FINITE GROUPS HAVING EXACTLY ONE NON-LINEAR IRREDUCIBLE CHARACTER

KAORU MOTOSE

Recently, A.S. Muktibodh [12, 11, 10] considered a 2-Con-Cos group G defined as follows: the commutator subgroup G' of a finite group G consist of two conjugate classes C_a and $C_1 = \{1\}$, and cosets $G'x$ are conjugate classes C_x of $x \in G \setminus G'$. In this paper, we replace "2-Con-Cos" by "concos".

These groups are just groups having exactly one non-linear irreducible character because the number of irreducible characters is equal to the number of conjugate classes, $G' \neq \{1\}$ contains at least two conjugate classes and a coset $G'x$ of G' contains at least one conjugate class C_x.

In §1, we shall prove these groups are isomorphic to affine groups over finite fields or central products of some dihedral groups D of order 8 and quaternion groups Q, and conversely.

After my talk, Professor Y. Ninomiya informed me that this characterization was known in some papers [14, 13, 2]. Further the paper [2] stated that more general information was considered in [8]. However I have arranged this characterization for some reasons that our proof is slight different from others, rather self contained and necessary for §2 and §3.

In §2, we determine $\mathbb{C}$-irreducible $\mathbb{R}$-representations of concos groups.

In §3, we shall show concos groups appear in the proof of Hurwitz theorem concerning quadratic forms. We also determine $\mathbb{C}$-irreducible $\mathbb{R}$-representations of slight different groups in the proof of this theorem.

All representations and characters are considered over $\mathbb{C}$.

1. Characterization of concos groups

First we show elementary properties of concos groups from the definition.

Lemma 1. Let G be concos. Then we have

(1) *If N is a normal subgroup of G then $N = \{1\}$ or $N \supset G'$.*
(2) *G' is an elementary abelian p-group.*
(3) *Exactly one non-linear irreducible character η of G has the next values and we can see from these values that η is faithful.*

$$\eta(1)^2 = |G/G'|(|G'| - 1), \quad \eta(x) = -\frac{|G/G'|}{\eta(1)} \ \text{ for } x \in G' \setminus \{1\},$$

and $\eta(x) = 0$ for $x \in G \setminus G'$.

The detailed version of this paper will be submitted for publication elsewhere.

Proof. (1) If N contains $b \neq 1$, then N contains C_b. In case $b \in G'$, $N \supset C_b = C_a$ and so $N \supset G'$. In case $b \notin G'$, $N \supset C_b = G'b$ and so $N = Nb^{-1} \supset G'$.

(2) G' is a p-group because $G' = \{1\} \cup C_a$ and G' contains an element of prime order p. Thus G'' is a normal subgroup of G properly contained in a p-group G' and so $G'' = 1$ from (1).

(3) Let ρ_G and $\rho_{G/G'}$ are regular characters of G and G/G', respectively. Then we have

$$\rho_G = \rho_{G/G'} + \eta(1)\eta.$$

Using this we obtain our assertion.

The next theorem follows from Lemma 1 (1) and (2).

Theorem 2 ([14, 13, 2, 12]). Let G be concos. Then we have the next groups and conversely.

1. G is the central product QD^{r-1} or D^r where D is the dihedral group of order 8 and Q is the quaternion group of order 8.
2. G is the next permutation group over a finite field $\mathbb{F}_q$ of order $q > 2$.

$$G = \{x \to \alpha x + \beta \mid \alpha \in \mathbb{F}_q^* \text{ and } \beta \in \mathbb{F}_q\}.$$

2. Real representations of concos groups

Let Ψ be a $\mathbb{C}$-irreducible representation of a finite group G and let χ be a character afforded by Ψ. We set

$$\nu(\chi) = \frac{1}{|G|} \sum_{x \in G} \chi(x^2).$$

If χ is linear, then $\nu(\chi) = (\chi, \bar{\chi})$, where $(\chi, \bar{\chi})$ is the inner product of χ and $\bar{\chi}$ is the complex conjugate of χ. Thus it is easy to see that $\nu(\chi) = 1, 0$ and also that (1) $\nu(\chi) = 1$ if and only if $\chi = \bar{\chi}$ and (2) $\nu(\chi) = 0$ if and only if $\chi \neq \bar{\chi}$.

Frobenius and Schur proved in [4] (see [3]) that $\nu(\chi) = 1, 0, -1$ and

(1) $\nu(\chi) = 1$ if and only if Ψ is equivalent to an $\mathbb{R}$-representation.
(2) $\nu(\chi) = 0$ if and only if $\chi \neq \bar{\chi}$.
(3) $\nu(\chi) = -1$ if and only if $\chi = \bar{\chi}$ but Ψ is not equivalent to an $\mathbb{R}$-representation.

Let d be a fixed element of a finite group G and let s_d be the number of elements $x \in G$ such that $x^2 = d$. There is the formula [3, p. 22 (3.6)] about s_d as follows:

$$s_d = \sum_{\lambda \in \Lambda} \nu(\lambda)\lambda(d)$$

where Λ is the set of irreducible characters of G.

The next lemma is useful on $\mathbb{C}$-irreducible $\mathbb{R}$-representations of concos groups.

Lemma 3. Let G be a concos group, $G' = \{1\} \cup C_a$ and let η be exactly one non-linear irreducible character. Then we have

$$s_1 - s_a = \frac{|G|}{\eta(1)} \nu(\eta).$$

In the next proposition we can see $\mathbb{C}$-irreducible $\mathbb{R}$-representations of concos 2-groups. We also can see the numbers of elements of orders 4, 2 in these groups. Our counting method is different from [5, pp. 205-207]. Therefore this gives a different proof about that D^r and QD^{r-1} are not isomorphic (see Remark. (2)).

Proposition 4. Let G be an extra special 2-group D^r or QD^{r-1} of order 2^n, where $n = 2r + 1$. Then elements in G are of order 1 or 2 or 4. Let R be the $\mathbb{C}$-irreducible representation of degree 2^r and η is a character afforded by R. Let s be the number of elements of order 2 or 1 and let t be the number of elements of order 4. Then we have

(1) In case $G = D^r$, R is equivalent to an $\mathbb{R}$-representation, $s = 2^{n-1} + 2^r$ and $t = 2^{n-1} - 2^r$.

(2) In case $G = QD^{r-1}$, R is not equivalent to an $\mathbb{R}$-representation but $\eta = \bar{\eta}$, $s = 2^{n-1} - 2^r$ and $t = 2^{n-1} + 2^r$.

Remark.

(1) The groups D and Q have the same character table. Hence group algebras $\mathbb{C}D$ and $\mathbb{C}Q$ over $\mathbb{C}$ are isomorphic. But two group algebras over $\mathbb{R}$ are not isomorphic. In fact,

$$\mathbb{R}D \cong \mathbb{R}^{(4)} \oplus (\mathbb{R})_2 \text{ and } \mathbb{R}Q \cong \mathbb{R}^{(4)} \oplus \mathbb{H}$$

where $\mathbb{H}$ is the quaternion algebra over $\mathbb{R}$.

(2) D^r is not isomorphic to QD^{r-1} because $\mathbb{C}$-irreducible $\mathbb{R}$-representations of degree 2^r are different (see [5, pp. 205-206]).

Here we state about $\mathbb{R}$-representations of affine groups over finite fields.

Proposition 5. Let G be a permutation group on finite field $\mathbb{F}_q$, where q is a power of a prime p, defined by

$$G = \{x \to ax + b \mid a \in \mathbb{F}_q^*,\ b \in \mathbb{F}_q\}.$$

Let s be the number of elements x with $x^2 = 1$ and let t be the number of elements $x \in G$ such that $x^2 = u_1$ where $u_1 : x \to x + 1$. Then in case $p \neq 2$, $s = |G'| + 1$ and $t = 1$ and in case $p = 2$, $s = |G'|$ and $t = 0$. The $\mathbb{C}$-irreducible representation of degree $|G'| - 1$ is equivalent to an $\mathbb{R}$-representation.

3. Theorem of Hurwitz

The converse of the next theorem is well known. In case $n = 1$, it is trivial. In case $n = 2, 4$, we have

1. $(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$
2. $(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) =$
   $(x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4)^2 + (x_1y_2 + x_2y_1 + x_3y_4 - x_4y_3)^2 +$
   $(x_1y_3 - x_2y_4 + x_3y_1 + x_4y_2)^2 + (x_1y_4 + x_2y_3 - x_3y_2 + x_4y_1)^2.$

In case $n = 8$, it is also known. The next theorem is very interested to suggest that algebras over the real number field can be constructed.

Theorem 6 (Hurwitz [7,1,6]). In polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$, if the next equation is satisfied for $z_s = \sum_{kt} c_{st}^{(k)} x_k y_t$, then $n = 1, 2, 4, 8$

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) = z_1^2 + z_2^2 + \cdots + z_n^2.$$

A key point in the above theorem is to prove $n = 1, 2, 4, 8$ if the next group H_n has a faithful representation of degree n, namely, there is such a group H_n in $GL(n, \mathbb{C})$.

$$H_n = \{(-I)^{s_0} B_1^{s_1} B_2^{s_2} \cdots B_{n-1}^{s_{n-1}} \mid s_k = 0, 1\}$$

where $B_k^2 = -I,\ B_k B_\ell = -B_\ell B_k$ for $k \neq \ell$.

However we can show that 2-groups H_n are realized in $GL(m, \mathbb{C})$ for some m. Therefore we shall state about $\mathbb{C}$-irreducible $\mathbb{R}$-representations of 2-groups H_n.

Lemma 7. The group H_n has two irreducible characters η_1 and η_2 for an even integer n. Let s be the number of elements $x \in H_n$ with $x^2 = 1$ and let t be the number of elements $x \in H_n$ of order 4. Then we obtain.

(1) $s + t = 2^n$.

(2) $\nu(\eta_1) = \nu(\eta_2) = 2^{-\frac{n+2}{2}}(s - t)$.

(3)

$$\nu(\eta_1) = \begin{cases} 1 & \text{for } s > 2^{n-1}, \\ -1 & \text{for } s < 2^{n-1}, \\ 0 & \text{for } s = 2^{n-1}. \end{cases}$$

(4) $s = 2^{n-1} + 2^{\frac{n}{2}}\nu(\eta_1)$ and $t = 2^{n-1} - 2^{\frac{n}{2}}\nu(\eta_1)$.

(5) $\displaystyle \frac{s}{2} = \sum_{k \equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{k}$ and $\displaystyle \frac{t}{2} = \sum_{k \equiv 1,2 \bmod 4}^{n-1} \binom{n-1}{k}$.

Proof. (1) is clear since every element of H_n is of order 1, 2, 4.

(2) follows from $\eta_1(1) = \eta_2(1) = 2^{\frac{n-2}{2}}$, $\eta_1(-1) = \eta_2(-1) = -2^{\frac{n-2}{2}}$ and the next equations

$$\nu(\eta_1) = \frac{1}{|H_n|}(s\eta_1(1) + t\eta_1(-1)) = \frac{\eta_1(1)}{|H_n|}(s - t) = \nu(\eta_2).$$

(3) and (4) follow easily from (1) and (2).

(5) follows from the equations

$$(B_{t_1} B_{t_2} \cdots B_{t_k})^2 = \begin{cases} I & \text{for } k \equiv 0, 3 \bmod 4, \\ -I & \text{for } k \equiv 1, 2 \bmod 4. \end{cases}$$

We proved our assertion

Using (5) in the above lemma, we can find value of $\nu(\eta_1)$. For this purpose, we consider the next equation

$$(1 + i)^m = \{\sqrt{2}(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4})\}^m = 2^{\frac{m}{2}}(\cos\frac{m\pi}{4} + i \sin\frac{m\pi}{4})$$

where $i = \sqrt{-1}$. Comparing imaginary parts between left and right sides in the above equation, we have the next formula

$$\sum_{r=0}^{[\frac{m-1}{2}]} (-1)^r \binom{m}{2r+1} = 2^{\frac{m}{2}} \sin\frac{m\pi}{4}$$

where [ ] is the Gauss symbol. In particular, we have for $m = 4k + 2$.

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} (-1)^r \binom{m}{2r+1} = (-1)^{\frac{m-2}{4}} 2^{\frac{m}{2}} \text{ (see [9, } p.\text{11]).} \qquad (\P)$$

The next proposition state about $\mathbb{C}$-irreducible $\mathbb{R}$-representations of groups H_n.

Proposition 8. Let η be a non-linear irreducible character of H_n. Then in case n is odd, H_n is concos and so we already know about $\mathbb{C}$-irreducible $\mathbb{R}$-representations of groups H_n. In case n is even, we have

$$\nu(\eta) = \begin{cases} 0 & \text{for } n \equiv 2 \bmod 4, \\ -1 & \text{for } n \equiv 4 \bmod 8, \\ 1 & \text{for } n \equiv 0 \bmod 8. \end{cases}$$

Proof. In case $n \equiv 2 \bmod 4$, noting that $k \equiv 0, 3 \bmod 4$ is equivalent to $n - 1 - k \equiv 1, 2 \bmod 4$ for $0 \leq k \leq n - 1$, we obtain easily

$$\begin{aligned} \frac{s}{2} &= \sum_{k \equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{k} = \sum_{k \equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{n-1-k} \\ &= \sum_{\ell \equiv 1,2 \bmod 4}^{n-1} \binom{n-1}{\ell} = \frac{t}{2}. \end{aligned}$$

In the another cases, using the above formula ($\P$), we have the our assertions.

$$\begin{aligned} \frac{s}{2} &= \sum_{k \equiv 0,3 \bmod 4}^{n-1} \binom{n-1}{k} = \sum_{k \equiv 0,3 \bmod 4}^{n-2} \left\{ \binom{n-2}{k-1} + \binom{n-2}{k} \right\} \\ &= 2^{n-2} - \sum_{\ell=0}^{\left[\frac{n-3}{2}\right]} (-1)^\ell \binom{n-2}{2\ell+1} = \begin{cases} 2^{n-2} + 2^{\frac{n-2}{2}} & \text{for } n \equiv 0 \bmod 8, \\ 2^{n-2} - 2^{\frac{n-2}{2}} & \text{for } n \equiv 4 \bmod 8. \end{cases} \end{aligned}$$

where $0 \leq k \leq n - 1$ and $\binom{n-2}{-1} = \binom{n-2}{n-1} = 0$.

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K. Motose
Department of Mathematical Sciences
Faculty of Science and Technology
Hirosaki University
Hirosaki 036-8561, Japan
E-mail address: skm@cc.hirosaki-u.ac.jp