrm{Ext}^1_A(T, X)\cdots
\end{array}$$

Similarly, applying the functor

$$\mathrm{Hom}_B(\Lambda(\varphi^T,\psi^T), ?) \cong \begin{array}{c} \mathrm{Hom}_B(B, ?) \\ \oplus \\ \mathrm{Hom}_B(T \otimes_A \mathrm{Hom}_A(T, M), ?) \\ \oplus \\ \mathrm{Hom}_B(\mathrm{D}B, ?) \end{array}$$

to the exact sequence

$$0 \longrightarrow W_X \otimes_A \mathrm{D}T \xrightarrow{W_f \otimes \mathrm{D}T} W_Y \otimes_A \mathrm{D}T \xrightarrow{W_g \otimes \mathrm{D}T} W_Z \otimes_A \mathrm{D}T \longrightarrow 0$$

we have two exact sequences

$$0 \longrightarrow \mathrm{Hom}_B(B, W_X \otimes_A \mathrm{D}T) \longrightarrow \mathrm{Hom}_B(B, W_Y \otimes_A \mathrm{D}T) \longrightarrow \mathrm{Hom}_B(B, W_Z \otimes_A \mathrm{D}T) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}_B(N, W_X \otimes_A \mathrm{D}T) \longrightarrow \mathrm{Hom}_B(N, W_Y \otimes_A \mathrm{D}T) \longrightarrow \mathrm{Hom}_B(N, W_Z \otimes_A \mathrm{D}T) \longrightarrow 0,$$

where $N = T \otimes_A \mathrm{Hom}_A(T, M)$, and the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}_B(\mathrm{D}B, W_X \otimes_A \mathrm{D}T) & \longrightarrow & \mathrm{Hom}_B(\mathrm{D}B, W_Y \otimes_A rmDT) & \longrightarrow & \mathrm{Hom}_B(\mathrm{D}B, W_Z \otimes_A \mathrm{D}T) & \\
& \cong\downarrow & & \cong\downarrow & & \cong\downarrow & \\
0 \longrightarrow & \mathrm{Hom}_A(T, W_X) & \xrightarrow[\mathrm{Hom}(T,W_f)]{} & \mathrm{Hom}_A(T, W_Y) & \xrightarrow[\mathrm{Hom}(T,W_g)]{} & \mathrm{Hom}_A(T, W_Z) \to \cdots &
\end{array}$$

Then, combining those diagrams, we get the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}er(X) & \xrightarrow{\mathcal{K}er(f)} & \mathcal{K}er(Y) & \xrightarrow{\mathcal{K}er(g)} & \mathcal{K}er(Z) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_Z \otimes_A \mathrm{D}T) & \longrightarrow & \mathrm{Ext}^1_A(T, W_X) \to \cdots \\
 & & \downarrow\lambda_X & & \downarrow\lambda_Y & & \downarrow\lambda_Z & & \\
0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \longrightarrow & \mathrm{Ext}^1_A(T, X) \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

On the other hand, from the exact sequence $0 \to V_X \to W_X \xrightarrow{\gamma_X} X \to 0$ with $V_X \in \mathcal{C}(T_A)$, we have an isomorphism $\mathrm{Ext}^1(T, \gamma_X) : \mathrm{Ext}^1_A(T, W_X) \xrightarrow{\approx} \mathrm{Ext}^1_A(T, X)$. Therefore, to prove the surjectivity of the map $\mathcal{K}er(g) : \mathcal{K}er(Y) \to \mathcal{K}er(Z)$, it is enough to show that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_B(\Lambda(\varphi^T, \psi^T), W_Z \otimes_A \mathrm{D}T) & \longrightarrow & \mathrm{Ext}^1_A(T, W_Z) \\
\lambda_Z \downarrow & & \downarrow \mathrm{Ext}^1(T,\gamma_X) \\
\mathrm{Hom}_{\Lambda(\varphi,\psi)}(\Theta, Z) & \longrightarrow & \mathrm{Ext}^1_A(T, X)
\end{array}$$

is commutative. It is easyto see that the commutativity of the above diagram is equivalent to the following two assertions:

(1) The composition maps

$$Z \otimes_A \mathrm{Hom}_A(T, M) \xrightarrow{\alpha^*_Z} \mathrm{Ext}^1_A(T, Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, X)$$

and

$$Z \otimes_A \mathrm{D}T \xrightarrow{\beta^*_Z} \mathrm{Hom}_A(T, Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, X)$$

are the zero maps, where $\mathrm{Hom}_A(T, Z) \xrightarrow{\Delta} \mathrm{Ext}^1_A(T, X)$ satands for the connecting homomorphism corresponding to the exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.

(2) The diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_B(T \otimes_A \mathrm{D}T, W_Z \otimes_A \mathrm{D}T) & \xleftarrow[\cong]{(? \otimes \mathrm{D}T)} & \mathrm{Hom}_A(T, W_Z) & \xrightarrow{\Delta} & \mathrm{Ext}^1_A(T, W_X) \\
\zeta \uparrow \cong & & & & \cong \downarrow \mathrm{Ext}^1(T,\gamma_X) \\
\mathrm{Hom}_A(T, W_Z) & \xrightarrow[\mathrm{Hom}(T,\gamma_Z)]{} & \mathrm{Hom}_A(T, Z) & \xrightarrow[\Delta]{} & \mathrm{Ext}^1_A(T, X)
\end{array}$$

is commutative, where the vertical map ζ in the left hand side is the composition

$$\mathrm{Hom}_A(T, W_Z) \xrightarrow[\cong]{\mathrm{Hom}(T,\eta^{\mathrm{D}T}_{W_Z})} \mathrm{Hom}_A(T, \mathrm{Hom}_B(\mathrm{D}T, W_Z \otimes_A \mathrm{D}T) \xrightarrow[\cong]{\mathrm{can}} \mathrm{Hom}_B(T \otimes_A \mathrm{D}T, W_Z \otimes_A \mathrm{D}T)$$

and the map $\mathrm{Hom}_A(T, W_Z) \overset{\Delta}{\to} \mathrm{Ext}^1_A(T, W_X)$ stands for the connecting homomorphism corresponding to the exact sequence $0 \to W_X \overset{W_f}{\to} W_Y \overset{W_g}{\to} W_Z \to 0$.

Proof of the assertion (1): For any element $y \in Y$ and $u \in \mathrm{Hom}_A(T, M)$, the element $\Delta(\alpha^*_Z(g(y) \otimes u)) \in \mathrm{Ext}^1_A(T, X)$ is determined by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \alpha^*_Z(g(y)\otimes u) & & \\
0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0
\end{array}$$

and it is easily verified that the homomorphism $\alpha^*_Z(g(y) \otimes u)$ is lifted to the homomorphism $\alpha^*_Y(y \otimes u)$ through the surjective map g. Therefore, the upper sequence in the diagram splits and we have $\Delta \cdot \alpha^*_Z = 0$. We can prove $\Delta \cdot \beta^*_Z = 0$ in the same way.

Proof of the assertion (2): It is checked that the map ζ coincides with $(? \otimes \mathrm{D}T)$: $\mathrm{Hom}_A(T, W_Z) \to \mathrm{Hom}_B(T \otimes_A \mathrm{D}T, W_Z \otimes_A \mathrm{D}T)$. Hence, the commutativity of the diagram follows from the naturality of the connecting homomorphisms. **q.e.d**

Theorem 3. *The stable equivalence functor $\mathcal{K}er$: $\underline{\mathrm{mod}}$–$\Lambda(\varphi, \psi) \to \underline{\mathrm{mod}}$–$\Lambda(\varphi^T, \psi^T)$ is an equivalence of triangulated categories.*

Proof. Applying Lemma 2 to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & E(X_1) & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1) & \longrightarrow & 0 \\
 & & f \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & X_2 & \longrightarrow & C_f & \longrightarrow & \Omega^{-1}_{\Lambda(\varphi,\psi}(X_1) & \longrightarrow & 0
\end{array}$$

we have the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}er(X_1) & \longrightarrow & \mathcal{K}er(E(X_1)) & \longrightarrow & \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)) & \longrightarrow & 0 \\
 & & \mathcal{K}er(f) \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{K}er(X_2) & \longrightarrow & \mathcal{K}er(C_f) & \longrightarrow & \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)) & \longrightarrow & 0
\end{array}$$

We know that the module $\mathcal{K}er(Q)$ over the algebra $\Lambda(\varphi^T, \psi^T)$ is projective for any projective module Q over the algebra $\Lambda(\varphi, \psi)$ by the construction. Therefore, we see that the equality

$$\mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1)) = \Omega^{-1}_{\Lambda(\varphi^T,\psi^T)}(\mathcal{K}er(X_1))$$

holds and the sequence

$$\mathcal{K}er(X_1) \xrightarrow{\mathcal{K}er(f)} \mathcal{K}er(X_2) \longrightarrow \mathcal{K}er(C_f) \longrightarrow \mathcal{K}er(\Omega^{-1}_{\Lambda(\varphi,\psi)}(X_1))$$

is again a distinguished triangle in the stable category $\underline{\text{mod}}-\Lambda(\varphi^T, \psi^T)$. This completes the proof. **q.e.d.**

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