ON A TENSOR PRODUCT OF SQUARE MATRICES IN JORDAN CANONICAL FORMS

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Abstract. Let K be an algebraically closed field of characteristic $p \geqslant 0$. We shall consider the problem of finding out a Jordan canonical form of $J(a, s) \otimes_K J(b, t)$, where $J(a, s)$ means the Jordan block with eigenvalue $a \in K$ and size s.

1. Introduction

To construct graded local Frobenius algebras over an algebraically closed field K, it is important to find out a Jordan canonical form (simply, JCF) of tensor product of square matrices. In fact, it is known that any graded local Frobenius algebra is of the form of $\Lambda(\varphi, \gamma) = T(V)/R(\varphi, \gamma)$, where V is a finite dimensional K-vector space, γ an element of $GL(V)$, and $\varphi : V^{\otimes n} \to K$ a K-linear map satisfying several conditions. Further, if we decompose as $(V, \gamma) = \bigoplus_i (V_i, \gamma_i)$, then the conditions of φ can be described in terms of each $\varphi_{i_1 \dots i_r} : V_{i_1} \otimes \cdots \otimes V_{i_r} \to K$. Then, we have to find out a Jordan canonical form of $\gamma_{i_1} \otimes \cdots \otimes \gamma_{i_r}$ as an element in $GL(V_{i_1} \otimes \cdots \otimes V_{i_r})$. (For detail, refer to T. Wakamatsu [2]).

Let K be an algebraically closed field of characteristic $p \geqslant 0$, and $J(a, s)$, $J(b, t)$ Jordan blocks over K. We shall consider the problem of finding out a JCF of $J(a, s) \otimes J(b, t)$, where $\otimes$ means $\otimes_K$. And then we may assume $s \leqslant t$.

In the case of $ab \neq 0$, our problem is reduced to the problem of finding the indecomposable decomposition of R as a $K[\theta]$-module, where R means the polynomial ring $K[x, y]$ with relation ($x^s = 0 = y^t$) and $\theta = x + y$. In Theorem 3, we show that we can find out s homogeneous elements $\omega_0, \omega_1, \dots, \omega_{s-1}$ such that

$$R \cong \bigoplus_{i=0}^{s-1} K[\theta]\omega_i$$

as $K[\theta]$-modules, where the degree of ω_i is i (for each $0 \leqslant i \leqslant s-1$). Applying this result, we show an algorithm for computing a JCF of $J(a, s) \otimes J(b, t)$ in Theorem 15. In the case of $ab = 0$, we give the complete solution of our problem in Theorem 9.

A. Martsinkovsky and A. Vlassov [1] gave the solution of this problem in the case of $p = 0$.

2. Main Result

2.1. **The indecomposable decomposition that gives a JCF of** $J(a, s) \otimes J(b, t)$**.** To find out a JCF of $J(a, s) \otimes J(b, t)$, we have to find its eigenvalues, the number of Jordan

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blocks, and the sizes of Jordan blocks. It is clear the eigenvalue of $J(a,\, s) \otimes J(b,\, t)$ is only ab.

We consider the indecomposable decomposition of

$$\frac{K[X]}{((X-a)^s)} \otimes \frac{K[Y]}{((Y-b)^t)}$$

as a $K[X \otimes Y]$-module. By replacing variables and so on, we have the following:

(1) $ab \neq 0$:

$$\left(\frac{K[X]}{((X-a)^s)} \otimes \frac{K[Y]}{((Y-b)^t)}\right)_{K[X \otimes Y]} \cong \left(\frac{K[X,\, Y]}{(X^s,\, Y^t)}\right)_{K[X+Y]}.$$

(2) $a = 0,\, b \neq 0$:

$$\left(\frac{K[X]}{(X^s)} \otimes \frac{K[Y]}{((Y-b)^t)}\right)_{K[X \otimes Y]} \cong \left(\frac{K[X,\, Y]}{(X^s,\, Y^t)}\right)_{K[X]}.$$

(3) $a \neq 0,\, b = 0$:

$$\left(\frac{K[X]}{((X-a)^s)} \otimes \frac{K[Y]}{(Y^t)}\right)_{K[X \otimes Y]} \cong \left(\frac{K[X,\, Y]}{(X^s,\, Y^t)}\right)_{K[Y]}.$$

(4) $a = 0 = b$:

$$\left(\frac{K[X]}{(X^s)} \otimes \frac{K[Y]}{(Y^t)}\right)_{K[X \otimes Y]} \cong \left(\frac{K[X,\, Y]}{(X^s,\, Y^t)}\right)_{K[XY]}.$$

We put $x = \bar{X},\, y = \bar{Y} \in K[X,\, Y]/(X^s,\, Y^t)$, and $R = K[x,\, y]$.

Lemma 1. *Our problem is reduced to the problem of finding the indecomposable decomposition of R as a $K[\theta]$-module, where θ means $x + y$ (if $ab \neq 0$), x ($a = 0,\, b \neq 0$), y ($a \neq 0,\, b = 0$), and xy ($a = 0 = b$).*

We discuss on the assumption $ab \neq 0$, i.e. $\theta = x + y$, unless otherwise stated.

It is clear R is a finite dimensional graded K-algebra. In fact, we denote by R_i the subset of R consisting of all homogeneous elements with degree i, then we have $R = \bigoplus_{i=0}^{s+t-2} R_i$. And we immediately know $\dim_K R_i$ are as follows $(1,\, 2,\, \ldots,\, s,\, s,\, \ldots,\, s,\, s-1,\, \ldots,\, 1)$ for $0 \leqslant i \leqslant s+t-2$.

The subalgebra $K[\theta]$ of R is uniserial, and hence is a quasi-Frobenius. We denote by n the *nilpotency* of θ (i.e. $\theta^n \neq 0$, but $\theta^{n+1} = 0$), and then we can choose $\langle 1,\, \theta,\, \cdots,\, \theta^n \rangle$ as a K-basis of $K[\theta]$. By easy calculation, we have the following inequality on n:

Lemma 2. *We have $t - 1 \leqslant n \leqslant s + t - 2$. In particular, $n = s + t - 2$ if $p = 0$.*

Since the algebra $K[\theta]$ is uniserial, any indecomposable summand M of $R_{K[\theta]}$ can be of written as $K[\theta]\omega$ for some element ω in R. Hence we can write the indecomposable decomposition of $R_{K[\theta]}$ such as:

$$R = \bigoplus_{i=1}^{r} K[\theta]\omega_i \quad (\omega_i \in R). \tag{2.1}$$

We shall call each element ω_i a *generator* (for an indecomposable summand of $R_{K[\theta]}$), and the set $\{\omega_1, \ldots, \omega_r\}$, which consists of the generators in (2.1), a *generating set* (for the indecomposable decomposition of $R_{K[\theta]}$). Although a generating set is not unique, we can choose some generating set that helps us to consider our problem:

Theorem 3. *There exists a generating set $\{\omega_0, \omega_1, \ldots, \omega_{s-1}\}$ whose generator ω_i is an i-th degree homogeneous element. Hence,*

$$R = \bigoplus_{i=0}^{s-1} K[\theta]\omega_i \quad (\omega_i \in R_i).$$

We prepare some lemmas and notation for the proof of Theorem 3.

For a uniserial $K[\theta]$-submodule M of R generated by some homogeneous elements of R, we denote by $\sigma(M)$ the socle degree of M as a $K[\theta]$-module; i.e. $\sigma(M) = d$ if $\operatorname{soc}_{K[\theta]}(M) \subseteq R_d$. For example, $\sigma(K[\theta]) = n$, and $\sigma(K[\theta]x) = n+1$ if $\theta^n x \neq 0$. The following lemmas are easily checked:

Lemma 4. *Let α, β be homogeneous elements of R. If $\sigma(K[\theta]\alpha) \neq \sigma(K[\theta]\beta)$, then $K[\theta]\alpha \cap K[\theta]\beta = \{0\}$ holds. Hence $K[\theta]\alpha + K[\theta]\beta = K[\theta]\alpha \oplus K[\theta]\beta$.*

Lemma 5. *Let κ be a homogeneous element of R. If $d := \sigma(K[\theta]\kappa) < s+t-2$, then $\kappa x^{s+t-2-d} \neq 0$ holds. Hence,*

$$\sum_{i=0}^{s+t-2-d} K[\theta]\kappa x^i = \bigoplus_{i=0}^{s+t-2-d} K[\theta]\kappa x^i.$$

The multiplication map $\times\theta^j : R_i \to R_{i+j}$ is a K-linear map. We denote by $K(i, i+j)$ the kernel of this map.

Lemma 6. *For each $0 \leqslant i \leqslant s-1$, we have the following:*

(1) *The map $\times\theta^{t-1-i} : R_i \to R_{t-1}$ is injective.*
(2) *The map $\times\theta^{s+t-1-2i} : R_i \to R_{s+t-1-i}$ is not injective.*

Hence, for an elemant κ_i in $K(i, s+t-1-i) \subseteq R_i$, we have

$$\theta^{s+t-2-1-2i}\kappa_i = 0, \quad \textit{but} \quad \theta^{t-1-i}\kappa_i \neq 0.$$

We now prove Theorem 3:

The proof of Theorem 3. We put $n_0 = n$ and $m_0 = s+t-2-n_0$. If $m_0 > 0$, then we have

$$\sum_{i_0=0}^{m_0} K[\theta]x^{i_0} = \bigoplus_{i_0=0}^{m_0} K[\theta]x^{i_0} \subseteq R$$

by Lemma 5. If this direct sum coincides with R, then we finish the proof. Suppose not. By Lemma 6, we can take an element $\kappa_{(1)} \in K(m_0+1, n_0)$ and then we have $t-1 \leqslant \sigma(K[\theta]\kappa_{(1)}) \leqslant n_0 - 1$. We put $n_1 = \sigma(K[\theta]\kappa_{(1)})$ and $m_1 = (n_0-1) - n_1$. If $m_1 > 0$, then we have

$$\left(\bigoplus_{i_0=0}^{m_0} K[\theta]x^{i_0}\right) + \left(\sum_{i_1=0}^{m_1} K[\theta]\kappa_{(1)}x^{i_1}\right) = \bigoplus_{i_0=0}^{m_0} K[\theta]x^{i_0} \oplus \bigoplus_{i_1=0}^{m_1} K[\theta]\kappa_{(1)}x^{i_1} \subseteq R$$

from Lemma 5. Thus, we can construct the direct sum of $K[\theta]$-submodules of R. However, since R is finite dimensional, this construction will be over in finite steps. And it is clear that this construction finishes just when s-th direct summand is constructed. By Krull-Schmidt theorem, this decomposition is the indecomposable decomposition of $R_{K[\theta]}$. (And this argument does work when some m_i is zero.) □

Remark 7. (1) This proof gives concretely the indecomposable summands of $R_{K[\theta]}$ such as:

$$\begin{aligned} &K[\theta],\ K[\theta]x,\ \ldots,\ K[\theta]x^{m_0}, \\ &\qquad K[\theta]\kappa_{(1)},\ K[\theta]\kappa_{(1)}x,\ \ldots,\ K[\theta]\kappa_{(1)}x^{m_1}, \\ &\qquad\qquad \cdots\cdots \\ &\qquad\qquad K[\theta]\kappa_{(r-1)},\ K[\theta]\kappa_{(r-1)}x,\ \ldots,\ K[\theta]\kappa_{(r-1)}x^{m_{r-1}}, \end{aligned}$$

where $\kappa_{(i)}$ means some elnement in $K(m_{i-1}+1,\ n_{i-1})$ and $m_i = (n_{i-1}-1) - n_i$, $n_i = \sigma(K[\theta]\kappa_{(i)})$. Thus, these $\kappa_{(i)}$, m_i, n_i are determined by the following order:

$$n = n_0 \to m_0 \to \kappa_{(1)} \to n_1 \to m_1 \to \kappa_{(2)} \to \cdots \to n_{i-1} \to m_{i-1} \to \kappa_{(i)} \to \cdots .$$

(Then we define $n_{-1} = s+t-1$, $m_{-1} = 0$, and $\kappa_{(0)} = 1_R$ for convenience).

(2) We have to discuss on whether the value of $n_i = \sigma(K[\theta]\kappa_{(i)})$ varies by the choice of an element $\kappa_{(i)} \in K(m_{i-1}+1,\ n_{i-1})$. However, we immediately find that the sequence $(n_0,\ n_1, \ldots,\ n_{r-1})$ is unique by the uniqueness of the indecomposable decomposition of $R_{K[\theta]}$. Therefore we can choose κ_i free.

(3) Theorem 3 declares the number of Jordan blocks of $J(a,\ s) \otimes J(b,\ t)$ is s if $ab \neq 0$.

Definition 8. Thus, the particular indecomposable summands

$$(K[\theta] =)\, K[\theta]\kappa_{(0)},\ K[\theta]\kappa_{(1)},\ \ldots,\ K[\theta]\kappa_{(r-1)}$$

of $R_{K[\theta]}$ characterize the indecomposable decomposition of $R_{K[\theta]}$. So, we shall call each $K[\theta]\kappa_{(i)}$ a *leading module* (of $R_{K[\theta]}$). And we call the number of the indecomposable summands of $R_{K[\theta]}$ whose lengths are equal to that of $K[\theta]\kappa_{(i)}$ the *leading degree* of $K[\theta]\kappa_{(i)}$.

By this result, if there are r leading modules $K[\theta]\kappa_{(0)}$, $K[\theta]\kappa_{(1)}$, …, $K[\theta]\kappa_{(r-1)}$, then we have

$$J(a,\ s) \otimes J(b,\ t) \equiv \bigoplus_{i=0}^{r-1} J(ab,\ \ell_i)^{\oplus d_i},$$

where ℓ_i and d_i mean the length and leading degree of $K[\theta]\kappa_{(i)}$ respectively.

In the case of $ab = 0$, the algebra $K[\theta]$ is also uniserial. Hence we can apply a similar argument of the proof of Theorem 3.

Theorem 9. *If $ab = 0$. Then, for any characteristic p, we have the following:*

(1) $a = 0,\ b \neq 0$: *By taking $\{1,\ y,\ \ldots,\ y^{t-1}\}$ as a generating set;*

$$J(0,\ s) \otimes J(b,\ t) \equiv J(0,\ s)^{\oplus t}.$$

(2) $a \neq 0,\ b = 0$: *By taking $\{1,\ x,\ \ldots,\ x^{s-1}\}$;*

$$J(a,\ s) \otimes J(0,\ t) \equiv J(0,\ t)^{\oplus s}.$$

(3) $a = 0 = b$: *By taking* $\{1, x, \ldots, x^{s-1}, y, y^2, \ldots, y^{t-1}\}$;

$$J(0, s) \otimes J(0, t) \equiv J(0, s)^{\oplus t-s+1} \oplus \bigoplus_{i=1}^{s-1} J(0, s-i)^{\oplus 2}.$$

2.2. **An algorithm for computing a JCF of $J(a, s) \otimes J(b, t)$.** Next, we show there exists a good way to compute a JCF of $J(a, s) \otimes J(b, t)$. To compute it, we find the lengths and the leading degrees of the leading modules.

For each $0 \leqslant i \leqslant s-1$, we define a function such as

$$D_p(i) = \begin{cases} 0 & (\text{if the map } \times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i} \text{ is bijective}) \\ 1 & (\text{if the map } \times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i} \text{ is not bijective}) \end{cases}.$$

And we put

$$\varDelta_p = (D_p(0), D_p(1), \ldots, D_p(s-1)).$$

Remark 10. By Lemma 6 (1), we have known the map $\times\theta^{t-s} : R_{s-1} \to R_{t-1}$ is always injective (hence, bijective) independently of the value of characteristic p. So $D_p(s-1) = 0$ holds.

By Theorem 3, we may assume that R is of the form of $\bigoplus_{i=0}^{s-1} K[\theta]\omega_i$, i.e. any base of R is of the form of $\theta^j\omega_i$. This procedures the following lemmas:

Lemma 11. *If an indecomposable summand $K[\theta]\omega_i$ is a leading module and $D_p(i) = 0$. Then we have the following:*

(1) $\sigma(K[\theta]\omega_i) = s+t-2-i$. *Hence the length and the leading degree of $K[\theta]\omega_i$ are $s+t-1-2i$ and one respectively.*
(2) *The next indecomposable summand $K[\theta]\omega_{i+1}$ is a leading module if $i+1 < s$.*

Lemma 12. *If an indecomposable summand $K[\theta]\omega_i$ is a leading module, $D_p(i) = D_p(i+1) = \cdots = D_p(i+f-1) = 1$, and $D_p(i+f) = 0\,(f > 0)$. Then we have the following:*

(1) $\sigma(K[\theta]\omega_i) = s+t-2-i-f$. *Hence the length and the leading degree of $K[\theta]\omega_i$ are $s+t-1-2i-f$ and $f+1$ respectively.*
(2) *The indecomposable summand $K[\theta]\omega_{i+f+1}$ is a leading module if $i+f+1 < s$.*

Since the indecomposable summand $K[\theta]\omega_0$ is a leading module, we can apply Lemma 11 and 12 to the components of an arbitrary $\varDelta_p$ inductively. Thus, via the sequence $\varDelta_p$, we can compute the lengths and the leading degrees of the leading modules concretely:

Theorem 13. *We can compute a JCF of $J(a, s) \otimes J(b, t)$ by using the sequence $\varDelta_p$.*

We can compute the determinant $D(i)$ of the linear map $\times\theta^{s+t-2-2i} : R_i \to R_{s+t-2-i}$ by using elementary techniques of linear algebra:

Theorem 14. *For each $0 \leqslant i \leqslant s-1$, we have*

$$D(i) = \prod_{k=0}^{i} \frac{\binom{s+t-2-2i+k}{t-1-i}}{\binom{t-1-i+k}{t-1-i}}.$$

By Theorem 13 and 14, we get an algorithm for computing a JCF of $J(a, s) \otimes J(b, t)$:

Theorem 15. *We can compute a JCF of $J(a, s) \otimes J(b, t)$ by taking the following steps:*
Step 1: Computing $D(i)$ for each $0 \leqslant i \leqslant s-1$.
Step 2: Computing the sequence Δ_p. $D_p(i) = 0$ iff $D(i) \not\equiv 0 \,(\mathrm{mod}\ p)$.
Step 3: Applying Theorem 13.

Example 16. Let us compute a JCF of $J(a, 4) \otimes J(b, 5)\,(ab \neq 0)$. The determinants $D(i)$ are

$$D(0) = \frac{\binom{7}{4}}{\binom{4}{4}} = 5 \cdot 7, \quad D(1) = \frac{\binom{5}{3}\binom{6}{3}}{\binom{3}{3}\binom{4}{3}} = 2 \cdot 5^2, \quad D(2) = \frac{\binom{3}{2}\binom{4}{2}\binom{5}{2}}{\binom{2}{2}\binom{3}{2}\binom{4}{2}} = 2 \cdot 5, \quad D(3) = 1.$$

So the sequence Δ_p is

$$\begin{aligned} \Delta_p &= (0, 0, 0, 0) \ \ (p \neq 2, 5, 7), \\ \Delta_2 &= (0, 1, 1, 0), \\ \Delta_5 &= (1, 1, 1, 0), \\ \Delta_7 &= (1, 0, 0, 0). \end{aligned}$$

Therefore

$$J(a, 4) \otimes J(b, 5) \equiv \begin{cases} J(ab, 8) \oplus J(ab, 6) \oplus J(ab, 4) \oplus J(ab, 2) & (p \neq 2, 5, 7) \\ J(ab, 8) \oplus J(ab, 4)^{\oplus 3} & (p = 2) \\ J(ab, 5)^{\oplus 4} & (p = 5) \\ J(ab, 7)^{\oplus 2} \oplus J(ab, 4) \oplus J(ab, 2) & (p = 7) \end{cases}.$$

If $p = 0$ or $p > s + t - 2$, then the determinants $D(i)$ are clearly all non-zero. Hence:

Corollary 17. *If $p = 0$ or $p > s + t - 2$, then*

$$J(a, s) \otimes J(b, t) \equiv \bigoplus_{i=0}^{s-1} J(ab, s + t - 1 - 2i).$$

References

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