DUAL OJECTIVITY OF QUASI-DISCRETE MODULES AND LIFTING MODULES

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Abstract. In [3], K.Oshiro and his students introduced "ojectivity (generalized injectivity)", a new concept of relative injectivity, and using this injectivity we obtained some results for direct sums of extending modules. Afterward, S.H.Mohamed and B.J.Müller [9] defined a dual concept of ojectivity as follows:

Definition. M is said to be *N-dual ojective (or generalized N-projective)* if, for any epimorphism $g : N \to X$ and any homomorphism $f : M \to X$, there exist decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$, such that $gh_1 = f|_{M_1}$ and $fh_2 = g|_{N_2}$.

The concept of relative dual ojectivity is a generalization of relative projectivity and this projectivity has an important meaning for the study of direct sums of lifting modules (cf. [6], [9]).

In this paper we introduce some results on "dual ojectivity" and apply it to direct sums of quasi-discrete modules.

1. Introduction

A module M is said to be *lifting* if, it satisfies the following property: For any submodule X of M, there exists a decomposition $M = X^* \oplus X^{**}$ such that $X^* \subseteq X$ and the kernel X/X^* of the canonical epimorphism $M/X^* \to M/X$ is a small submodule of M/X^*, equivalently, $X \cap X^{**}$ is a small submodule of X^{**}. In [9], S.H.Mohamed and B.J.Müller defined dual ojective module. This projectivity plays an important role in the study of direct sums of lifting modules (cf. [6], [9]). Since the structure of dual ojectivity is complicated, it is difficult to see whether dual ojectivity pass to a (finite) direct sum. This problem is not easy even in the case each module is quasi-discrete.

In this paper we consider this problem and apply it to direct sums of quasi-discrete modules.

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules. A submodule S of a module M is said to be a *small* submodule, if $M \neq K + S$ for any proper submodule K of M and we write $S \ll M$ in this case. Let M be a module and let N and K be submodules of M with $K \subseteq N$. K is said to be a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M in this case. Let X be a submodule of M. X is called *co-closed* submodule in M if X has not a proper co-essential submodule in M. X' is called a *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$ in M.

The detailed version of this paper will be submitted for publication elsewhere.

A module M has the *finite internal exchange property* if, for any finite direct sum decomposition $M = M_1 \oplus \cdots \oplus M_n$ and any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i = 1, \cdots, n)$ such that $M = X \oplus \overline{M_1} \oplus \cdots \oplus \overline{M_n}$.

A module M is said to be a *lifting* module if, for any submodule X, there exists a direct summand X^* of M such that $X^* \subseteq_c X$ in M.

Let $\{M_i \mid i \in I\}$ be a family of modules and let $M = \oplus_I M_i$. M is said to be a *lifting module for* the decomposition $M = \oplus_I M_i$ if, for any submodule X of M, there exist $X^* \subseteq M$ and $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $X^* \subseteq_c X$ in M and $M = X^* \oplus (\oplus_I \overline{M_i})$, that is, M is a lifting module and satisfies the internal exchange property in the direct sum $M = \oplus_I M_i$.

Let X be a submodule of a module M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$, if and only if Y is minimal with $M = X + Y$. Note that supplement Y of X in M is co-closed in M. A module M is $(\oplus-)$*supplemented* if, for any submodule X of M, there exists a submodule (direct summand) Y of M such that Y is supplement of X in M. A module M is called *amply supplemented* if, X contains a supplement of Y in M whenever $M = X + Y$. We note that

$$\text{lifting} \Rightarrow \text{amply supplemented} \Rightarrow \text{supplemented}.$$

Now we consider the following condition:

$(\sharp)$ Any submodule of M has a co-closure in M.

Note that a module M is amply supplemented if and only if M is supplemented with a condition $(\sharp)$ (cf. [2], [5]).

The reader can refer to [1], [4], [8], [11] and [12] for research on lifting modules, quasi-discrete modules and exchange properties.

2. Generalized Projectivity

A module A is said to be *B-dual ojective* (*generalized B-projective*) if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there exist decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $g \circ h_1 = f|_{A_1}$ and $f \circ h_2 = g|_{B_2}$ (cf. [9]). Note that every B-projective modules is B-dual ojective.

Now we introduce some properties of the dual ojectivity.

Proposition 2.1 (cf. [9]). Let B^* be a direct summand of B. If A is B-dual ojective, then A is B^*-dual ojective.

Proposition 2.2 (cf. [6, Proposition 2.2]). Let A be a module with the finite internal exchange property and let A^* be a direct summand of A. If A is B-dual ojective, then A^* is B-dual ojective.

Proposition 2.3 (cf. [6, Proposition 2.3]). Let $M = A \oplus B$ be supplemented with $(\sharp)$ and let A^* be a direct summand of A. If A is B-dual ojective, then A^* is B-dual ojective.

A ring R is said to be *right perfect* if any right R-module has projective cover. By [10, Theorem 1.3], any submodule N of a module M over a right perfect ring has co-closure of N in M. Thus the following is immediate from Proposition 2.3.

Corollary 2.4. Let R be a right perfect ring, A, B be R-modules and A^* be a direct summand of A. If A is B-dual ojective, then A^* is B-dual ojective.

A module A is said to be *im-small B-projective* if, for any epimorphism $g : B \to X$ and any homomorphism $f : A \to X$ with $\mathrm{Im} f \ll X$, there exists a homomorphism $h : A \to B$ such that $g \circ h = f$ (cf. [5]).

Proposition 2.5. (1) Let A be a module and let $\{B_i \mid i = 1, \cdots, n\}$ be a family of modules. Then A is im-small $\oplus_{i=1}^{n} B_i$-projective if and only if A is im-small B_i-projective $(i = 1, \cdots, n)$.
(2) Let I be any set and let $\{A_i \mid i \in I\}$ be a family of modules. Then $\oplus_I A_i$ is im-small B-projective if and only if A_i is im-small B-projective for all $i \in I$.

Proposition 2.6 (cf. [6, Proposition 2.5]). Let A be any module and let B be a lifting module. If A is B-dual ojective, then A is im-small B-projective.

The concept of relative dual ojectivity has an important meaning for the study of direct sums of lifting modules.

Theorem 2.7 (cf. [6, Theorem 3.7]). Let $M_1, \cdots, M_n$ be lifting modules with the finite internal exchange property and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.
(1) M is lifting with the finite internal exchange property.
(2) M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.
(3) M_i and $\oplus_{j \neq i} M_j$ are relative dual ojective.

3. Direct sums of quasi-discrete modules

A lifting module M is said to be *quasi-discrete* if M satisfies the following condition (D):

(D) If M_1 and M_2 are direct summands of M such that $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Any quasi-discrete module has the internal exchange property [10, Theorem 3.10].

Lemma 3.1 (cf. [7]). Let N be a quasi-discrete module and let $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. Assume that M_i is generalized N-projective $(i = 1, \cdots, n)$. Then, for any epimorphism $f : M \to X$ with $\ker f \ll M$ and any epimorphism $g : N \to X$ with $\ker g \ll N$, there exist decompositions $M = \overline{M} \oplus \overline{\overline{M}}$, $N = \overline{N} \oplus \overline{\overline{N}}$ and epimorphisms $\varphi : \overline{\overline{M}} \to \overline{N}$, $\psi : \overline{\overline{N}} \to \overline{M}$ such that $f|_{\overline{\overline{M}}} = g \circ \varphi$ and $g|_{\overline{\overline{N}}} = f \circ \psi$.

By the using lemma above, we can obtain the following propositions.

Proposition 3.2 (cf. [7]). Let N be a quasi-discrete module and $M = M_1 \oplus \cdots \oplus M_n$ be lifting for $M = M_1 \oplus \cdots \oplus M_n$. If M_i is N-dual ojective $(i = 1, \cdots, n)$, then M is N-dual ojective.

Proposition 3.3 (cf. [7]). Let M be a quasi-discrete module and $N = N_1 \oplus \cdots \oplus N_m$ be lifting for $N = N_1 \oplus \cdots \oplus N_m$. If N_i and M are relative dual ojective ($i = 1, \cdots, m$), then M is N-dual ojective.

The following is immediate from Propositions 3.2, 3.3, Theorem 2.7 and induction.

Theorem 3.4. Let $M_1, \cdots, M_n$ be quasi-discrete modules and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following conditions are equivalent.
(1) M is lifting with the (finite) internal exchange property.
(2) M is lifting for $M = M_1 \oplus \cdots \oplus M_n$.
(3) M_i is M_j-dual ojective ($i \neq j$).

A module H is said to be *hollow* if it is an indecomposabe lifting module.

Corollary 3.5. Let $H_1, \cdots, H_n$ be hollow modules and put $M = H_1 \oplus \cdots \oplus H_n$. Then the following conditions are equivalent.
(1) M is lifting with the (finite) internal exchange property.
(2) M is lifting for $M = H_1 \oplus \cdots \oplus H_n$.
(3) H_i is H_j-dual ojective ($i \neq j$).

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