INTEGRALITY OF EIGENVALUES OF CARTAN MATRICES IN FINITE GROUPS

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Abstract. Let C_B be the Cartan matrix of a p-block B of a finite group G. We show that there is a unimodular eigenvector matrix U_B of C_B over a discrete valuation ring R, if all eigenvalues of C_B are integers when B is a cyclic block, a tame block, a p-block of a p-solvable group, the principal 2-block with abelian defect group or the principal 3-block with elementary abelian defect group of order 9.

Keywords: Cartan matrix; Eigenvalue; Eigenvector matrix; Block; Finite group

1. Introduction

Let G be a finite group, let F be an algebraically closed field of characteristic $p > 0$, and let B be a block of the group algebra FG with defect group D. Let C_B be the Cartan matrix of B and $\rho(B)$ the Frobenius-Perron eigenvalue (i.e. the largest eigenvalue) of C_B. Let (K, R, F) be a p-modular system, where R is a complete discrete valuation ring of rank one with $R/(\pi) \simeq F$ for a unique maximal ideal (π) and K is the quotient field of R with characteristic 0. Let us denote the number $l(B)$ of irreducible Brauer characters in B simply by l.

We studied on integrality of eigenvalues of the Cartan matrix of a finite group in [3], [9],[10]. Recently C.C.Xi and D.Xiang showed that integrality of all eigenvalues of the Cartan matrix of a cellular algebra is closely related to its semisimplicity in Theorem 1.1 of [11]. Let R_B and E_B be the set of all eigenvalues (i.e. the spectrum) and of $\mathbb{Z}$-elementary divisors of C_B, respectively.

First we show some known properties of the Cartan matrix C_B of a finite group (e.g. see [6]).

(C1) $C_B = (D_B)^T D_B$, where D_B is the *decomposition matrix* of B.
(C2) C_B is nonnegative integral, indecomposable and symmetric.
(C3) C_B is positive definite (this comes from (1)).
(C4) $\det C_B = p^r \geq |D|$.

Secondly we show some known properties of $E_B = \{e_1, \dots, e_l\}$ (e.g. see [6]).

(E1) Every e_i is a power of p, there is a unique largest $e_1 = |D| \in E_B$ and others $e_i < |D|$ for all $i > 1$.

The detailed version of this paper will be submitted for publication elsewhere.

(E2) Every $e_i = |C_G(x_i)|_p$ for some p-regular element $x_i \in G$.

(E3) $\prod_{i=1}^{l} e_i = \det C_B$.

(E4) If two blocks B and B' are derived equivalent, then there is a perfect isometry between the set of $\mathbb{Z}$-linear combination of ordinary irreducible characters of B and that of B'. Therefore $C_{B'} = V^T C_B V$ for some $V \in \mathrm{GL}(l, \mathbb{Z})$ and so we have $E_B = E_{B'}$ (see [2, 4.2 Proposition]).

Comparing with elementary divisors, properties of eigenvalues are not well known and they seem complicated and sensitive. We show some known properties of $R_B = \{\rho_1, \dots, \rho_l\}$.

(R1) $\rho_i' s$ need not be integers. But there is a unique largest eigenvalue $\rho_1 = \rho(B) \in R_B$ such that $\rho_i < \rho(B)$ for all $i > 1$. It can occur both cases $\rho(B) < |D|$ and $\rho(B) > |D|$ (see Examples 1 and 2 below).

(R1) For any $\rho \in R_B$ there is an algebraic integer λ such that $\rho\lambda = |D|$. In particular, if $\rho \in R_B$ is a rational integer, then $\rho = p^s \mid |D|$.

(R3) $\prod_{i=1}^{l} \rho_i = \det C_B$.

(R4) For two blocks B and B', R_B and $R_{B'}$ are not necessarily equal even if B and B' are derived equivalent (see Examples 1 and 2 below. It is known that the principal 2-blocks of S_4 and S_5 are derived equivalent). But of course, if B and B' are Morita equivalent, then $C_B = C_{B'}$ and so $R_B = R_{B'}$.

We show some examples of the Cartan matrices C for symmetric groups of small degree.

Example 1 $S_4,\ p = 2,\ C = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}$. There is only one block B_1 and $l(B_1) = 2$. Then $R_{B_1} = \{\rho_1 = \frac{7+\sqrt{17}}{2} < |D| = 8,\ \rho_2\}$, $E_{B_1} = \{8, 1\}$.

Example 2 $S_5,\ p = 2,\ C = \begin{pmatrix} 8 & 4 & \\ 4 & 3 & \\ & & 2 \end{pmatrix}$. There are two blocks B_1, B_2, and $l(B_1) = 2, l(B_2) = 1$. Then $R_{B_1} = \{\rho_1 = \frac{11+\sqrt{89}}{2} > |D| = 8, \rho_2\}$, $R_{B_2} = \{2\}$, and $E_{B_1} = \{8, 1\},\ E_{B_2} = \{2\}$

Example 3 $S_4,\ p = 3,\ C = \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$. There are three blocks B_1, B_2, B_3 and $l(B_1) = 2,\ l(B_2) = l(B_3) = 1$. Then $R_{B_1} = \{3, 1\}, R_{B_2} = \{1\}, R_{B_3} = \{1\}$ and $E_{B_1} =$

$\{3,1\}, E_{B_2} = \{1\}, E_{B_3} = \{1\}$. In this case, all eigenvalues are rational integers (see Conjecture below).

2. Questions and Conjecture

It is fundamental to ask the following about integrality of eigenvalues of the Cartan matrix of a finite group G.

- When are eigenvalues of C_B of G rational integers?
- What relations are there between eigenvalues and elementary divisors?
- What do eigenvalues and eigenvectors represent?

We had the following very strong conjecture studying many examples and some typical blocks.

Conjecture. Let C_B be the Cartan matrix of a block B of FG with defect group D for a finite group G. Let $\rho(B)$ be the Frobenius-Perron eigenvalue. Then the following are equivalent.

(a) $\rho(B) \in \mathbb{Z}$.
(b) $\rho(B) = |D|$.
(c) $R_B = E_B$.
(d) All eigenvalues are rational integers.

Considering the condition (d) ((d) itself does not have so deep meanings), we had the notion U_B an *eigenvector matrix* of C_B whose rows consist of linearly independent l eigenvectors of C_B over the field of real numbers $\mathbb{R}$. We have the following question for U_B.

Question. When all eigenvalues are rational integers, can we take a unimodular eigenvector matrix U_B over a complete discrete valuation ring R?

If Question is answered affirmatively, then for example, we find that the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ cannot be the Cartan matrix of a finite group, because this matrix never has a unimodular eigenvector matrix when $p = 2$. In fact, this is the Cartan matrix of a Brauer tree algebra whose tree consists of three vertices such that both end points are exceptional with multiplicity 2. So Question is not true for the Cartan matrix of a general algebra.

3. Some results

We show some evidences for Conjecture and Question. The following proposition is the most fundamental result and a starting point for this research.

Proposition 1 ([3, Proposition 2]). Assume that a defect group D of B is a normal subgroup of G. Then $R_B = E_B$. In fact, the following condition $(*)$ holds.

$$(*)\ C_B\Phi_B = \Phi_B \ \mathrm{diag}\{|C_D(x_1)|, \ldots, |C_D(x_l)|\}$$

where $\Phi_B = (\varphi_i(x_j))$ is the Brauer character table of B, and $\{x_1, \ldots, x_l\}$ is a complete set of representatives of p-regular classes associated with B.

Remark of Proof. We consider a block decomposition of the formula in [1, p.419, ↑ 7]. At this time, we associate a complete set of p-regular classes to B, furthemore we should arrange the first l_1 classes to $\overline{B}_1, \ldots$, the last l_r classes to $\overline{B}_r$, where $\overline{B} = \overline{B}_1 + \cdots + \overline{B}_r$ is a block decomposition of $\overline{B}$ which is the homomorphic image of B by the canonical algebra epimorphism $\tau : FG \to F\overline{G}$, for a normal p-subgroup Q and $\overline{G} := G/Q$. In our case, $Q = D$. This means each $\overline{B}_i$ is of defect 0 and so $l_1 = \cdots = l_r = 1$. Thus $C_{\overline{B}} = I_l$ is the identity matrix. So we have the formula $(*)$ above. As a consequence we may admit any choice of block association of p-regular classes with B.

It is known that $\det \Phi_B \not\equiv 0 \pmod{\pi}$ and then Φ_B is a unimodular matrix over R (see [6, Theorem V 11.6]). $(*)$ implies each $|C_D(x_i)|$ is an eigenvalue of C_B and $\varphi^{(i)} = (\varphi_1(x_i), \ldots, \varphi_l(x_i))^T$ is its eigenvector, when $D \lhd G$. Furthermore, we can take Φ_B as a unimodular eigenvector matrix U_B of C_B.

Then we have the following lemma as a direct corollary of Proposition.

Lemma. Assume that a block B of FG is Morita equivalent to the Brauer correspondent b of B which is a block of $FN_G(D)$. Then we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

In the following we state some theorems about integrality of $\rho(B)$ most of which satisfy the condition mentioned in above Lemma.

Theorem 1 ([3],[10]). If D is cyclic (i.e. B is a finite type), then the following are equivalent.
(1) $\rho(B) \in \mathbb{Z}$
(2) $\rho(B) = |D|$
(3) $R_B = E_B$
(4) $B \sim b$ (Morita equivalent), where b is the Brauer correspondent block in $FN_G(D)$
(5) The Brauer tree of B is the star with the exceptional vertex at the center if it exists.
In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

Theorem 2 ([3],[10]). If B is a tame block (not finite type, i.e. $p = 2$ and $D \simeq$ a dihedral, a generalized quaternion or a semidihedral 2-group), then the following are equivalent.
(1) $\rho(B) \in \mathbb{Z}$
(2) $\rho(B) = |D|$
(3) $R_B = E_B$

(4) $B \sim b$ (Morita equivalent), where b is the Brauer correspondent block in $FN_G(D)$
(5) One of the following holds.
(i) $l = 1, C_B = (|D|)$

(ii) $l = 3, D \simeq E_4$, $C_B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

(iii) $l = 3, D \simeq Q_8$, $C_B = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$.

In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

Theorem 3 ([3],[10]). If B is a p-block of a p-solvable group G, then the following are equivalent.
(1) $\rho(B) = |D|$
(2) $R_B = E_B$

In this case, B and its Brauer correspondent b are not necessarily Morita equivalent. For example, let $G = \mathrm{SL}(2,3)\cdot E_{27}$, $p = 3$ and let B be a unique non-principal block. Then $l(B) = 1$, and the number of ordinary irreducible characters $k(B) = 13$, but $k(b) = 17$, respectively. So B and b are not Morita equivalent. However, we can take Φ_β as a unimodular eigenvector matrix U_B of C_B, where β is a block of a subgroup of G or of a factor group of a central extension of a subgroup of G.

We cannot prove yet that if $\rho(B) \in \mathbb{Z}$, then $\rho(B) = |D|$ for a block B of a p-solvable group. In the following two results we are inspired by many author's results proving Broué's abelian defect group conjecture for $p = 2$ and 3 to be true (see e.g. [2, 4, 7, 8]). In abelian defect group case, our question yields a special case of Broué's abelian defect group conjecture.

Theorem 4 ([5], [10]). If $p = 2$, $\tilde{B}$ and $\tilde{b}$ are the principal blocks of $\tilde{G}$ and $N_{\tilde{G}}(D)$ respectively, with abelian defect group D, then the following are equivalent.
(1) $\rho(\tilde{B}) \in \mathbb{Z}$
(2) $\rho(\tilde{B}) = |D|$
(3) $R_{\tilde{B}} = E_{\tilde{B}}$
(4) $\tilde{B} \sim \tilde{b}$ Morita equivalent (even stronger Puig equivalent)
(5) For a finite group $\tilde{G}$ with an abelian Sylow 2-subgroup D and $O(\tilde{G}) = 1$, the following holds. Let $G := O'(\tilde{G})$. Then

$$G = G_1 \times \ldots \times G_r \times S,$$

where $G_i \simeq \mathrm{PSL}(2, q_i),\ 3 < q_i \equiv 3 \pmod 8$ for $1 \leq i \leq r$ and S is an abelian 2-group.
In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

Theorem 5 ([10]). If $p=3$, $\tilde{B}$ and $\tilde{b}$ are the principal blocks of $\tilde{G}$ and $N_{\tilde{G}}(D)$ respectively, with elementary abelian defect group D of order 9, then the following are equivalent.

(1) $\rho(\tilde{B}) \in \mathbb{Z}$

(2) $\rho(\tilde{B}) = |D|$

(3) $R_{\tilde{B}} = E_{\tilde{B}}$

(4) $\tilde{B} \sim \tilde{b}$ Morita equivalent (even stronger Puig equivalent)

(5) Let $\tilde{G}$ be a finite group with an elementary abelian Sylow 3-subgroup D of order 9 and $O_{3'}(\tilde{G}) = 1$. Let $G := O^{3'}(\tilde{G})$. Then G satisfies the following (i) or (ii).

(i) $G = X \times Y$ for simple groups X, Y with a cyclic Sylow 3-subgroup of order 3, respectively.

(ii) G is one of the following simple groups.

(a) $\mathrm{PSU}(3, q^2),\ 2 < q \equiv 2 \text{ or } 5 \pmod 9$

(b) $\mathrm{PSp}(4, q),\ q \equiv 4 \text{ or } 7 \pmod 9$

(c) $\mathrm{PSL}(5, q),\ q \equiv 2 \text{ or } 5 \pmod 9$

(d) $\mathrm{PSU}(4, q^2),\ q \equiv 4 \text{ or } 7 \pmod 9$

(e) $\mathrm{PSU}(5, q^2),\ q \equiv 4 \text{ or } 7 \pmod 9$

In this case, we can take Φ_b as a unimodular eigenvector matrix U_B of C_B.

We use Koshitani-Kunugi's method in [4] to prove (5) $\to$ (4) in Theorems 4 and 5. Also we use the following fundamental Proposition to prove (1) $\to$ (5) in Theorems 4 and 5. The last statements of Theorems 4 and 5 are clear from Lemma.

Proposition 2 ([10]). Assume $H \lhd G$ and $|G : H| = q$ (a prime $\neq p$). Let b be a p-block of H. Let B be any p-block of G covering b. Then $\rho(B) = \rho(b)$.

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