ON S-COHN-JORDAN EXTENSIONS

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Abstract. Let a monoid S act on a ring R by injective endomorphisms. A series of results relating various algebraic properties of R and that of the S-Cohn-Jordan ring extension $A(R;S)$ of R are presented. For example: primeness, Goldie conditions and other finiteness conditions are considered. Some problems and possible applications will be also discussed.

1. Introduction

Let R be an associative unital ring and $\sigma\colon R \to R$ an injective endomorphism of R. Jordan [2] constructed a minimal, in a sense that $A = \sum_{i=0}^{\infty} \sigma^{-i}(R)$, over-ring A of R such that σ extends to an automorphism of A. Then he began systematic studies of relations between various algebraic properties of R and that of A. The motivation for such studies was the observation that this knowledge can often be used for reducing the investigation of a skew polynomial ring $R[x;\sigma]$ of endomorphism type, to the case of the skew polynomial ring $A[x;\sigma]$ of automorphism type, which is much easier to handle. Examples of such approach one can also find in [6] and [7].

Instead of looking at the action of a single endomorphism σ on R one can consider the action of a monoid. Let S denote a monoid which acts on R by injective endomorphisms. That is, a homomorphism $\phi\colon S \to \mathrm{End}(R)$ is given, such that $\phi(s)$ is a monomorphism, for any $s \in S$. We say that an over-ring $A(R;S)$ of R is an S-Cohn-Jordan extension of R if it is a minimal over-ring of R such that the action of S on R extends to the action of S on $A(R;S)$ by automorphisms (Cf. Definition 1).

A classical result of Cohn (see Theorem 7.3.4 [1]) says that if the monoid S possesses a group $S^{-1}S$ of left quotients, then $A(R;S)$ exists, moreover it is uniquely determined up to an R-isomorphism.

The above mentioned theorem of Cohn was originally formulated in much more general context of Ω-algebras, not just rings. The construction of $A(R;S)$ was given as a limit of a suitable directed system.

The possibility of enlarging an object and replacing the action of endomorphisms by the action of automorphisms is a powerful tool, similar to a localization. Perhaps this was the reason that the theorem of Cohn was formulated and reproved in various algebraic contexts (see for example [3], [9], [10], [11], [12]).

The aim of the paper is to present a series of results relating various algebraic properties of R and that of the S-Cohn-Jordan extension $A(R;S)$ of R. For example: primeness, Goldie conditions and other finiteness conditions are considered. Most of the presented

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results can be found in full details in [7]. Some problems, questions and applications are also discussed.

2. Properties of S-Cohn-Jordan Extensions

Henceforth R stands for an associative ring, $\phi\colon S \to \mathrm{End}(R)$ denotes the action of a monoid S on R by injective endomorphisms. For any $s \in S$, the endomorphism $\phi(s) \in \mathrm{End}(R)$ will be denoted by ϕ_s.

Definition 1. An over-ring $A(R;S)$ of R is called an S-Cohn-Jordan extension of R if:

(1) the action of S on R extends to an action of S (also denoted by ϕ) on $A(R;S)$ by automorphisms, i.e. ϕ_s is an automorphism of $A(R;S)$, for any $s \in S$.
(2) every element $a \in A(R;S)$ is of the form $a = \phi_s^{-1}(b)$, for some suitable $b \in R$ and $s \in S$.

Henceforth, as in the above definition, ϕ_s will also denote the automorphism $\phi(s)$ of $A(R;S)$, where $s \in S$.

As it was mentioned in the introduction, the extension $A(R;S)$ exists provided the monoid S possesses a group of left quotients. Recall that this is the case exactly when the monoid S is left and right cancellative and satisfies the left Ore condition, that is, for any $s_1, s_2 \in S$, there exist $t_1, t_2 \in S$ such that $t_1 s_1 = t_2 s_2$.

In the case $S = \langle \sigma \rangle$ is a cyclic monoid, Jordan recognized $A(R;S)$ as a subring of the left localization of the Ore extension $R[x;\sigma]$ with respect to the set of all powers of the indeterminate x.

When the monoid S possesses a group of left quotients, then one can construct $A(R;S)$ in a similar way as Jordan did. Namely, let us consider the skew semigroup ring $R\#_\phi S$. One can check that elements of S are regular in $R\#_\phi S$ and S is a left Ore set in $R\#_\phi S$. In particular, we can consider the left localization $T = S^{-1}(R\#_\phi S)$ of $R\#_\phi S$. For any $s \in S$ and $r \in R$, we have $sr = \phi_s(r)s$ in $R\#_\phi S$. Thus one can think of $s^{-1}Rs$ as the preimage $(\phi_s)^{-1}(R)$ of R in T. The Goldie condition implies that $A = \bigcup_{s\in S} s^{-1}Rs \subseteq T$ is a subring of T. In fact it is easy to see that $A = A(R;S)$ (Cf. Lemma 2.1 [7]), in this case.

Suppose that $A(R;S)$ exists and the action of S on R is faithful, in the sense that ϕ is an injection. Then the action of S on $A(R;S)$ is also faithful. This means that the monoid S embeds in a group (the group of automorphisms of $A(R;S)$). However even in this case, conditions for existence of $A(R;S)$ seem to be not clear and we may formulate the following:

Problem 2. Suppose that S acts faithfully on R.

(1) What are the necessary and sufficient conditions for the existence of $A(R;S)$?
(2) Assume that $A(R;S)$ exists. What are the necessary and sufficient conditions for uniqueness of $A(R;S)$?
(3) Let T be a submonoid of S. Suppose that $A(R;S)$ exists. When does $A(R;T)$ exist? If so, is it naturally embedded in $A(R;S)$?

The Definition 1 reminds somehow the definition of a left localization of R with respect to a multiplicatively closed set. In this way, the analogue of a common left denominator for a finite set X of elements of $A(R;S)$, should be an element ϕ_s, for some $s \in S$, such that $\phi_s(X) \subseteq R$. It is easy to see that, for any finite subset X of $A(R;S)$, such element ϕ_s do exists. This suggests that the relations between some algebraic properties of R and its S-Cohn-Jordan extension $A(R;S)$ should be similar to those between R and its localization. This is indeed the case. In particular we have:

Proposition 3. (Cf. [7])

(1) *Let $\mathcal{T}$ denote one of the following classes of rings: the class of all division, simple, von Neumann regular, prime, semiprime rings, rings having finite block theory. If $R \in \mathcal{T}$ then $A(R;S) \in \mathcal{T}$.*

(2) *Let $\mathcal{P}$ denote one of the following classes of rings: the class of all domains, reduced rings, $n \times n$ matrix rings, commutative or, more generally, rings satisfying a fixed polynomial identity. Then $A(R;S) \in \mathcal{P}$ if and only if $R \in \mathcal{P}$.*

The properties listed in the statement (1) of the above proposition do not pass down from $A(R;S)$ to R. Indeed, the following easy example shows that $A(R;S)$ can be a field with R being not simple.

Example 4. Let $A = K(x_i \mid i \in \mathbb{Z})$ be the field of rational functions over a field K in commuting indeterminates $\{x_i\}_{i\in\mathbb{Z}}$ and $S = \langle\sigma\rangle$, where σ is the K-automorphism of A given by $\sigma(x_i) = x_{i+1}$, for $i \in \mathbb{Z}$. Let us set $R = K(x_i \mid i \geq 1)[x_0] \subseteq A$. Then S acts in a natural way on R and for any $a \in A$, there exists $n \geq 1$ such that $\sigma^n(a) \in K(x_i \mid i \geq 1) \subseteq R$. This means that $A = A(R;S)$.

Example 1.10 [7] offers a prime ring R such that $A(R;S)$ is not semiprime. Example 1.15 [7] shows that there exists a ring R having infinitely many central orthogonal idempotents, while $A(R;S)$ has no nontrivial central idempotents, i.e. $A(R;S)$ has a finite block theory but R does not.

Theorem 5. (Cf. [7]) *Suppose $A(R;S)$ exists. Then:*

(1) *$A(R;S)$ is semiprime if and only if for any nonzero left ideal I of R, there exists $s \in S$ such that $(R\phi_s(I))^2 \neq 0$.*

(2) *Suppose that R is left noetherian. Then R is prime (semiprime) if and only if $A(R;S)$ is prime (semiprime).*

When R is one-sided noetherian, then there exists a finite common bound on the cardinality of sets of orthogonal idempotents of R (as otherwise R would have infinite left Goldie dimension). Thus Proposition 1.14 [7] yields immediately the following:

Theorem 6. *Suppose that R is a one-sided noetherian ring and $A(R;S)$ exists. Then R has finite block theory if and only if $A(R;S)$ has finite block theory. Moreover if $\bigoplus_{i=1}^n e_iR$ is a decomposition of R into indecomposable blocks, then $\bigoplus_{i=1}^n e_iA(R;S)$ is a block decomposition of $A(R;S)$.*

Much more can be said about the relations of R and that of $A(R;S)$, provided the monoid S has a group of left quotients. The idea, which goes back to Jordan [2], is to compare left ideals I of $A(R;S)$ with its orbits $\{\phi_s(I)\cap R \mid s \in S\}$ in R. An important role is also played by S-closed left ideals J of R, i.e. left ideals J such that $A(R;S)J \cap R = J$.

The following theorem (Cf. Theorem 2.19 and Corollary 2.20 of [7]) offers complete characterization of artinian property of $A(R; S)$.

Theorem 7. *Suppose that S possesses a group of left quotients. Then:*

(1) *The ring $A(R; S)$ is left artinian if and only if there exists a finite bound on lengths of chains of S-closed left ideals of R. Moreover, if one of the equivalent conditions holds, then the length of $A(R; S)$ as a left $A(R; S)$-module is equal to the length of the longest chain of S-closed left ideals of R.*

(2) *If R is left artinian then so is $A(R; S)$.*

In the case S is a cyclic monoid, the above theorem was proved in [2]. Surprisingly, the proof of the theorem in the general case seems to be easier than the arguments used in the case S is a cyclic monoid.

Making use of Theorems 5, 6, 7 and some localizations technics one can prove the following two results (Cf. [7]):

Theorem 8. *Suppose S possesses a group of left quotients. If the ring R is left artinian, then:*

(1) *R is a semisimple ring if and only if $A(R; S)$ is a semisimple ring.*

(2) *If $R = \bigoplus_{i=1}^{k} e_i R$, with $e_i R = M_{n_i}(B_i)$, is a block decomposition of the semisimple ring R, then $A(R; S) = \bigoplus_{i=1}^{k} e_i A(R; S)$ is a block decomposition of $A(R; S)$ and $e_i A(R; S) = M_{n_i}(D_i)$ for some division ring. Moreover, for $1 \leq i \leq k$, the division ring D_i is an extension of B_i.*

In the case $S = \langle \sigma \rangle$ being a cyclic monoid the above theorem was known in special cases. Namely, the first statement was proved in [2], the second one appeared in [4].

From now on $Q(R)$ will denote the classical left quotient ring of a semiprime left Goldie ring R and udim R will stand for the left uniform dimension of R.

Theorem 9. *Suppose S possesses a group of left quotients. Let R be a semiprime left Goldie ring. Then $A(R; S)$ is also a semiprime left Goldie ring. Moreover $Q(A(R; S)) = A(Q(R); S)$ and* udim $R =$ udim $A(R; S)$.

Contrary to the artinian property, the situation with the noetherian property of $A(R; S)$ seems to be not clear at all. Even when S is a cyclic monoid, one can find examples of rings R and $A(R; S)$ showing that one of those rings is left noetherian but the other is not left noetherian. Nevertheless Jordan [2] succeeded to give necessary and sufficient conditions for $A(R; S)$ to be left noetherian, in the case S is a cyclic monoid. The characterization was given in terms of properties of the lattice of S-closed left ideals of R.

Problem 10. To characterize the left noetherian property of the S-Cohn-Jordan extension $A(R; S)$ in terms of properties of R and the action of S.

If $G = S^{-1}S$ is the group of left quotients of S, then we have seen that $A(R; S)$ can be considered as a subring of the left localization $S^{-1}(R\#_\phi S)$ of $R\#_\phi S$. Using this approach, one can see that there is a natural isomorphism between $A(R; S)\#_\phi G$ and $S^{-1}(R\#_\phi S)$. Since the left noetherian property of a ring is preserved under left localization with respect to a left Ore set, we have:

Proposition 11. *Suppose S possesses a group of left quotients. If the ring $R\#_\phi S$ is left noetherian, then $A(R;S)$ is also left noetherian.*

3. Examples of Applications

As it was briefly mentioned at the end of the previous section, when S has the group G of left quotients, then $R\#_\phi S \subseteq A(R;S)\#_\phi S \subseteq A(R;S)\#_\phi G = S^{-1}(R\#_\phi S)$. This means that problems concerning the skew semigroup rings $R\#_\phi S$ can often be reduced to the skew group ring $A(R;S)\#_\phi S$. The following theorem is an example of such application.

Theorem 12. (Cf. [7]) *Let S be a monoid having a poly-infinite cyclic group of left quotients. Suppose that S acts on a semiprime (prime) left Goldie ring R by injective endomorphisms. Then the skew semigroup ring $R\#_\phi S$ is a semiprime (prime) left Goldie ring and* udim $(R\#_\phi S) =$ udim R.

The idea of the proof of the above theorem is as follows. By Theorem 9, $A(R;S)$ is a semiprime left Goldie ring and the assumption imposed on the group $G = S^{-1}S$ of left quotients of the monoid S imply that $A(R;S)\#_\phi G$ is a semiprime left Goldie ring. $R\#_\phi S$ is a subring of $A(R;S)\#_\phi G$ such that $S^{-1}(R\#_\phi S) = A(R;S)\#_\phi G$. Thus the localization $S^{-1}(R\#_\phi S)$ is a semiprime left Goldie ring. Hence the ring $R\#_\phi S$ is also semiprime left Goldie.

It was proved in [5] that the property of being a semiprime left Goldie ring lifts from a ring R to its Ore extension $R[x;\sigma,\delta]$, where σ is an automorphism and δ a σ-derivation of R. This result was extended in [4] to the following theorem.

Theorem 13. *Let R be a semiprime left Goldie ring, σ, δ an injective endomorphism and a σ-derivation of R, respectively. Then $R[x;\sigma,\delta]$ is also a semiprime left Goldie ring and* udim $R[x;\sigma,\delta] =$ udim $R =$ udim $A(R;\langle\sigma\rangle)$.

One of the key ingredient in the proof of the above theorem was the use of the $\langle\sigma\rangle$-Cohn-Jordan extension $A(R;\langle\sigma\rangle)$ and Theorem 9.

Mushrub in [9] investigated the left uniform dimension of skew polynomial rings $R[x;\sigma]$, where σ denotes an injective endomorphism of the ring R. He proved, in particular, that udim $R[x;\sigma] =$ udim $A(R;\langle\sigma\rangle)$ (for a short proof see Lemma 3.2 [4]). He also constructed examples showing that:

1. For any $n \in \mathbb{N}$, there is a commutative ring R (not semiprime) with an injective endomorphism σ, such that udim $R = n$ and udim $R[x;\sigma] = 1$.
2. There exists a domain R of infinite left uniform dimension and an injective endomorphism σ of R such that udim $R[x;\sigma] = 1$.

The following question comes from [9].

Question 14. (Mushrub) Let R be a semiprime ring of finite left Goldie dimension. Suppose that σ is an injective endomorphism of R. Is udim $R =$ udim $R[x;\sigma]$?

As we recorded earlier, udim $R[x;\sigma] =$ udim $A(R;\langle\sigma\rangle)$, for any injective endomorphism σ of R. Thus the above question of Mushrub can be be read as a question: Is udim $R =$ udim $A(R;\langle\sigma\rangle)$? Therefore, the following question can be viewed as a generalization of Question 14.

Question 15. Suppose that R is a semiprime ring of finite left Goldie dimension acted by a monoid S which has a group of left quotients. Is udim $R =$ udim $A(R;S)$?

The left uniform dimension is preserved under left localizations with respect to Ore sets of regular elements (Cf. Lemma 2.2.12 [8]). This implies that udim $R\#_\phi S =$ udim $S^{-1}(R\#_\phi S) =$ udim $A(R;S)\#_\phi G$. Thus Theorem 12 yields that Question 15 has a positive answer if the group $G = S^{-1}S$ is poly-infinite cyclic and R satisfies the ACC on left annihilators. This also means that Question 14 has a positive answer if one additionally assume that R has the ACC on left annihilators. The last fact was observed earlier in [4].

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