AUSLANDER-REITEN CONJECTURE ON GORENSTEIN RINGS

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Abstract. The Auslander-Reiten conjecture is related closely to the Nakayama conjecture. In this lecture, we consider the Auslander-Reiten conjecture for a Gorenstein rings.

1. Introduction

The Nakayama's 1958 conjecture (NC) is a one of most famous and important conjecture in ring theory.

(NC) Let $0 \to_\Lambda \Lambda \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of an artin algebra Λ. If all I^j are projective, then Λ is self-injective.

Auslander and Reiten conjectured the generalized Nakayama conjecture (GNC) in [3]

(GNC) Let $0 \to_\Lambda \Lambda \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of an Artin algebra Λ. For any indecomposable injective Λ-module I, I is a direct summand of some I^j.

They showed that (GNC) holds for all artin algebras if and only if the following conjecture (ARC') holds for all artin algebras.

(ARC') For an Artin algebra Λ, if M is a finitely generated Λ-module and $\mathrm{Ext}^i_\Lambda(M, M \oplus \Lambda) = 0$ $(\forall i > 0)$, then M is projective.

M. Auslander, S. Ding, and Ø. Solberg widened the context to algebras over commutative local rings [2].

(ARC) For a commutative Noetherian local ring R, if M is a finitely generated R-module and $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ $(\forall i > 0)$, then M is free.

They showed in [2] that if R is a complete intersection, then R satisfies (ARC). We shall show the following main theorem.

Theorem 1. *Let R be a Gorenstein ring. If R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$ with* $\mathrm{ht}\, p \leq 1$, *then R_p satisfies* (ARC) *for all $p \in \mathrm{Spec}R$.*

The detailed version of this paper will be submitted for publication elsewhere.

2. Main Results

Through in this paper, we denote by R the d-dimensional commutative Gorenstein local ring with the unique maximal ideal $\mathfrak{m}$. We also denote by $\operatorname{mod} R$ the category of finitely generated R-modules and by $\operatorname{CM} R$ the full subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay modules.

We give a following condition to consider the Auslander-Reiten conjecture.

(ARC) For $M \in \operatorname{mod} R$, suppose $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ $(i > 0)$, then M is free.

The main theorem of this paper is following;

Theorem 1. *If R_p satisfies* (ARC) *for all $p \in \operatorname{Spec} R$ with $\operatorname{ht} p \leq 1$, then R_p satisfies* (ARC) *for all $p \in \operatorname{Spec} R$.*

It is difficult to check the freeness of modules in general. We give a following theorem to check the freeness of vector bundles.

Theorem 2. *We assume $\dim R = d \geq 2$. Let $M \in \operatorname{CM} R$ be a vector bundle. Suppose $\operatorname{Ext}^{d-1}_R(M, M) = 0$, then M is free.*

We say M is a *vector bundle* if M_p is a free R_p-module for all prime ideal p which is not maximal ideal $\mathfrak{m}$. We want to omit the assumption M is a vector bundle in Theorem 2. But there is a counterexample if M is not a vector bundle.

Example 3. Let k be a field. We set $R = k[x, y, z]/(xy)$ be a 2-dimensional hypersurface and $M = R/(x)$. In this case, we can check that $\operatorname{Ext}^i_R(M, M) = 0$ if and only if i is odd. In particular, we see that $\operatorname{Ext}^{2-1}_R(M, M) = 0$ even if M is not free.

We prepare a lemma to show Theorem 2.

Lemma 4. [9, Lemma 3.10.] *Let R be a d-dimensional Cohen-Macaulay local ring and ω be a canonical module. We denote by $(-)^\vee$ the canonical dual $\operatorname{Hom}_R(-, \omega)$. For vector bundles M and $N \in \operatorname{CM} R$, we have a following isomorphism;*

$$\operatorname{Ext}^d_R(\underline{\operatorname{Hom}}(N, M), \omega) \cong \operatorname{Ext}^{d+1}_R(M, (\operatorname{tr} N)^\vee)$$

Here, $\underline{\operatorname{Hom}}(N, M)$ is the set of stable homomorphisms.

Proof of Theorem 2. Let $M \in \operatorname{CM} R$ be a vector bundle and we assume $\operatorname{Ext}^{d-1}_R(M, M) = 0$.

We take a minimal free resolution of M;

$$F_\bullet : \cdots \to F_1 \to F_0 \to M \to 0.$$

Apply $(-)^* := \operatorname{Hom}_R(-, R)$, we get exact sequence;

$$0 \to M^* \to F_0^* \to F_1^* \to \operatorname{tr} M \to 0.$$

Since R is Gorenstein and M is maximal Cohen-Macaulay, we have $\Omega^2 M \cong (\operatorname{tr} M)^* (\cong (\operatorname{tr} M)^\vee)$. Therefore, we have

$$\begin{aligned}\operatorname{Ext}_R^{d+1}(M,(\operatorname{tr}N)^\vee) &\cong \operatorname{Ext}_R^{d+1}(M,(\operatorname{tr}N)^*)\\ &\cong \operatorname{Ext}_R^{d+1}(M,\Omega^2 M)\\ &\cong \operatorname{Ext}_R^{d-1}(M,M)=0.\end{aligned}$$

Since M is vector bundle,

$$\underline{\operatorname{Hom}}_R(M,M)_p \cong \underline{\operatorname{Hom}}_{R_p}(M_p,M_p)=0\ (\forall p\neq\mathfrak{m}).$$

Thus we have $\underline{\operatorname{Hom}}_R(M,M)$ has finite length and we have

$$\begin{aligned}\underline{\operatorname{Hom}}_R(M,M) &\cong \operatorname{Ext}_R^d(\operatorname{Ext}_R^d(\underline{\operatorname{Hom}}_R(M,M),R),R)\\ &\cong \operatorname{Ext}_R^d(\operatorname{Ext}_R^{d+1}(M,(\operatorname{tr}M)^\vee),R)=0\end{aligned}$$

Thus we get M is free. □

Proof of Theorem 1. We put $\mathfrak{P} := \{\ p\in \operatorname{Spec}R \mid R_p$ does not satisfy (ARC) $\}$ and assume $\mathfrak{P}\neq\phi$. Let q be a minimal element in $\mathfrak{P}$ and replace R with R_q. By the minimalty, R is a $d(\geq 2)$-dimrnsional Gorenstein local ring which does not satisfy (ARC) but R_p satisfy (ARC) for all prime $p\neq\mathfrak{m}$. There exists $M\in \operatorname{mod}R$ s.t. $\operatorname{Ext}_R^i(M,M\oplus R)=0$ $(\forall i>0)$ but M is not free. Since $\operatorname{Ext}_R^i(M,R)=0$ $(i>0)$, M is maximal Cohen-Macaulay. For any $p\neq\mathfrak{m}$, $\operatorname{Ext}_{R_p}^i(M_p,M_p\oplus R_p)=0$ $(\forall i>0)$ and R_p satisfies (ARC), we have M_p is a free R_p-module. Thus we get M is vector bundle. Furthermore, $\operatorname{Ext}_R^{d-1}(M,M)=0$ implies M is free. ($\because$Theorem 2.) Therefore we get contradiction and we have $\mathfrak{P}=\phi$. □

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