LONG EXACT SEQUENCES COMING FROM TRIANGLES

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Abstract. Suppose we are given a homological functor, from a triangulated to an abelian category. It takes triangles to long exact sequences. It turns out that not every long exact sequence can occur; there are restrictions.

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0. Introduction

Suppose $\mathcal{A}$ is a sufficiently nice abelian category, so that it has a derived category $\mathbf{D}(\mathcal{A})$. This will happen, for example, if $\mathcal{A}$ has enough projectives, or if it has enough injectives; for details see Hartshorne [2] or Verdier [3, 4]. Given a distinguished triangle in $\mathbf{D}(\mathcal{A})$

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \ ,$$

we can form the long exact sequence in cohomology. We deduce in $\mathcal{A}$ a long exact sequence

$$\cdots \longrightarrow H^{-1}(Z) \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow \cdots$$

We can wonder what long exact sequences can be obtained this way.

It is clear that any sequence of length four is obtainable. If we have an exact sequence in $\mathcal{A}$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

then it is very easy to deal with it; consider B and C as objects of $\mathcal{A} \subset \mathbf{D}(\mathcal{A})$, and complete the morphism $B \longrightarrow C$ into a triangle in $\mathbf{D}(\mathcal{A})$. The reader can easily check that the long exact sequence, obtained from the functor H applied to this triangle, is nothing other than

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0 \ .$$

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The remarkable fact, which I do not fully understand, is what comes next. It turns out that not all sequences of length five

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

are the long exact sequences of triangles. Any exact sequence of length five defines a class in $\mathrm{Ext}^3_{\mathcal{A}}(E,A)$, or equivalently a morphism $E \longrightarrow \Sigma^3 A$ in $\mathbf{D}(\mathcal{A})$. It turns out that the sequence will be the long exact sequence of a triangle if and only if this class in $\mathrm{Ext}^3_{\mathcal{A}}(E,A)$ vanishes. In this article I will only prove the necessity, but the sufficiency is easy enough.

More is true. Given any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in the derived category $\mathbf{D}(\mathcal{A})$, we can look at its long exact sequence in cohomology. It can be chopped into bits of length five, for example

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0\ ,$$

where K is the kernel of $H^0(u) : H^0(X) \longrightarrow H^0(Y)$, while Q is cokernel of $H^0(v) : H^0(Y) \longrightarrow H^0(Z)$. We will prove that, for every such length-five bit, the corresponding element in $\mathrm{Ext}^3_{\mathcal{A}}(Q,K)$ vanishes. I know that this vanishing is necessary, but have no idea whether it suffices. In other words, I do not know whether it characterizes the long exact sequences coming from triangles in $\mathbf{D}(\mathcal{A})$.

In the proof we will be slightly more general. We will start with an arbitrary triangulated category $\mathcal{T}$, possessing a t–structure; the reader is referred to Beilinson, Bernstein and Deligne [1] for the definitions and elementary properties of t–structures. We will let $\mathcal{A}$ be the heart of the t–structure. We will assume that $\mathcal{T}$ is nice enough so that the inclusion $\mathcal{A} \longrightarrow \mathcal{T}$ factors through a triangulated functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$; this is a very weak hypothesis, usually satisfied. We recall that, for any pair of objects $A, B \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(A,B) = \mathrm{Hom}_{\mathcal{T}}(A,B)\,, \qquad \mathrm{Hom}_{\mathbf{D}^b(\mathcal{A})}(A,\Sigma B) = \mathrm{Hom}_{\mathcal{T}}(A,\Sigma B)\,.$$

The reason for the first equality is that $\mathcal{A}$ embeds fully faithfully in both $\mathbf{D}^b(\mathcal{A})$ and $\mathcal{T}$, and the second equality is because both groups classify extensions $0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0$ in $\mathcal{A}$. But it is perfectly possible for a non-zero morphism $\alpha : A \longrightarrow \Sigma^n B$, in the category $\mathbf{D}^b(\mathcal{A})$, to map to zero in $\mathcal{T}$; all we learn, from the discussion above, is that this can only happen if $n \geq 2$.

What we will prove, in the generality of triangulated categories $\mathcal{T}$ with t–structures, is the following. Given any triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in the category $\mathcal{T}$, we can still look at its long exact sequence in cohomology. It can still be chopped into bits of length five, for example

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0\ .$$

Every such length-five bit corresponds to an element in $\mathrm{Ext}^3_{\mathcal{A}}(Q, K)$, that is to a morphism $\alpha : Q \longrightarrow \Sigma^3 K$ in $\mathbf{D}^b(\mathcal{A})$. We will prove that the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$ must take α to zero.

1. The proof

Before all else we need to fix our conventions for this section.

Notation 1.1. Let $\mathcal{T}$ be a triangulated category with a t–structure. Let $\mathcal{A}$ be the heart of this t–structure. Assume that the category $\mathcal{T}$ is "natural" enough so that the embedding of $\mathcal{A}$ into $\mathcal{T}$ extends to a triangulated functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$. We fix these assumptions throughout the section.

Let us also fix the notation that $H : \mathcal{T} \longrightarrow \mathcal{A}$ will be the homological functor sending an object $X \in \mathcal{T}$ to the truncation $H(X) = \big(X^{\leq 0}\big)^{\geq 0}$. We will let $H^n(X) = H(\Sigma^n X)$.

With these conventions, we are ready to state and prove our main observation:

Lemma 1.2. *Let $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ be a triangle in $\mathcal{T}$, and suppose that*

(1) X and Y lie in $\mathcal{T}^{\leq 1} \cap \mathcal{T}^{\geq 0}$.
(2) Z lies in $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$.

This implies that the functor H sends the triangle to the long exact sequence

$$(*) \qquad 0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z) \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0$$

with all the other terms vanishing. In the abelian category $\mathcal{A}$, this 5–term exact sequence defines a class in $\mathrm{Ext}^3_{\mathcal{A}}\big(H^1(Y), H^0(X)\big)$. This class can also be viewed as a morphism $\alpha : H^1(Y) \longrightarrow \Sigma^3 H^0(X)$, in the derived category $\mathbf{D}^b(\mathcal{A})$.

We assert that, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, the image of α vanishes.

Proof. Consider the commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^{\geq 1} & \longrightarrow & Y^{\geq 1} \end{array} \qquad .$$

We may complete to a 3×3 diagram, were the rows and columns are triangles

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X^{\geq 1} & \longrightarrow & Y^{\geq 1} & \longrightarrow & I & \longrightarrow & \Sigma X^{\geq 1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X^{\leq 0} & \longrightarrow & \Sigma Y^{\leq 0} & \longrightarrow & \Sigma K & \longrightarrow & \Sigma^2 X^{\leq 0} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma Z & \longrightarrow & \Sigma^2 X
\end{array} ,$$

and the proof is by studying this 3×3 diagram. In the second row, we have that $\Sigma\big(X^{\geq 1}\big) = H^1(X)$ and $\Sigma\big(Y^{\geq 1}\big) = H^1(Y)$ are both in $\mathcal{A} \subset \mathcal{T}$, and that the morphism $\Sigma\big(X^{\geq 1}\big) \longrightarrow \Sigma\big(Y^{\geq 1}\big)$ is surjective; it is the morphism $H^1(X) \longrightarrow H^1(Y)$ in the long exact sequence (*) of the lemma. The second row reduces to the short exact sequence in $\mathcal{A}$

$$0 \longrightarrow I \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0 ,$$

and the map $Y^{\geq 1} \longrightarrow I$ is the image, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, of the morphism in $\mathbf{D}^b(\mathcal{A})$ defining the extension $0 \longrightarrow I \longrightarrow H^1(X) \longrightarrow H^1(Y) \longrightarrow 0$.

So much for the second row. Now look at the commutative diagram

$$\begin{array}{ccccc}
Z & \longrightarrow & \Sigma X & & \\
\downarrow & & \downarrow & & \\
I & \longrightarrow & \Sigma X^{\geq 1} & \longrightarrow & \Sigma Y^{\geq 1}
\end{array} .$$

If we apply to it the functor H, we discover the diagram

$$\begin{array}{ccccc}
H^0(Z) & \longrightarrow & H^1(X) & & \\
\downarrow & & \| & & \\
H^0(I) & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y)
\end{array} .$$

Both Z and I lie in the heart $\mathcal{A}$, and the diagram above identifies for us the map $Z \longrightarrow I$ as the factorization of the morphism from $Z = H^0(Z)$ to $H^1(X)$ through the kernel of $H^1(X) \longrightarrow H^1(Y)$, which is the image of $Z \longrightarrow H^1(X)$. Now the column

$$K \longrightarrow Z \longrightarrow I \longrightarrow \Sigma K$$

is a triangle, which reduces to the short exact sequence $0 \longrightarrow K \longrightarrow Z \longrightarrow I \longrightarrow 0$ in $\mathcal{A} \subset \mathcal{T}$. We also learn that the map $I \longrightarrow \Sigma K$ is the image, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, of the morphism in $\mathbf{D}^b(\mathcal{A})$ corresponding to the extension.

Next consider the commutative square

$$\begin{array}{ccc} Y^{\leq 0} & \longrightarrow & K \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array} .$$

If we apply the functor H we learn that the map $Y^{\leq 0} \longrightarrow K$, which is a map between objects in $\mathcal{A}$, is just the factorization through K of the morphism $H^0(Y) \longrightarrow H^0(Z) = Z$. The triangle

$$X^{\leq 0} \longrightarrow Y^{\leq 0} \longrightarrow K \longrightarrow \Sigma X^{\leq 0}$$

is therefore nothing fancy; it is simply the exact sequence $0 \longrightarrow H^0(X) \longrightarrow H^0(Y) \longrightarrow K \longrightarrow 0$ in $\mathcal{A}$. Moreover, the map $K \longrightarrow \Sigma X^{\leq 0}$ is just exactly the image, under the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$, of the morphism in $\mathbf{D}^b(\mathcal{A})$ corresponding to the extension.

What we have learned so far is that three of the six triangles, in our 3×3 diagram, amount to short exact sequences in $\mathcal{A}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(X) & \longrightarrow & H^0(Y) & \longrightarrow & K & \longrightarrow & 0 \\ 0 & \longrightarrow & K & \longrightarrow & Z & \longrightarrow & I & \longrightarrow & 0 \\ 0 & \longrightarrow & I & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y) & \longrightarrow & 0 \end{array} .$$

Moreover, the differentials of these triangles are the classes of the three extensions, and are also part of our 3×3 diagram. The composite of these three differentials is the map

$$\begin{array}{ccccc} Y^{\geq 1} & \longrightarrow & I & & \\ & & \downarrow & & \\ & & \Sigma K & \longrightarrow & \Sigma^2 X^{\leq 0} \end{array} ,$$

which the reader will find in our diagram. The commutativity of

$$\begin{array}{ccccc} Y^{\geq 1} & \longrightarrow & I & \longrightarrow & \Sigma X^{\geq 1} \\ & & \downarrow & & \downarrow \\ & & \Sigma K & \longrightarrow & \Sigma^2 X^{\leq 0} \end{array} ,$$

coupled with the vanishing of $Y^{\geq 1} \longrightarrow I \longrightarrow \Sigma X^{\geq 1}$, tells us that this composite vanishes. In the category $\mathcal{T}$ the three extensions compose to zero. □

Proposition 1.3. *Let the conventions be as in Notation 1.1. Suppose* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ *is a triangle in* $\mathcal{T}$. *Complete* $H^0(X) \longrightarrow H^0(Y) \longrightarrow H^0(Z)$ *to an exact sequence*

$$0 \longrightarrow K \longrightarrow H^0(X) \xrightarrow{H^0(u)} H^0(Y) \xrightarrow{H^0(v)} H^0(Z) \longrightarrow Q \longrightarrow 0 ,$$

where K must be the kernel of $H^0(u) : H^0(X) \longrightarrow H^0(Y)$, while Q is forced to be the cokernel of $H^0(v) : H^0(Y) \longrightarrow H^0(Z)$. The sequence defines an element in $\mathrm{Ext}^3_{\mathcal{A}}(Q, K)$, or equivalently a morphism $\alpha : Q \longrightarrow \Sigma^3 K$ in $\mathbf{D}^b(\mathcal{A})$.

We assert that the functor $F : \mathbf{D}^b(\mathcal{A}) \longrightarrow \mathcal{T}$ takes α to zero.

Proof. Consider the commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^{\geq 0} & \longrightarrow & Y^{\geq 0} \end{array} \qquad .$$

It may be extended to a morphism of triangles, which we will write

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array} \qquad .$$

That is $X' = X^{\geq 0}$ and $Y' = Y^{\geq 0}$. We have

(1) X' and Y' belong to $\mathcal{T}^{\geq 0}$, while Z' belongs to $\mathcal{T}^{\geq -1}$.

(2) The three maps

$$H^0(X) \longrightarrow H^0(X')\,, \qquad H^0(Y) \longrightarrow H^0(Y')\,, \qquad H^0(Z) \longrightarrow H^0(Z')$$

are all isomorphisms. For $X' = X^{\geq 0}$ and $Y' = Y^{\geq 0}$ this is obvious, by the definition of the functor H in terms of truncations. For Z' consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} H^0(X) & \longrightarrow & H^0(Y) & \longrightarrow & H^0(Z) & \longrightarrow & H^1(X) & \longrightarrow & H^1(Y) \\ \rho\downarrow & & \sigma\downarrow & & \tau\downarrow & & \Sigma\rho\downarrow & & \Sigma\sigma\downarrow \\ H^0(X') & \longrightarrow & H^0(Y') & \longrightarrow & H^0(Z') & \longrightarrow & H^1(X') & \longrightarrow & H^1(Y') \end{array} \qquad .$$

We know that ρ, σ, $\Sigma\rho$ and $\Sigma\sigma$ are isomorphisms. The 5–lemma permits us to conclude that so is τ.

Now apply the dual construction; consider the commutative square

$$\begin{array}{ccc} (Y')^{\leq 0} & \longrightarrow & (Z')^{\leq 0} \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array} \qquad .$$

We can extend to a morphism of triangles

$$\begin{array}{ccccccc} X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & \Sigma X'' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array} \quad ;$$

as before, this means $Y'' = (Y')^{\leq 0}$ and $Z'' = (Z')^{\leq 0}$. We leave it as an exercise to the reader to check that

(1) X'' belongs to $\mathcal{T}^{\leq 1} \cap \mathcal{T}^{\geq 0}$, and Y'' belong to $\mathcal{A} = \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$, while Z'' belongs to $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq -1}$.

(2) The three maps

$$H^0(X'') \longrightarrow H^0(X'), \qquad H^0(Y'') \longrightarrow H^0(Y'), \qquad H^0(Z'') \longrightarrow H^0(Z')$$

are all isomorphisms.

The proposition now follows from Lemma 1.2, applied to the triangle $\Sigma^{-1}Z'' \longrightarrow X'' \longrightarrow Y'' \longrightarrow Z''$. □

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