ON CONTRAVARIANTLY FINITE SUBCATEGORIES OF FINITELY GENERATED MODULES

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ABSTRACT. This paper studies contravariantly finite resolving subcategories of the category of finitely generated modules over a commutative ring. The main theorem of this paper implies that there exist only three contravariantly finite resolving subcategories over a henselian Gorenstein local ring. It also implies the theorem of Christensen, Piepmeyer, Striuli and Takahashi.

Key Words: contravariantly finite subcategory, resolving subcategory, Gorenstein ring, Cohen-Macaulay ring, maximal Cohen-Macaulay module, totally reflexive module.

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INTRODUCTION

The notion of a contravariantly finite subcategory (of the category of finitely generated modules) was first introduced over artin algebras by Auslander and Smalø [6] in connection with studying the problem of which subcategories admit almost split sequences. The notion of a resolving subcategory was introduced by Auslander and Bridger [3] in the study of modules of Gorenstein dimension zero, which are now also called totally reflexive modules. There is an application of contravariantly finite resolving subcategories to the study of the finitistic dimension conjecture [5].

This paper deals with contravariantly finite resolving subcategories over commutative rings. Let R be a commutative noetherian henselian local ring. We denote by $\operatorname{mod} R$ the category of finitely generated R-modules, by $\mathcal{F}(R)$ the full subcategory of free R-modules, and by $\mathcal{C}(R)$ the full subcategory of maximal Cohen-Macaulay R-modules. The subcategory $\mathcal{F}(R)$ is always contravariantly finite, and so is $\mathcal{C}(R)$ provided that R is Cohen-Macaulay. The latter fact is known as the Cohen-Macaulay approximation theorem, which was shown by Auslander and Buchweitz [4].

In this paper, we shall prove the following amazing theorem; the category of finitely generated modules over a henselian Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Theorem A. *If R is Gorenstein, then all the contravariantly finite resolving subcategories of $\operatorname{mod} R$ are $\mathcal{F}(R)$, $\mathcal{C}(R)$ and $\operatorname{mod} R$.*

This theorem especially says that if R is a commutative selfinjective local ring, then there are no contravariantly finite resolving subcategories other than $\mathcal{F}(R)$ and $\operatorname{mod} R$.

The main theorem of this paper asserts the following: let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that the residue field of R has a right $\mathcal{X}$-approximation. Assume that

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there exists an R-module $G \in \mathcal{X}$ of infinite projective dimension with $\mathrm{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Let M be an R-module such that each $X \in \mathcal{X}$ satisfies $\mathrm{Ext}_R^i(X, M) = 0$ for $i \gg 0$. Then M has finite injective dimension. From this result, we will prove the following two theorems. Theorem A will be obtained from Theorem B. The assertion of Theorem C is a main result of [10], which has been a motivation for this paper. (Our way of obtaining Theorem C is quite different from the original proof given in [10].)

Theorem B. *Let $\mathcal{X} \neq \mathrm{mod}\, R$ be a contravariantly finite resolving subcategory of $\mathrm{mod}\, R$. Suppose that there is an R-module $G \in \mathcal{X}$ of infinite projective dimension such that $\mathrm{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Then R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.*

Theorem C (Christensen-Piepmeyer-Striuli-Takahashi)**.** *Suppose that there is a nonfree R-module in $\mathcal{G}(R)$. If $\mathcal{G}(R)$ is contravariantly finite in $\mathrm{mod}\, R$, then R is Gorenstein.*

Here, $\mathcal{G}(R)$ denotes the full subcategory of totally reflexive R-modules. A totally reflexive module, which is also called a module of Gorenstein dimension (G-dimension) zero, was defined by Auslander [2] as a common generalization of a free module and a maximal Cohen-Macaulay module over a Gorenstein local ring. Auslander and Bridger [3] proved that the full subcategory of totally reflexive modules over a left and right noetherian ring is resolving. The other details of totally reflexive modules are stated in [3] and [9].

If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$, and so $\mathcal{G}(R)$ is contravariantly finite by virtue of the Cohen-Macaulay approximation theorem. Thus, Theorem C can be viewed as the converse of this fact. Theorem C implies the following: let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R-module and are only finitely many nonisomorphic indecomposable totally reflexive R-modules. Then R is an isolated simple hypersurface singularity. For the details, see [10].

Conventions

In the rest of this paper, we assume that all rings are commutative and noetherian, and that all modules are finitely generated. Unless otherwise specified, let R be a henselian local ring. The unique maximal ideal of R and the residue field of R are denoted by $\mathfrak{m}$ and k, respectively. We denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules. By a *subcategory* of $\mathrm{mod}\, R$, we always mean a full subcategory of $\mathrm{mod}\, R$ which is closed under isomorphisms. Namely, in this paper, a subcategory $\mathcal{X}$ of $\mathrm{mod}\, R$ means a full subcategory such that every R-module which is isomorphic to some R-module in $\mathcal{X}$ is also in $\mathcal{X}$.

1. Contravariant finiteness of totally reflexive modules

In this section, we will state background materials which motivate the main results of this paper. We start by recalling the definition of a totally reflexive module.

Definition 1. We denote by $(-)^*$ the R-dual functor $\mathrm{Hom}_R(-, R)$. An R-module M is called *totally reflexive* (or *of Gorenstein dimension zero*) if

(1) the natural homomorphism $M \to M^{**}$ is an isomorphism, and
(2) $\mathrm{Ext}_R^i(M, R) = \mathrm{Ext}_R^i(M^*, R) = 0$ for any $i > 0$.

We introduce three subcategories of $\operatorname{mod} R$ which will often appear throughout this paper.

We denote by $\mathcal{F}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all free R-modules, by $\mathcal{G}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all totally reflexive R-modules, and by $\mathcal{C}(R)$ the subcategory of $\operatorname{mod} R$ consisting of all maximal Cohen-Macaulay R-modules. By definition, $\mathcal{F}(R)$ is contained in $\mathcal{G}(R)$. If R is Cohen-Macaulay, then $\mathcal{G}(R)$ is contained in $\mathcal{C}(R)$. If R is Gorenstein, then $\mathcal{G}(R)$ coincides with $\mathcal{C}(R)$.

Next, we recall the notion of a right approximation over a subcategory of $\operatorname{mod} R$.

Definition 2. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$.

(1) Let $\phi : X \to M$ be a homomorphism of R-modules with $X \in \mathcal{X}$. We say that ϕ is a *right $\mathcal{X}$-approximation* (of M) if the induced homomorphism $\operatorname{Hom}_R(X', \phi) : \operatorname{Hom}_R(X', X) \to \operatorname{Hom}_R(X', M)$ is surjective for any $X' \in \mathcal{X}$.

(2) We say that $\mathcal{X}$ is *contravariantly finite* (in $\operatorname{mod} R$) if every R-module has a right $\mathcal{X}$-approximation.

The following result is well-known.

Theorem 3 (Auslander-Buchweitz)**.** *Let R be a Cohen-Macaulay local ring. Then $\mathcal{C}(R)$ is contravariantly finite.*

Corollary 4. *If R is Gorenstein, then $\mathcal{G}(R)$ is contravariantly finite.*

The converse of this corollary essentially holds:

Theorem 5. [10] *Suppose that there is a nonfree totally reflexive R-module. If $\mathcal{G}(R)$ is contravariantly finite in* $\operatorname{mod} R$*, then R is Gorenstein.*

This theorem yields the following corollary, which is a generalization of [12, Theorem 1.3].

Corollary 6. *Let R be a non-Gorenstein local ring. If there is a nonfree totally reflexive R-module, then there are infinitely many nonisomorphic indecomposable totally reflexive R-modules.*

Combining this with [13, Theorems (8.15) and (8.10)] (cf. [11, Satz 1.2] and [8, Theorem B]), we obtain the following result.

Corollary 7. *Let R be a homomorphic image of a regular local ring. Suppose that there is a nonfree totally reflexive R-module but there are only finitely many nonisomorphic indecomposable totally reflexive R-modules. Then R is a simple hypersurface singularity.*

2. Contravariantly finite resolving subcategories

In this section, we will give the main theorem of this paper and several results it yields. One of them implies Theorem 5, which is the motive fact of this paper.

First of all, we recall the definition of the syzygies of a given module. Let M be an R-module and n a positive integer. Let

$$F_\bullet = (\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$$

be a minimal free resolution of M. We define the *nth syzygy* - ${}^nM$ of M as the image of the homomorphism d_n. We set - ${}^0M = M$.

We recall the definition of a resolving subcategory.

Definition 8. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if it satisfies the following four conditions.

(1) $\mathcal{X}$ contains R.
(2) $\mathcal{X}$ is closed under direct summands: if M is an R-module in $\mathcal{X}$ and N is a direct summand of M, then N is also in $\mathcal{X}$.
(3) $\mathcal{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N are in $\mathcal{X}$, then M is also in $\mathcal{X}$.
(4) $\mathcal{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M and N are in $\mathcal{X}$, then L is also in $\mathcal{X}$.

Now we state the main theorem in this paper.

Theorem 9. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *such that the residue field k has a right $\mathcal{X}$-approximation. Assume that there exists an R-module $G \in \mathcal{X}$ of infinite projective dimension such that $\operatorname{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Let M be an R-module such that each $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, M) = 0$ for $i \gg 0$. Then M has finite injective dimension.*

We shall prove Theorem 9 in the next section. In the rest of this section, we will state and prove several results by using Theorem 9. We begin with two corollaries which are immediately obtained.

Corollary 10. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *which is contained in the subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0$ for $i \gg 0 \}$ of* $\operatorname{mod} R$*. Suppose that in $\mathcal{X}$ there is an R-module of infinite projective dimension. If k has a right $\mathcal{X}$-approximation, then R is Gorenstein.*

Proof. Each module X in $\mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Hence Theorem 9 implies that R has finite injective dimension as an R-module. □

Corollary 11. *Let $\mathcal{X}$ be one of the following.*

(1) $\mathcal{G}(R)$.
(2) *The subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0$ for $i > n \}$ of* $\operatorname{mod} R$*, where n is a non-negative integer.*
(3) *The subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0$ for $i \gg 0 \}$ of* $\operatorname{mod} R$.

Suppose that in $\mathcal{X}$ there is an R-module of infinite projective dimension. If k has a right $\mathcal{X}$-approximation, then R is Gorenstein.

Proof. The subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is resolving. Since $\mathcal{X}$ is contained in the subcategory $\{ M \mid \operatorname{Ext}_R^i(M, R) = 0 \text{ for } i \gg 0 \}$, the assertion follows from Corollary 10. □

Remark 12. Corollary 11 implies Theorem 5. Indeed, any nonfree totally reflexive module has infinite projective dimension by [9, (1.2.10)].

For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, let $\mathcal{X}^{\perp}$ (respectively, ${}^{\perp}\mathcal{X}$) denote the subcategory of $\operatorname{mod} R$ consisting of all R-modules M such that $\operatorname{Ext}_R^i(X, M) = 0$ (respectively,

$\mathrm{Ext}_R^i(M,X) = 0)$ for all $X \in \mathcal{X}$ and $i > 0$. Applying Wakamatsu's lemma to a resolving subcategory, we obtain the following lemma.

Lemma 13. *Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. If an R-module M has a right $\mathcal{X}$-approximation, then there is an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$.*

By using this lemma and the theorem which was formerly called "Bass' conjecture", we obtain another corollary of Theorem 9.

Corollary 14. *Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ such that k has a right $\mathcal{X}$-approximation and that k is not in $\mathcal{X}$. Assume that there is an R-module $G \in \mathcal{X}$ with $\mathrm{pd}_R\, G = \infty$ and $\mathrm{Ext}_R^i(G,R) = 0$ for $i \gg 0$. Then R is Cohen-Macaulay and $\dim R > 0$.*

Before giving the next corollary of Theorem 9, we establish an easy lemma without proof.

Lemma 15. (1) *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\operatorname{mod} R$. Then, $k \in \mathcal{X}$ if and only if $\mathcal{X} = \operatorname{mod} R$.*

(2) *Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. Suppose that every R-module in ${}^\perp(\mathcal{X}^\perp)$ admits a right $\mathcal{X}$-approximation. Then $\mathcal{X} = {}^\perp(\mathcal{X}^\perp)$.*

(3) *Let M and N be nonzero R-modules. Assume either that M has finite projective dimension or that N has finite injective dimension. Then one has an equality*

$$\sup\{\, i \mid \mathrm{Ext}_R^i(M,N) \neq 0 \,\} = \operatorname{depth} R - \operatorname{depth}_R M.$$

Now we can show the following corollary. There are only two contravariantly finite resolving subcategories possessing such G as in the corollary.

Corollary 16. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of $\operatorname{mod} R$. Assume that there is an R-module $G \in \mathcal{X}$ with $\mathrm{pd}_R\, G = \infty$ and $\mathrm{Ext}_R^i(G,R) = 0$ for $i \gg 0$. Then either of the following holds.*

(1) *$\mathcal{X} = \operatorname{mod} R$,*

(2) *R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.*

Proof. Suppose that $\mathcal{X} \neq \operatorname{mod} R$. Then k is not in $\mathcal{X}$. By Corollary 14, R is Cohen-Macaulay.

First, we show that $\mathcal{C}(R)$ is contained in $\mathcal{X}$. For this, let M be a maximal Cohen-Macaulay R-module. We have only to prove that M is in ${}^\perp(\mathcal{X}^\perp)$. Let N be a nonzero R-module in $\mathcal{X}^\perp$. Theorem 9 implies that N is of finite injective dimension. Since M is maximal Cohen-Macaulay, we have $\sup\{\, i \mid \mathrm{Ext}_R^i(M,N) \neq 0 \,\} = 0$. Therefore $\mathrm{Ext}_R^i(M,N) = 0$ for all $N \in \mathcal{X}^\perp$ and $i > 0$. It follows that M is in ${}^\perp(\mathcal{X}^\perp)$, as desired.

Next, we show that $\mathcal{X}$ is contained in $\mathcal{C}(R)$. We have an exact sequence $0 \to Y \to X \to k \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^\perp$ by Lemma 13. Since k is not in $\mathcal{X}$, the module Y is nonzero. By Theorem 9, Y has finite injective dimension. For a nonzero R-module X' in $\mathcal{X}$, we have equalities $0 \geq \sup\{\, i \mid \mathrm{Ext}_R^i(X',Y) \neq 0 \,\} = \operatorname{depth} R - \operatorname{depth}_R X' = \dim R - \operatorname{depth}_R X'$. Therefore X' is a maximal Cohen-Macaulay R-module, as desired. $\square$

Next, we study contravariantly finite resolving subcategories all of whose objects X satisfy $\mathrm{Ext}_R^{\gg 0}(X,R) = 0$. We start by considering special ones among such subcategories.

Proposition 17. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that every R-module in $\mathcal{X}$ has finite projective dimension. Then either of the following holds.*

(1) $\mathcal{X} = \mathcal{F}(R)$,

(2) *R is regular and* $\mathcal{X} = \operatorname{mod} R$.

Proof. If $\mathcal{X} = \operatorname{mod} R$, then our assumption says that all R-modules have finite projective dimension. Hence R is regular. Assume that $\mathcal{X} \neq \operatorname{mod} R$. Then there is an R-module M which is not in $\mathcal{X}$. There is an exact sequence $0 \to Y \to X \to M \to 0$ with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 13. Note that $Y \neq 0$ as $M \notin \mathcal{X}$. Fix a nonzero R-module $X' \in \mathcal{X}$. We have $\operatorname{Ext}_R^i(X', Y) = 0$ for all $i > 0$, and hence $\operatorname{pd}_R X' = \sup\{ i \mid \operatorname{Ext}_R^i(X', Y) \neq 0 \} = 0$ by the Auslander-Buchsbaum formula. Hence X' is free. This means that $\mathcal{X}$ is contained in $\mathcal{F}(R)$. On the other hand, $\mathcal{X}$ contains $\mathcal{F}(R)$ since $\mathcal{X}$ is resolving. Therefore $\mathcal{X} = \mathcal{F}(R)$. ☐

Combining Proposition 17 with Corollary 16, we can get the following.

Corollary 18. *Let $\mathcal{X}$ be a contravariantly finite resolving subcategory of* $\operatorname{mod} R$. *Suppose that every module $X \in \mathcal{X}$ is such that $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$. Then one of the following holds.*

(1) $\mathcal{X} = \mathcal{F}(R)$,

(2) *R is Gorenstein and* $\mathcal{X} = \mathcal{C}(R)$,

(3) *R is Gorenstein and* $\mathcal{X} = \operatorname{mod} R$.

Proof. The corollary follows from Proposition 17 in the case where all R-modules in $\mathcal{X}$ are of finite projective dimension. So suppose that in $\mathcal{X}$ there exists an R-module of infinite projective dimension. Then Corollary 16 shows that either of the following holds.

(1) $\mathcal{X} = \operatorname{mod} R$,

(2) R is Cohen-Macaulay and $\mathcal{X} = \mathcal{C}(R)$.

By the assumption that every $X \in \mathcal{X}$ satisfies $\operatorname{Ext}_R^i(X, R) = 0$ for $i \gg 0$, we have $\operatorname{Ext}_R^i(k, R) = 0$ for $i \gg 0$ in the case (i). In the case (ii), since $\text{-}\ {}^{d}k$ is in $\mathcal{X}$ where $d = \dim R$, we have $\operatorname{Ext}_R^{i+d}(k, R) \cong \operatorname{Ext}_R^i(\text{-}\ {}^{d}k, R) = 0$ for $i \gg 0$. Thus, in both cases, the ring R is Gorenstein. ☐

Finally, we obtain the following result from Corollary 18 and Theorem 3. It says that the category of finitely generated modules over a Gorenstein local ring possesses only three contravariantly finite resolving subcategories.

Corollary 19. *Let R be a Gorenstein local ring. Then all the contravariantly finite resolving subcategories of* $\operatorname{mod} R$ *are* $\mathcal{F}(R)$, $\mathcal{C}(R)$ *and* $\operatorname{mod} R$.

3. Proof of the main theorem

Let M be an R-module. Take a minimal free resolution $F_\bullet = (\cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$ of M. We define the *transpose* $\operatorname{Tr} M$ of M as the cokernel of the R-dual homomorphism $d_1^* : F_0^* \to F_1^*$ of d_1. The transpose $\operatorname{Tr} M$ has no nonzero free summand.

For an R-module M, let M^*M be the ideal of R generated by the subset

$$\{ f(x) \mid f \in M^*, x \in M \}$$

of R. Note that M has a nonzero free summand if and only if $M^*M = R$.

Proposition 20. *Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$ *and* $0 \to Y \xrightarrow{f} X \to M \to 0$ *an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. Let $G \in \mathcal{X}$, set $H = \mathrm{Tr}\text{-}\ G$, and suppose that $(H^*H)M = 0$. Let $0 \to K \xrightarrow{g} F \xrightarrow{h} H \to 0$ be an exact sequence of R-modules with F free. Then the induced sequence*

$$0 \longrightarrow K \quad_R Y \xrightarrow{g \quad_R Y} F \quad_R Y \xrightarrow{h \quad_R Y} H \quad_R Y \longrightarrow 0$$

is exact, and the map $h \quad_R Y$ factors through the map $F \quad_R f : F \quad_R Y \to F \quad_R X$.

Proof. We can show that there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H \quad_R Y & \xrightarrow{\delta} & H \quad_R X & \xrightarrow{\varepsilon} & H \quad_R M & \longrightarrow & 0 \\
 & & \alpha \big\downarrow \cong & & \beta \big\downarrow & & \gamma \big\downarrow 0 & & \\
0 & \longrightarrow & \operatorname{Hom}_R(H^*, Y) & \xrightarrow{\zeta} & \operatorname{Hom}_R(H^*, X) & \xrightarrow{\eta} & \operatorname{Hom}_R(H^*, M) & \longrightarrow & 0
\end{array}$$

with exact rows, and see that δ is a split monomorphism. Thus, the homomorphism $h \quad_R Y$ factors through the homomorphism $F \quad_R f$. We have isomorphisms $\operatorname{Tor}_1^R(H, Y) = \operatorname{Tor}_1^R(\mathrm{Tr}\text{-}\ G, Y) \cong \underline{\operatorname{Hom}}_R(\text{-}\ G, Y) = 0$, which completes the proof of the proposition. $\square$

Now we can prove the following, which will play a key role in the proof of Theorem 9.

Proposition 21. *Let $\mathcal{X}$ be a subcategory of* $\operatorname{mod} R$ *which is closed under syzygies. Let $0 \to Y \to X \to M \to 0$ be an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$. Suppose that there is an R-module $G \in \mathcal{X}$ with $\operatorname{pd}_R G = \infty$ and $\operatorname{Ext}_R^i(G, R) = 0$ for $i \gg 0$. Put $H_i = \mathrm{Tr}\text{-}\ (\text{-}\ ^iG)$ and assume that $((H_i)^*H_i)M = 0$ for $i \gg 0$. Let $D = (D^j)_{j \geq 0} : \operatorname{mod} R \to \operatorname{mod} R$ be a contravariant cohomological δ-functor. If $D^j(X) = 0$ for $j \gg 0$, then $D^j(Y) = D^j(M) = 0$ for $j \gg 0$.*

Proof. Replacing G with $\text{-}\ ^iG$ for $i \gg 0$, we may assume that $\operatorname{Ext}_R^i(G, R) = 0$ for all $i > 0$ and that $((H_i)^*H_i)M = 0$ for all $i \geq 0$. Let $F_\bullet = (\cdots \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1} \xrightarrow{d_{i-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0)$ be a minimal free resolution of G. Dualizing this by R, we easily see that $H_i \cong (\text{-}\ ^{i+3}G)^*$ and $\text{-}\ H_i \cong (\text{-}\ ^{i+2}G)^*$ for $i \geq 0$. By Proposition 20, for each integer $i \geq 0$ we have an exact sequence

$$0 \to (\text{-}\ ^{i+2}G)^* \quad_R Y \to (F_{i+2})^* \quad_R Y \xrightarrow{f_i} (\text{-}\ ^{i+3}G)^* \quad_R Y \to 0$$

such that f_i factors through $(F_{i+2})^* \quad_R X$. The homomorphism $D^j(f_i)$ factors through $D^j((F_{i+2})^* \quad_R X)$, which vanishes for $j \gg 0$. Hence $D^j(f_i) = 0$ for $j \gg 0$, and we obtain an exact sequence

$$0 \to D^j((F_{i+2})^* \quad_R Y) \to D^j((\text{-}\ ^{i+2}G)^* \quad_R Y) \xrightarrow{\varepsilon_{i,j}} D^{j+1}((\text{-}\ ^{i+3}G)^* \quad_R Y) \to 0$$

for $i \geq 0$ and $j \gg 0$. Thus, there is a sequence

$$D^j((\text{-}\ ^{i+2}G)^* \quad_R Y) \xrightarrow{\varepsilon_{i,j}} D^{j+1}((\text{-}\ ^{i+3}G)^* \quad_R Y) \xrightarrow{\varepsilon_{i+1,j+1}} D^{j+2}((\text{-}\ ^{i+4}G)^* \quad_R Y) \xrightarrow{\varepsilon_{i+2,j+2}} \cdots$$

of surjective homomorphisms of R-modules, and $\varepsilon_{i,j}$ is an isomorphism. It follows that $D^j((F_{i+2})^* \quad_R Y) = 0$ for $i \geq 0$ and $j \gg 0$. Thus we have $D^j(Y) = 0$ for $j \gg 0$, and $D^j(M) = 0$ for $j \gg 0$. $\square$

Now we can prove our main theorem.

Proof of Theorem 9. Since k admits a right $\mathcal{X}$-approximation, there exists an exact sequence $0 \to Y \to X \to k \to 0$ of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{X}^{\perp}$ by Lemma 13. For an integer $i \geq 0$, put $H_i = \mathrm{Tr}\text{-}\ (\text{-}\ ^{i}G)$. The module H_i has no nonzero free summand. We have $(H_i)^{*}H_i \neq R$. Hence $((H_i)^{*}H_i)k = 0$ for $i \geq 0$. Applying Proposition 21 to the contravariant cohomological δ-functor $D = (\mathrm{Ext}_R^j(\quad, M))_{j \geq 0}$, we obtain $D^j(k) = 0$ for $j \gg 0$. Namely, we have $\mathrm{Ext}_R^j(k, M) = 0$ for $j \gg 0$, which implies that M has finite injective dimension. $\square$

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