A CONSTRUCTION OF LOCAL QF-RINGS WITH RADICAL CUBED ZERO

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ABSTRACT. The purpose of this paper is to give a construction of local QF-rings with Jacobson radical cubed zero. From our construction, we can foresee that there are many QF-rings which are not finite dimensional algebras over fields. Needless to say, local QF-rings together with local Nakayama rings are important artinian rings in the sense that these rings are parts of QF-rings and Nakayama rings. Furthermore, as we mention, local QF-rings are important for the study on the Faith conjecture, since the Faith conjecture is not solved even for local semiprimary one-sided selfinjective rings with Jacobson radical cubed zero.

1. INTRODUCTION

There are many open problems on QF-rings. The two most famous, longstanding, unsolved problems are the Nakayama conjecture and the Faith conjecture. One may refer to Nicholson-Yousif [16] for the Faith conjecture, as well as for several more recent questions on QF-rings.

The Faith conjecture. Is a semiprimary right self-injective ring a QF-ring? Faith conjectured "no" in his book [6].

The Faith conjecture is not solved even for a local semiprimary ring with radical cubed zero. Thus we record:

Problem 1. Is a semiprimary local right self-injective ring with radical cubed zero a QF-ring?

The following result gives some information on this question.

Fact 1. (Baba-Oshiro [2]) *If R is a semiprimary ring, then R is a right self-injective ring if and only if R is a right simple-injective ring. In particular, if R is a local semiprimary ring with radical J cubed zero, then R is right self-injective if and only if ${}_RJ^2, J^2_R$ are simple and, for any maximal right submodule M of J, there exists $a \in J \setminus J^2$ satisfying $aM = 0$.*

We now provide a careful analysis of Problem 1 and translate this problem into a problem on two-sided vector spaces over division rings.

In order to do so, let R be a local semiprimary ring with $J^2 \neq 0$ and $J^3 = 0$, where $J := J(R)$ denotes the radical of R. Let D denote the division ring R/J and put $\overline{J} = J/J^2$.

The detailed version of this paper will be submitted for publication elsewhere.

Then, $\overline{J}$ and J^2 are (D,D)-bispaces. We denote by $Soc^\ell(R)$ and $Soc^r(R)$ the left and the right socle of R, respectively and by $l_R(A)$ and $r_R(A)$ the left and the right annihilator of a subset A of R, respectively.

We now record some properties on R.

Fact 2. (1) *If $_RSoc^\ell(R)$ and $Soc^r(R)_R$ are simple, then $r_Rl_R(A) = A$ and $l_Rr_R(B) = B$ for any finitely generated right submodule A and for any finitely generated left submodule B of J, $J^2 = Soc^\ell(R) = Soc^r(R)$, and $_DJ^2$ and J^2_D are one-dimensional spaces.*

(2) *If J_R is finitely generated and $_RSoc^\ell(R)$ and $Soc^r(R)_R$ are simple, then R is QF. For this QF-ring R, we can make a new QF-ring T of graded type as follows: Consider the (D,D)-bispace $T = D \times \overline{J} \times J^2$. In T, we define a multiplication by setting*

$$t_1t_2 = (d_1d_2, d_1\overline{a_2} + \overline{a_1}d_2, d_1s_2 + s_1d_2 + a_1a_2)$$

for $t_1 = (d_1, \overline{a_1}, s_1)$ and $t_2 = (d_2, \overline{a_2}, s_2) \in T$, where $\overline{a_i} = a_i + J^2 \in J/J^2 =: \overline{J}$. Then, T is a QF-ring with $J(T) = 0 \times \overline{J} \times J^2$, $J(T)^2 = 0 \times 0 \times J^2$ and $J(T)^3 = 0$. (In general, $R \not\cong T$.)

Fact 3. *Assume that R_R is (simple-)injective. Then*

(1) *$_RSoc^\ell(R)$ and $Soc^r(R)_R$ are simple.*
(2) *For any maximal submodule M of J_R, $aM = 0$ for some $a \in J \setminus S$.*
(3) *$r_Rl_R(A) = A$ for any finitely generated submodule A of J_R and $l_Rr_R(B) = B$ for any finitely generated submodule B of $_RJ$.*
(4) *Put $J^* = \mathrm{Hom}_R(\overline{J}_R, J^2_R)$. Then, for any $a \in J$, the map $a \to (a)_L$ (left multiplication) gives an (R,R)-bimodule isomorphism and a (D,D)-bispace isomorphism: $_R\overline{J}_R \cong {}_RJ^*_R$ and $_D\overline{J}_D \cong {}_DJ^*_D$.*
(5) *Put $\alpha = \dim(\overline{J}_D)$. If α is finite, then R is QF, while if α is infinite, then $\dim({}_D\overline{J}) = (\#D)^\alpha = \#R > \alpha$; in particular, if $\alpha = \aleph_0$ and $\#D = \aleph$, then $\dim({}_D\overline{J}) = \aleph$, where $\#A$ denotes the cardinal number of a set A.*

Most known information on R emanates from Fact 2 and Fact 3. In particular, (4) in Fact 3 is important for investigating Problem 1.

2. Local *QF*-rings

We now give a construction of local *QF*-rings.

In this section, let D be a division ring and let $_DV_D$ be a (D,D)-bispace. We put

$$T = D \times V \times (V \otimes_D V).$$

Then, T is a (D,D)-bispace. In T, we define a multiplication as follows:

$$t_1t_2 = (d_1d_2, d_1v_2 + v_1d_2, d_1x_2 + x_1d_2 + v_1 \otimes v_2)$$

for $t_1 = (d_1, v_1, x_1)$ and $t_2 = (d_2, v_2, x_2) \in T$. It is easy to see that T is a local semiprimary ring with radical cubed zero and that

$$J(T) = 0 \times V \times V \otimes_D V \quad \text{and} \quad J(T)^2 = 0 \times 0 \times V \otimes_D V.$$

We identify $D \times 0 \times 0$, $0 \times V \times 0$ and $0 \times 0 \times V \otimes_D V$ with D, V and $V \otimes_D V$, respectively.

We note the following:

Proposition 4. (1) *Assume that there exists a (D, D)-bisubspace I of $V \otimes_D V$ with $\dim((V \otimes_D V)/I_D) = \dim((V \otimes_D V)/{}_D I) = 1$ and $vD \otimes_D V \not\subset I$ and $V \otimes_D Dv \not\subset I$ for any $0 \neq v \in V$. Then, I is an ideal of T, $J(T/I)^2 = Soc^r(T/I) = Soc^\ell(T/I)$, and $J(T/I)^2$ is simple as a left T/I-module and as a right T/I-module.*

(2) *Assume that $\dim(V_D)$ is finite and such a (D, D)-bisubspace I in (1) exists. Then, T/I is a local QF-ring with radical cubed zero.*

Proof. (1) is easily seen and (2) follows from Fact 1. □

Let pD be a one-dimensional right vector space and let $\rho \in \mathrm{Aut}(D)$. Then, pD becomes a one-dimensional left vector space by defining $dp = p\rho(d)$ for $d \in D$. We denote such a (D, D)-bispace by pD^ρ. We also put

$$V^* = \mathrm{Hom}_D(V_D, pD^\rho_D).$$

Then, V^* is canonically a (D, D)-bispace. Here we assume the following:

Assumption A: There exists a (D, D)-bispace isomorphism $\theta : V \to V^*$.

Since the map $(V, V) \to pD^\rho$ given by $(v, w) \mapsto \theta(v)(w)$ is a bilinear (D, D) onto map, the map

$$\lambda : V \otimes_D V \to pD^\rho \quad \text{by} \quad \sum_i v_i \otimes w_i \mapsto \sum_i \theta(v_i)(w_i)$$

is a (D, D)-bispace onto homomorphism. As is easily seen, $\mathrm{Ker}\,\lambda$ is an ideal of the ring $T = D \times V \times (V \otimes_D V)$. We put

$$D\langle V, \theta, \rho, pD^\rho\rangle = T/\mathrm{Ker}\,\lambda.$$

Let $w \in \lambda^{-1}(p)$ be fixed and put $s = w + \mathrm{Ker}\,\lambda \in (V \otimes_D V)/\mathrm{Ker}\,\lambda$. Then we can show the following result.

Theorem 5. *Let R be the ring $D\langle V, \theta, \rho, pD^\rho\rangle$ above. Then the following hold.*

(1) *$J := J(R) = (V \times V \otimes_D V)/\mathrm{Ker}\,\lambda$, $J^2 = (V \otimes_D V)/\mathrm{Ker}\,\lambda = Rs = sR$, $J^3 = 0$ and $Soc^r(R) = Soc^\ell(R) = J^2$.*

(2) *${}_R J^2$ and J^2_R are simple.*

(3) *R is a right self-injective ring.*

(4) *R is QF if and only if $\dim(V_D)$ is finite.*

Proof. (1), (2) and (4) follow from Proposition 4. To show (3), let I be a maximal submodule of J_R. By Fact 1, it suffices to show that there exists $a \in J \setminus J^2$ satisfying $aI = 0$. Let X be a subspace of V_D with $X/\mathrm{Ker}\,\lambda = I$. Then, as X is a proper subspace of V, we can take $0 \neq v^* \in V^*$ such that $v^*(X) = 0$. Put $a = \theta^{-1}(v^*) + \operatorname{Ker}\lambda$. Then, $a \in J \setminus J^2$ and $aI = 0$, as desired. □

By Theorem 5, we can translate Problem 1 into the following:

Problem 2. Does there exist a division ring D and a (D, D)-bispace V such that $\dim(V_D) = \infty$ and ${}_DV_D \cong {}_DV_D^*$ ((D, D)-isomorphism)?

If such a space ${}_DV_D$ exists, Theorem 5 asserts the Faith conjecture is true, that is, we can construct a semiprimary right self-injective ring which is not *QF*. However, this problem is very difficult. In fact, if we try to solve this, we immediately encounter pathologies.

However, as a biproduct of the study on Problem 2, we can obtain an important way of constructing local *QF*-rings. We shall state this construction.

Lemma 6. *Let V be a bispace over a division ring D with $n = \dim({}_DV) = \dim(V_D) < \infty$. Then we can take $x_1, \ldots, x_n \in V$ satisfying $V = Dx_1 \oplus \cdots \oplus Dx_n = x_1D \oplus \cdots \oplus x_nD$.*

Proof. Let $x_1, \ldots, x_k, y, z \in V$ such that $x_1, \ldots, x_k, y$ and $x_1, \ldots, x_k, z$ are left and right independent over D, respectively. If $Dz \cap \sum_{i=1}^k Dx_i = 0$ or $yD \cap \sum_{i=1}^k x_iD = 0$, then $x_1, \ldots, x_k, z$ or $x_1, \ldots, x_k, y$ are left and right independent, respectively. If otherwise, i.e., $Dz \subset \sum_{i=1}^k Dx_i$ and $yD \subset \sum_{i=1}^k x_iD$, then $x_1, \ldots, x_k, y+z$ are left and right independent. By continuing this procedure, the statement is shown. □

Now, henceforth, let

$$V_D = x_1D \oplus \cdots \oplus x_nD$$

be a finite dimensional right vector space over a division ring D and let

$$\sigma = (\sigma_{ij}) : D \to (D)_n \quad \text{by} \quad d \mapsto \sigma(d) = \begin{pmatrix} \sigma_{11}(d) & \cdots & \sigma_{1n}(d) \\ & \cdots & \\ \sigma_{n1}(d) & \cdots & \sigma_{nn}(d) \end{pmatrix}$$

be a ring homomorphism, where $(D)_n$ is the ring of all $n \times n$ matrices over D. By using σ, we define a left D-operation on V as follows: For $d \in D$, $dx_i = \sum_{j=1}^n x_j\sigma_{ji}(d)$, namely,

$$d(x_1, \ldots, x_n) = (x_1, \ldots, x_n) \begin{pmatrix} \sigma_{11}(d) & \cdots & \sigma_{1n}(d) \\ & \cdots & \\ \sigma_{n1}(d) & \cdots & \sigma_{nn}(d) \end{pmatrix}.$$

Then, V_D becomes a (D, D)-bispace. We denote this bispace by $V\langle x_1, \ldots, x_n; \sigma\rangle$ or simply V^σ. We note that pD^ρ mentioned above is $pD\langle p; \rho\rangle$.

Proposition 7. *The following are equivalent:*

(1) $V^\sigma = Dx_1 \oplus \cdots \oplus Dx_n$.

(2) *There is a ring homomorphism* $\xi = (\xi_{ij}) : D \to (D)_n$ *such that for* $1 \le i, k \le n$, *the following formulas hold:*

$$\sum_{j=1}^{n} \sigma_{kj}(\xi_{ij}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i \end{cases}$$

and

$$\sum_{j=1}^{n} \xi_{jk}(\sigma_{ji}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i. \end{cases}$$

Proof. (1) ⇒ (2). Since $x_i d = \sum_j \xi_{ij}(d) x_j = \sum_j (\sum_k x_k \sigma_{kj}(\xi_{ij}(d)))$, we see that

$$\sum_{j} \sigma_{kj}(\xi_{ij}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i. \end{cases}$$

Similarly, the second formula is obtained.

(2) ⇒ (1). Since $\sum_j \xi_{ij}(d) x_j = \sum_j (\sum_k x_k \sigma_{kj}(\xi_{ij}(d))) = x_i \sum_j \sigma_{ij}(\xi_{ij}(d)) = x_i d$, we see that for any $d \in D$,

$$\xi(d) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} d,$$

from which $V^\sigma = Dx_1 + \cdots + Dx_n$.

Next, to show that $\{Dx_1, \ldots, Dx_n\}$ is an independent set, assume $d_1 x_1 + \cdots + d_n x_n = 0$ for $d_1, \ldots, d_n \in D$. Since $\sum_i (\sum_j x_j \sigma_{ji}(d_i)) = 0$, we see that $\sum_i \sigma_{ji}(d_i) = 0$ for $j = 1, \ldots n$ and hence

$$\begin{array}{rcll} \sigma_{11}(d_1) + \sigma_{12}(d_2) + \cdots + \sigma_{1n}(d_n) & = & 0 & \cdots (1) \\ \sigma_{21}(d_1) + \sigma_{22}(d_2) + \cdots + \sigma_{2n}(d_n) & = & 0 & \cdots (2) \\ \cdots \quad \cdots \quad \cdots & & & \\ \sigma_{n1}(d_1) + \sigma_{n2}(d_2) + \cdots + \sigma_{nn}(d_n) & = & 0 & \cdots (n). \end{array}$$

Thus, $0 = \xi_{1j} \times (1) + \xi_{2j} \times (2) + \cdots + \xi_{nj} \times (n) = \sum_i \xi_{ij}(\sigma_{ij}(d_j)) = d_j$; hence $d_j = 0$, as desired. □

By Proposition 7, we see that, if there is a ring homomorphism $\xi = (\xi_{ij}) : D \to (D)_n$ satisfying the formulas of the proposition, then $V^\sigma = Dx_1 \oplus \cdots \oplus Dx_n$. For this situation, we use $V\langle x_1, \ldots, x_n; \sigma, \xi\rangle$ instead of $V\langle x_1, \ldots, x_n; \sigma\rangle$. Moreover, we construct the ring R above for this bispace under Assumption A and denote it by $D\langle V, \sigma, \xi, \theta, \rho, pD^\rho\rangle$. Combining the proposition with Lemma 6, we have the following:

Theorem 8. *Assume that* $\dim({}_D V) = n < \infty$. *Then,* $R = D\langle V, \sigma, \xi, \theta, \rho, pD^\rho\rangle$ *is a local QF-ring with radical cubed zero.*

Now, we return to our Assumption A and construct such a (D, D)-bispace isomorphism $\theta : V \to V^*$ under some condition. Let $\xi = (\xi_{ij}) : D \to (D)_n$ be a ring homomorphism satisfying the formulas of Proposition 7(2) such that $\rho\xi_{ij} = \sigma_{ij}$ for all i, j. For each i, let $\alpha_i \in V^* = \mathrm{Hom}_D(V_D, pD^\rho_D)$ be defined by

$$\alpha_i : x_1 d_1 + \cdots + x_n d_n \mapsto p d_i,$$

where d_1, ..., $d_n \in D$. Then, ${}_DV^* = D\alpha_1 \oplus \cdots \oplus D\alpha_n$ and the map

$$\theta^* : {}_DV_D \to {}_DV^*_D \ \text{ by } \ d_1x_1 + \cdots + d_nx_n \to d_1\alpha_1 + \cdots + d_n\alpha_n$$

is a (D, D)-bispace isomorphism. Therefore, in this case, we can make a local QF-ring $D\langle V, \sigma, \xi, \theta^*, \rho, pD^\rho\rangle$. In particular, setting $\rho = id_D$, we can take σ as ξ above . Hence, we obtain the following, which is useful for making local QF-rings with radical cubed zero.

Theorem 9. *Let $V_D = x_1D \oplus \cdots \oplus x_nD$ be an n-dimensional vector space over a division ring D and let $\sigma = (\sigma_{ij}) : D \to (D)_n$ be a homomorphism satisfying the formulas: For $1 \le i, k \le n$,*

$$\sum_{j=1}^{n} \sigma_{kj}(\sigma_{ij}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i \end{cases}$$

and

$$\sum_{j=1}^{n} \sigma_{jk}(\sigma_{ji}(d)) = \begin{cases} d & k = i \\ 0 & k \neq i. \end{cases}$$

Then we can make a local QF-ring $D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$.

3. Examples of local QF-rings

Example 1. Let $V = x_1D \oplus \cdots \oplus x_nD$ be an n-dimensional vector space over a division ring D and let π be a ring automorphism of D. Consider a ring homomorphism

$$\sigma : D \to (D)_n \ \text{ by } \ d \ \mapsto \begin{pmatrix} \pi(d) & & 0 \\ & \ddots & \\ 0 & & \pi(d) \end{pmatrix}.$$

Then, by σ, V becomes a (D, D)-bispace and

$$\begin{pmatrix} \pi^{-1}(d) & & 0 \\ & \ddots & \\ 0 & & \pi^{-1}(d) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} d.$$

The map $\xi = (\xi_{ij}) : D \to (D)_n$ is given by

$$d \ \mapsto \begin{pmatrix} \pi^{-1}(d) & & 0 \\ & \ddots & \\ 0 & & \pi^{-1}(d) \end{pmatrix}.$$

It then follows from the argument above Theorem 9 that we can construct a local QF-ring $D\langle V, \sigma, \xi, \theta^*, \pi^2, pD^{\pi^2}\rangle$.

Example 2. Let $\mathbb{C}$ be the field of complex numbers, let $V = x_1\mathbb{C} \oplus x_2\mathbb{C}$ be a 2-dimensional vector space over $\mathbb{C}$ and consider a ring homomorphism

$$\sigma : \mathbb{C} \to (\mathbb{C})_2 \ \text{ by } \ a + bi \ \mapsto \ \begin{pmatrix} a & bi \\ bi & a \end{pmatrix}.$$

Then the map σ satisfies the formulas in Theorem 9. Hence we can make a local QF-ring $\mathbb{C}\langle V, \sigma, \sigma, \theta^*, id_{\mathbb{C}}, 1\mathbb{C}^{id_{\mathbb{C}}}\rangle$.

This example can be slightly generalized as the following:

Example 3. Let k be a commutative field and let $f(x) = x^n - a \in k[x]$ be irreducible with α a root, $D = k(\alpha)$ and $V = \sum_{i=1}^{n} \oplus x_i D$. Let a map

$$\sigma : D \to (D)_n \ \text{ by } \ \sum_{i=0}^{n-1} a_i\alpha^i \ \mapsto \ \begin{pmatrix} a_0 & a_1\alpha & a_2\alpha^2 & \cdots & a_{n-1}\alpha^{n-1} \\ a_{n-1}\alpha^{n-1} & a_0 & a_1\alpha & \cdots & a_{n-2}\alpha^{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_2\alpha^2 & \ddots & \ddots & \ddots & a_1\alpha \\ a_1\alpha & a_2\alpha^2 & \cdots & a_{n-1}\alpha^{n-1} & a_0 \end{pmatrix}.$$

Then, σ is a ring homomorphism satisfying the formulas in Theorem 9. Hence, for a given n-dimensional vector space V over D, we can make a local QF-ring $D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$.

Example 4. Let $\mathbb{H}$ be the quaternion algebra, let $V = x_1\mathbb{H} \oplus x_2\mathbb{H} \oplus x_3\mathbb{H} \oplus x_4\mathbb{H}$ be a 4-dimensional vector space over $\mathbb{H}$ and consider a ring homomorphism

$$\sigma : \mathbb{H} \to (\mathbb{H})_4 \ \text{ by } \ a + bi + cj + dk \ \mapsto \ \begin{pmatrix} a & bi & cj & dk \\ bi & a & dk & cj \\ cj & dk & a & bi \\ dk & cj & bi & a \end{pmatrix}.$$

Then, σ satisfies the formulas in Theorem 9. Hence we can make a local QF-ring $\mathbb{H}\langle V, \sigma, \sigma, \theta^*, id_{\mathbb{H}}, 1\mathbb{H}^{id_{\mathbb{H}}}\rangle$.

Further, using Theorem 9, we shall show two constructing ways of local QF-algebras R with radical cubed zero, one of which gives an example of a local QF-algebra which is not a finite dimensional algebra.

Example 5. Let E be a field and let π be an automorphism of E satisfying

(1) $\pi^2 = id_E$.

(2) $\alpha\pi(\alpha) + \beta\pi(\beta) = 0 \Rightarrow \alpha = 0$ and $\beta = 0$ for $\alpha, \beta \in E$.

Define a 2-dimensional vector space over E : Let $D = E \oplus E\boldsymbol{i} = \{\alpha + \beta\boldsymbol{i} \mid \alpha, \beta \in E\}$ with the product $\boldsymbol{i}\alpha = \pi(\alpha)\boldsymbol{i}$ for any $\alpha \in E$ (the addition, as well as the multiplication between elements of E being the natural ones). Then, D is a division ring, as it can be checked; see the product:

$$(\alpha + \beta\boldsymbol{i})(\pi(\alpha) - \beta\boldsymbol{i}) = \alpha\pi(\alpha) + \beta\pi(\beta)$$

and if $\alpha + \beta \boldsymbol{i} \neq 0$, then we have $\alpha\pi(\alpha) + \beta\pi(\beta) \neq 0$ by (3). Also, the center of D is $K := \{a \in E \mid \pi(a) = a\}$.

Let $V = x_1 D \oplus x_2 D$ be a 2-dimensional vector space over D and consider a ring homomorphism

$$\sigma : D \to (D)_2 \text{ by } \alpha + \beta \boldsymbol{i} \mapsto \begin{pmatrix} \alpha & \beta \boldsymbol{i} \\ \beta \boldsymbol{i} & \alpha \end{pmatrix}.$$

Then we see that σ satisfies the formulas in Theorem 9. Hence we can make a local *QF*-ring $R = D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$.

We shall give some examples of fields E satisfying (1) and (2) above.

(i) Let $E = \mathbb{C}$ or an arbitrary imaginary quadratic field (e.g. $\mathbb{Q}\left(\sqrt{-3}\right)$) and the map $\pi : E \to E$ defined by $\pi(\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ denotes the conjugate of α.

(ii) Let K be a field and π an automorphism of K satisfying the conditions (1) and (2). Moreover, let $E = K(x)$ be the field of rational functions in x over K. For $f = a_n x^n + \cdots + a_1 x + a_0 \in K[x]$, we put $\overline{f} = \pi(a_n)x^n + \cdots + \pi(a_1)x + \pi(a_0)$. Then the map $\overline{\pi} : E \to E$ given by $\overline{\pi}(f/g) = \overline{f}/\overline{g}$ is an automorphism of E. We see that the fixed field of $\overline{\pi}$ in E is $F(x)$, where F is the fixed field of π in K, and E and $\overline{\pi}$ satisfies (1) and (2) again.

Example 6. Let E be a division ring such that E is infinite dimensional over its center K and $x^2 \neq -1$ holds for any element $x \in E$.

Define a 2-dimensional vector space over E : Let $D = E \oplus E\boldsymbol{i} = \{\alpha + \beta\boldsymbol{i} \mid \alpha, \beta \in E\}$. Define the products $\boldsymbol{i}^2 = -1$ and $\boldsymbol{i}\alpha = \alpha\boldsymbol{i}$ for any $\alpha \in E$. Then, D becomes a ring (the addition, as well as the multiplication between elements of E being the natural ones). Furthermore, D is a division ring. Actually, let $d = \alpha + \beta\boldsymbol{i}$ be a non-zero element in D. If $\beta = 0$, then clearly $d^{-1} = \alpha^{-1}$. In case $\beta \neq 0$, it is easily checked that $(\alpha+\beta\boldsymbol{i})\cdot(\beta^{-1}\alpha-\boldsymbol{i})\beta^{-1}((\alpha\beta^{-1})^2+1)^{-1} = 1$ and $((\beta^{-1}\alpha)^2+1)^{-1}(\beta^{-1}\alpha-\boldsymbol{i})\beta^{-1}\cdot(\alpha+\beta\boldsymbol{i}) = 1$. This means that d is invertible.

Next, let $V = x_1 D \oplus x_2 D$ be a 2-dimensional vector space over D and consider a ring homomorphism

$$\sigma : D \to (D)_2 \text{ by } \alpha + \beta \boldsymbol{i} \mapsto \begin{pmatrix} \alpha & \beta \boldsymbol{i} \\ \beta \boldsymbol{i} & \alpha \end{pmatrix}.$$

Then we see that σ satisfies the formulas in Theorem 9. Hence we can make a local *QF*-ring $R = D\langle V, \sigma, \sigma, \theta^*, id_D, 1D^{id_D}\rangle$ and we can see that R is an infinite dimensional algebra with K (its center).

We shall give an example of a division ring E in Example 6. Consider the functional field $L = \mathbb{R}(x)$ over the field $\mathbb{R}$ of real numbers and let σ be an into monomorphism of L given by $f(x)/g(x) \to f(x^2)/g(x^2)$. Let $L[y; \sigma]$ be a skew-polynomial ring associated with σ. Although $L[y; \sigma]$ is a non-commutative domain, it has the quotient ring which is a division ring. We denote it by E. As is easily seen, the center of E is $\mathbb{R}$ and E is infinite dimensional over $\mathbb{R}$ and it holds that $a^2 \neq -1$ for any non-zero element $a \in E$.

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