HOCHSCHILD COHOMOLOGY AND STRATIFYING IDEALS

HIROSHI NAGASE

Abstract. Suppose B is an algebra with a stratifying ideal BeB generated by an idempotent e. We will establish long exact sequences relating the Hochschild cohomology groups of the three algebras B, B/BeB and eBe. This provides a common generalization of various known results, all of which are extending Happel's long exact sequence for one-point extensions. Applying one of these sequences to Hochschild cohomology algebras modulo nilpotent shows, in some cases, that these algebras are finitely generated.

1. Introduction

Hochschild cohomology is not functorial. Thus there is no natural way to relate Hochschild cohomology of an algebra to that of its quotient or subalgebras. Still it is natural to try to find a way relating cohomology of an algebra B to that of an 'easier' or 'smaller' algebra A, such as a quotient modulo an idempotent ideal or a centralizer subalgebra. One such situation is that of B being a one-point extension of A, which has been studied by Happel in [7]. More recently, Happel's long exact sequence has been generalized to the case of triangular matrix algebras, for example by Michelena and Platzeck in [10], Green and Solberg in [6] and Cibils, Marcos, Redondo and Solotar in [2]. On the other hand, in [11], de la Peña and Xi have generalized Happel's long exact sequence to the case of algebras with heredity ideals.

Here, these results will be extended further. A natural common generalization of both triangular algebras and algebras with heredity ideals are algebras with stratifying ideals; indeed, heredity ideals are stratifying and any triangular matrix algebra has stratifying ideals such that the quotients are the respective triangular parts. A stratifying ideal of a finite dimensional algebra B is generated by an idempotent e in B. By one of our long exact sequences in Theorem 6

$$\cdots \to \mathrm{Ext}^n_{B^e}(B/BeB, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(B/BeB) \oplus \mathrm{HH}^n(eBe) \to \cdots,$$

we can compare Hochschild cohomology groups of three algebras B, B/BeB and eBe. We will get this long exact sequence, and another two long exact sequences (Theorem 6), by using elementary homological methods based on a key observation in triangulated categories, see Lemma 1.

For any finite dimensional algebra B, we will apply our long exact sequence to the quotient of the Hochschild cohomology algebra $\mathrm{HH}^*(B)$ modulo the ideal $\mathcal{N}_B$ generated by homogeneous nilpotent elements. We denote by $\overline{\mathrm{HH}}^*(B)$ the graded factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

In [13], Snashall and Solberg conjectured that $\overline{\mathrm{HH}}^*(A)$ is a finitely generated algebra for any finite dimensional algebra A. Green, Snashall and Solberg have shown the conjecture

The detailed version of this paper will be submitted for publication elsewhere.

to hold true for self-injective algebras of finite representation type [4] and for monomial algebras [5].

In Corollary 7, for any algebra B with a stratifying ideal BeB, we get an injective graded algebra homomorphism

$$\overline{\mathrm{HH}}^*(B) \hookrightarrow \overline{\mathrm{HH}}^*(B/BeB) \times \overline{\mathrm{HH}}^*(eBe)$$

Applying this embedding, we verify the conjecture for the case of Brauer algebra $B_k(3,\delta)$ in Example 13. Moreover, we consider a condition when the above embedding induces an isomorphism $\overline{\mathrm{HH}}^*(B) \cong \overline{\mathrm{HH}}^*(eBe)$ of graded algebras. By using this isomorphism, we can produce many examples of finite dimensional algebras for which the conjecture holds, including an algebra which is neither self-injective nor monomial (Example 12).

2. A general lemma

Throughout this paper we assume that k is a commutative noetherian ring and algebras are associative unital k-algebras that are projective as k-modules. For any algebra A, $\operatorname{mod} A$ denotes the category of finitely generated left A-modules, A^e the enveloping algebra $A \quad {}_k A^{\mathrm{op}}$, $\mathrm{HH}^n(A)$ the n-th Hochschild cohomology group of A with coefficients in A itself and $\mathrm{HH}^*(A)$ the Hochschild cohomology algebra $\oplus_n \mathrm{HH}^n(A)$. It is known that $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A,A)$ as groups and $\mathrm{HH}^*(A) \cong \oplus_n \mathrm{Ext}^n_{A^e}(A,A)$ as graded algebras.

For simplicity we will use the language of triangulated categories only in Lemma 1 below. Let $\mathcal{T}$ be a triangulated category with a shift functor T. For any X and Y in $\mathcal{T}$ and any $n \in \mathbb{Z}$, we denote by $\mathcal{T}^n(X,Y)$ the morphism group $\mathcal{T}(X, T^nY)$ and by $\mathcal{T}^*(X,X)$ the graded ring $\oplus_{n\in\mathbb{Z}}\mathcal{T}^n(X,X)$. If $\mathcal{T}$ is a derived category $D(\operatorname{Mod} A^e)$ for some algebra A, then $\mathcal{T}^n(A,A) \cong \mathrm{Ext}^n_{A^e}(A,A) \cong \mathrm{HH}^n(A)$ as groups and $T^*(A,A) \cong \mathrm{HH}^*(A)$ as graded algebras. The following is the key lemma of this paper.

Lemma 1. *Let $\mathcal{T}$ be a triangulated category. Suppose there is a triangle $X \to Y \to Z \to$ in $\mathcal{T}$ such that $\mathcal{T}^n(X,Z) = 0$ for all $n \in \mathbb{Z}$.*

(1) *We have the following three long exact sequences:*
$\cdots \to \mathcal{T}^n(Y,X) \to \mathcal{T}^n(Y,Y) \to \mathcal{T}^n(Z,Z) \to \mathcal{T}^{n+1}(Y,X) \to \cdots$;
$\cdots \to \mathcal{T}^n(Z,Y) \to \mathcal{T}^n(Y,Y) \to \mathcal{T}^n(X,X) \to \mathcal{T}^{n+1}(Z,Y) \to \cdots$; *and*
$\cdots \to \mathcal{T}^n(Z,X) \xrightarrow{u_n} \mathcal{T}^n(Y,Y) \to \mathcal{T}^n(Z,Z) \oplus \mathcal{T}^n(X,X) \to \mathcal{T}^{n+1}(Z,X) \to \cdots$.

(2) *Let $u : \mathcal{T}^*(Y,Y) \to \mathcal{T}^*(Z,Z) \times \mathcal{T}^*(X,X)$ be the graded ring homomorphism induced from the third long exact sequence. Then $(\operatorname{Ker} u)^2$ vanishes.*

The following facts are well-known (see [1]).

Lemma 2. *Let X be an A-B-bimodule, Y a B-C-bimodule and Z an A-C-bimodule. Then there are the following isomorphisms:*

(1) *If $\mathrm{Tor}^B_i(X,Y) = 0$ and $\mathrm{Ext}^i_C(Y,Z) = 0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \quad {}_B Y, Z) \cong \mathrm{Ext}^n_{A-B}(X, \mathrm{Hom}_C(Y,Z)).$$

(2) *If $\mathrm{Tor}^B_i(X,Y) = 0$ and $\mathrm{Ext}^i_A(X,Z) = 0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \quad {}_B Y, Z) \cong \mathrm{Ext}^n_{B-C}(Y, \mathrm{Hom}_A(X,Z)).$$

3. Stratifying ideals

In this section we study Hochschild cohomology groups of algebras with stratifying ideals. The following definition is due to Cline, Parshall and Scott ([3], 2.1.1 and 2.1.2), who work with finite dimensional algebras over fields. We keep our general setup of algebras projective over a commutative noetherian ring.

Definition 3. Let B be an algebra and $e = e^2$ an idempotent. The two-sided ideal BeB generated by e is called a *stratifying ideal* if the following equivalent conditions (A) and (B) are satisfied:
(A) (a) The multiplication map $Be \quad {}_{eBe}\, eB \to BeB$ is an isomorphism.
(b) For all $n > 0$: $\mathrm{Tor}_n^{eBe}(Be, eB) = 0$.
(B) The epimorphism $B \to A := B/BeB$ induces isomorphisms

$$\mathrm{Ext}_A^*(X, Y) \simeq \mathrm{Ext}_B^*(X, Y)$$

for all A-modules X and Y

The following remark can be used to check if an ideal is stratifying.

Remark 4. Let e be an idempotent element in B. Then BeB is projective as a left (resp. right) B-module if and only if eB (resp. Be) is projective as a left (respectively right) eBe-module and the multiplication map $Be \quad {}_{eBe}\, eB \to BeB$ is an isomorphism.

Proof. Suppose that BeB is a projective left B-module. Then $Be \quad {}_k\, eB \twoheadrightarrow BeB$ splits in $\mathrm{mod}\, B$. Multiplying by e on the left hand side, $eBe \quad {}_k\, eB \twoheadrightarrow eB$ splits in $\mathrm{mod}\, eBe$. Thus eB is a projective left eBe-module. Let X be the kernel of the multiplication map $Be \quad {}_{eBe}\, eB \twoheadrightarrow BeB$. Multiplying by e on the left hand side, eX is the kernel of the multiplication map $eBe \quad {}_{eBe}\, eB \twoheadrightarrow eB$. But the latter multiplication map is an isomorphism, and therefore $eX = 0$. Applying the functor $\mathrm{Hom}_B(-, X)$ to the short exact sequence $0 \to X \to Be \quad {}_{eBe}\, eB \to BeB \to 0$, yields a short exact sequence

$$0 \to \mathrm{Hom}_B(BeB, X) \to \mathrm{Hom}_B(Be \quad {}_{eBe}\, eB, X) \to \mathrm{Hom}_B(X, X) \to 0,$$

because BeB is a projective left B-module. Since the middle term $\mathrm{Hom}_B(Be \quad {}_{eBe}\, eB, X) \cong \mathrm{Hom}_{eBe}(eB, eX) = 0$, we get $End_B(X) = 0$ and thus $X = 0$, so that the multiplication map $Be \quad {}_{eBe}\, eB \twoheadrightarrow BeB$ is an isomorphism.

The converse is shown by using the isomorphism

$$\mathrm{Hom}_B(BeB, -) \cong \mathrm{Hom}_B(Be \quad {}_{eBe}\, eB, -) \cong \mathrm{Hom}_{eBe}(eB, \mathrm{Hom}_B(Be, -)).$$

□

Heredity ideals are examples of stratifying ideals, thus our results will extend results obtained in [11]. On the other hand, for any triangulated algebra

$$B = \begin{pmatrix} A & 0 \\ M & C \end{pmatrix}$$

and the idempotent

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have $BeB = eB$, so that BeB is a stratifying ideal. Thus our results also will extend results of [2, 6, 10]. There are, however, plenty of other examples. Stratifying ideals and stratified algebras occur frequently in applications, for example in algebraic Lie theory in the context of Schur algebras and of blocks of the Bernstein-Gelfand-Gelfand category of a semisimple complex Lie algebra.

From now on, we assume that BeB is a stratifying ideal of B and we put $A := B/BeB$.

Proposition 5. *For any $i \geq 0$, the following hold:*

(1) $\operatorname{Ext}^i_{B^e}(BeB, A) = 0$.
(2) $\operatorname{Ext}^i_{B^e}(BeB, BeB) \cong \operatorname{Ext}^i_{(eBe)^e}(eBe, eBe)$.
(3) $\operatorname{Ext}^i_{A^e}(A, A) \cong \operatorname{Ext}^i_{B^e}(A, A)$.
(4) *The isomorphisms in (2) and (3) preserve Yoneda products.*

Theorem 6. *There are long exact sequences as follows:*

(1) $\cdots \to \operatorname{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
(2) $\cdots \to \operatorname{Ext}^n_{B^e}(A, B) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(eBe) \to \cdots$; *and*
(3) $\cdots \to \operatorname{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \xrightarrow{f} \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

Proof. By Lemma 1 and Proposition 5 . ☐

We remark that by using the partial recollement of bounded below derived categories

$$D^+(\operatorname{mod} B/BeB) \rightleftarrows D^+(\operatorname{mod} B) \rightleftarrows D^+(\operatorname{mod} eBe),$$

we also can get the long exact sequence (3).

We also note that Suarez-Alvarez [12] independently has obtained the first long exact sequence in Theorem 6 above by using different methods based on spectral sequences.

Recall the notation that $\mathcal{N}_B$ is the ideal of $\mathrm{HH}^*(B)$ which is generated by homogeneous nilpotent elements, and $\overline{\mathrm{HH}}^*(B)$ is the factor algebra $\mathrm{HH}^*(B)/\mathcal{N}_B$.

Corollary 7.

(1) *Let $f : \mathrm{HH}^*(B) \to \mathrm{HH}^*(A) \times \mathrm{HH}^*(eBe)$ be the graded algebra homomorphism in sequence (3) above. Then $(\operatorname{Ker} f)^2$ vanishes.*
(2) *The induced homomorphism $\overline{f} : \overline{\mathrm{HH}}^*(B) \to \overline{\mathrm{HH}}^*(A) \times \overline{\mathrm{HH}}^*(eBe)$ is injective.*

Proof. By Lemma 1 and statement (4) of Proposition 5.

☐

4. Examples

By adapting the well-known recursive constructions of quasi-hereditary algebras, we construct for any algebra C a new algebra B which is an extension of C and has a stratifying ideal. We will compare the Hochschild cohomology algebras of C and of B. For simplicity we will assume all algebras to be finite dimensional and split over a field k.

Let A and C be algebras, M a C-A-bimodule and N an A-C-bimodule. For any morphism $\mu : M \quad_A N \to \operatorname{rad} C$ of C-C-bimodules, we can define a split extension $\tilde{A}$ of A by $N \quad_C M$ (where $N \quad_C M$ multiplies trivially with itself) so that we get an algebra (with multiplication induced by μ)

$$B = \begin{pmatrix} \tilde{A} & N \\ M & C \end{pmatrix}.$$

For the idempotent

$$e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we observe that $A \cong B/BeB$, that $C \cong eBe$ and that the multiplication map $Be \quad_{eBe} eB \to BeB$ is an isomorphism. We keep the notation above in this section.

Lemma 8. *If* $\mathrm{Tor}^n_C(N, M) = 0$ *for any* $n \geq 1$*, then* BeB *is a stratifying ideal.*

Lemma 9. *Let* A *be the ground field* k*. If* ${}_CM$ *and* N_C *are projective* C*-modules, then* $\mathrm{pd}_{B^e} A \leq 2$.

Lemma 10. *Let* D *be a finite dimensional algebra, split over the field* k*.*

(1) *Let* n *be the number of blocks of* D*. Then* $\overline{\mathrm{HH}}^0(D) \cong k^n$ *as an algebra.*
(2) *If* $\mathrm{char} k \neq 2$*, then* $\overline{\mathrm{HH}}^*(D) \cong \overline{\mathrm{HH}}^{even}(D) := \oplus_{n \geq 0} \overline{\mathrm{HH}}^{2n}(D)$.

Proposition 11. *Let* A *be the ground field* k*. If* ${}_CM$ *and* N_C *are non-zero projective* C*-modules, the number of blocks of* C *is the same as that of* B*. If* $\mathrm{char} k \neq 2$*, then* $\overline{\mathrm{HH}}^*(B) \cong \overline{\mathrm{HH}}^*(C)$ *as graded algebras.*

Proof. By Lemma 10, it is enough to show that $\overline{\mathrm{HH}}^{even}(B) \cong \overline{\mathrm{HH}}^{even}(C)$. By Lemma 8, BeB is stratifying. By Lemma 9, Theorem 6 and $\mathrm{HH}^*(A) \cong k$, we have that $\mathrm{HH}^n(B) \cong \mathrm{HH}^n(C)$ for any $n \geq 3$ and $\mathrm{HH}^2(B) \to \mathrm{HH}^2(C)$ is surjective. Hence, by Corollary 7, $\overline{\mathrm{HH}}^n(B) \cong \overline{\mathrm{HH}}^n(C)$ for any $n \geq 2$. By Lemma 10, $\dim_k \overline{\mathrm{HH}}^0(B)$= the number of blocks of B equals the number of blocks of C=$\dim_k \overline{\mathrm{HH}}^0(C)$. Therefore $\overline{\mathrm{HH}}^{even}(B) \cong \overline{\mathrm{HH}}^{even}(C)$ as graded algebras. $\square$

The following example shows that we cannot drop the condition $\mathrm{char} k \neq 2$ in Proposion 11 above.

Example 12. Keep the notation in the previous section. Let A be the ground field k, C a truncated polynomial algebra $k[x]/(x^p)$. If $M = C$, $N = C$, $\mu : M \quad_A N \to \mathrm{rad}\, C$ is defined by $\mu(1 \quad 1) = x^q$ and $1 \leq q < p$, then B is given by the following quiver

$$1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \circlearrowright c$$

with two relations $c^p = 0$ and $ab = c^q$. Note that B is neither self-injective nor monomial unless $q = 1$. By Proposition 11, if $\mathrm{char} k \neq 2$, then $\overline{\mathrm{HH}}^*(B) \cong \overline{\mathrm{HH}}^*(C)$. Since $\overline{\mathrm{HH}}^*(C)$ is a finitely generated algebra (see [4]), so is $\overline{\mathrm{HH}}^*(B)$.

On the other hand, if $\mathrm{char} k = 2$, $q = 1$ and $p = 2$, then $\overline{\mathrm{HH}}^*(B) \cong k[x, z]/(x^3 - z^2)$ with deg x=2 and deg z=3 by [13] and $\overline{\mathrm{HH}}^*(C) \cong k[x]$ with deg x=1 by [8] or [4]. Hence $\overline{\mathrm{HH}}^*(B)$ is strictly contained in $\overline{\mathrm{HH}}^*(C)$, so that we cannot drop the condition $\mathrm{char} k \neq 2$ in Proposition 11.

Finally we give an example of an algebra occuring in algebraic Lie theory, see for instance [9] for the properties of Brauer algebras used in this example.

Example 13. Let B be a Brauer algebra $B_k(3, \delta)$, where δ is in k. B has a stratifying ideal BeB such that $eBe \cong k$ and $B/BeB \cong k\Sigma_3$, where Σ_3 is the symmetric group on three letters. By Corollary 7, there exists an embedding

$$\overline{\mathrm{HH}}^*(B) \hookrightarrow \overline{\mathrm{HH}}^*(k\Sigma_3) \times \overline{\mathrm{HH}}^*(k)$$

as a graded algebra homomorphism. Since $k\Sigma_3$ is a self-injective algebra of finite representation type, $\overline{\mathrm{HH}}^*(k\Sigma_3)$ is isomorphic to a product of some polynomial algebras in one variable $k[x]$ and some copies of the ground field k (see [4]). Because any graded subalgebra of a product of some polynomial algebras with one variable $k[x]$ is a finitely generated algebra, we get that $\overline{\mathrm{HH}}^*(B_k(3, \delta))$ is a finitely generated algebra.

References

1. H. Cartan and S. Eilenberg: *Homological algebra*, Princeton Landmarks in Mathematics (1973). Originally pubulished 1956.
2. C.Cibils, E.Marcos, M.J.Redondo and A.Solotar: *Cohomology of split algebras and of trivial extensions*, Glasgow Math. J. **45** (2003), no. 1, 21-40.
3. E.Cline, B.Parshall and L.Scott: *Stratifying endomorphism algebras*, Memoir AMS **591** (1996), i-vi, 1-119.
4. E.L. Green, N. Snashall and Ø. Solberg: *The Hochschild cohomology ring of a selfinjective algebra of finite representation type*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3387-3393.
5. E.L. Green, N. Snashall and Ø. Solberg: *The Hochschild cohomology ring modulo nilpotence of a monomial algebra*, J. Algebra Appl. **5** (2006), no. 2, 153–192.
6. E.L.Green and Ø.Solberg: *Hochschild cohomology rings and triangular rings*, Proceedings of the Ninth International Conference, Beijing, 2000, eds. D. Happel and Y. B. Zhang, Beijing Normal University Press, vol. II, (2002), 192-200.
7. D. Happel: *Hochschild cohomology of finite-dimensional algebras*, Lecture Notes in Math., **1404**, Springer, Berlin (1989), 108-126
8. T. Holm: *Hochschild cohomology rings of algebras $k[X]/(f)$*, Beiträge Algebra Geom. **41** (2000), no. 1, 291-301.
9. S. Koenig and C.C.Xi: *A characteristic free approach to Brauer algebras*, Trans. Amer. Math. Soc. **353** (2001), no. 4, 1489–1505.
10. S. Michelena and M.I. Platzeck: *Hochschild cohomology of triangular matrix algebras*, J. Algebra, **233** (2000), 502-525.
11. J. A. de la Peña and C. C. Xi: *Hochschild cohomology of algebras with homological ideals*, Tsukuba J. Math. **30** (2006), no. 1, 61-79.
12. M. Suarez-Alvarez: *Applications of the change-of-rings spectral sequence to the computation of Hochschild cohomology*, arXiv:0707.3210.
13. N. Snashall and Ø. Solberg: *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705-732.

Nara National College of Technology
22 Yata-cho, Yamatokoriyama,
Nara, 639-1080, JAPAN
E-mail address: nagase@libe.nara-k.ac.jp