ON COLOCAL PAIRS

YOSHITOMO BABA

Abstract. In [9, Theorem 3.1] K. R. Fuller characterized indecomposable injective projective modules over artinian rings using i-pairs. In [3] the author generalized this theorem to indecomposable projective quasi-injective modules and indecomposable quasi-projective injective modules over artiniain rings. In [2] the author and K. Oshiro studied the above Fuller's theorem minutely. Further in [12] M. Hoshino and T. Sumioka extended these results to perfect rings. In this paper we studies the results in [3] from the point of view of [2], [12].

1. ON FULLER'S THEOREM AND PAST RESULTS

Throughout this paper, we let R be a semiperfect ring. By M_R (resp. ${}_RM$) we stress that M is a unitary right (resp. left) R-module. For an R-module M, we denote the injective hull, the Jacobson radical, the socle, the top $M/J(M)$, and the composition length of M by $E(M)$, $J(M)$, $S(M)$, $T(M)$, and $|M|$, respectively. Further we denote the right (resp. left) annihilator of T in S by $r_S(T)$ (resp. $l_S(T)$).

Definition 1. Let M, N be R-modules. We say that M is N*-injective* if, for any submodule X of N and any R-homomorphism $\varphi : X \to M$, there exists $\tilde{\varphi} : N \to M$ with $\tilde{\varphi}|_X = \varphi$. And we say that M is N*-simple-injective* if, for any submodule X of N and any R-homomorphism $\varphi : X \to M$ with $\mathrm{Im}\varphi$ simple, there exists $\tilde{\varphi} : N \to M$ with $\tilde{\varphi}|_X = \varphi$.

Definition 2. Let e, f be primitive idempotents in R and let g be an idempotent in R. We say that R satisfies $\alpha_r[e, g, f]$ if $r_{gRf}l_{eRg}(X) = X$ for any right fRf-submodule X of gRf with $r_{gRf}(eRg) \subseteq X$. And we say that R satisfies $\alpha_l[e, g, f]$ if $l_{eRg}r_{gRf}(Y) = Y$ for any left eRe-submodule Y of eRg with $l_{eRg}(gRf) \subseteq Y$. Further we say that (eR, Rf) is an *injective pair* (abbreviated *i-pair*) if $S(eR_R) \cong T(fR_R)$ and $S({}_RRf) \cong T({}_RRe)$.

The following theorem is given by K. R. Fuller in [9]. By this theorem, indecomposable projective injective right R-modules over right artinian rings are characterized using i-pairs.

Theorem 3. (**Fuller**) *Let R be a right artinian ring and let e, f be primitive idempotents in R. Then the following are equivalent.*

The detailed version of this paper will be submitted for publication elsewhere.

(a) eR_R *is injective with* $S(eR_R) \cong T(fR_R)$.
(b) (eR, Rf) *is an i-pair.*
(c) R *satisfies* $\alpha_r[e,1,f]$ *and* $\alpha_l[e,1,f]$.

In [2] Theorem 3 is minutely studied by the author and K. Oshiro over semiprimary rings as follows.

Theorem 4. (**Baba, Oshiro**) *Let* R *be a semiprimary ring and let* e, f *be primitive idempotents in* R.

(I) *The following are equivalent.*
  (a) eR_R *is injective.*
  (b) (i) *There exists a primitive idempotent* f *in* R *with* (eR, Rf) *an i-pair.*
    (ii) R *satisfies* $\alpha_r[e,1,f]$.

(II) *Suppose that* (eR, Rf) *is an i-pair.*
  (1) *If ACC holds on right annihilator ideals, then*
    (i) $\alpha_r[e,1,f]$ *holds,*
    (ii) *the equivalent conditions in the following* (2) *hold.*
  (2) *The following are equivalent.*
    (a) $|{}_{eRe}eR| < \infty$.
    (b) $|Rf_{fRf}| < \infty$.
    (c) *Both* eR_R *and* ${}_RRf$ *are injective.*

Theorem 4 is further considered over perfect rings by M. Hoshino and T. Sumioka in [12]. And the following theorem is given.

Theorem 5. (**Hoshino, Sumioka**) *Let* R *be a left perfect ring and let* e, f *be primitive idempotents in* R.

(I) *The following are equivalent.*
  (a) eR_R *is R-simple-injective.*
  (b) *There exists a primitive idempotent* f *in* R *such that*
    (i) (eR, Rf) *is an i-pair.*
    (ii) R *satisfies* $\alpha_r[e,1,f]$.

(II) *Suppose that* (eR, Rf) *is an i-pair. Then the following are equivalent.*
  (a) $|{}_{eRe}eR| < \infty$.
  (b) $|Rf_{fRf}| < \infty$.
  (c) *Both* eR_R *and* ${}_RRf$ *are injective.*

On the other hand, in [3] the author generalized Theorem 3 to indecomposable projective quasi-injective modules and indecomposable quasi-projective injective modules over artiniain rings as follows.

Theorem 6. (**Baba**) *Let* R *be a semiprimary ring and let* e, f *be primitive idempotents in* R. *Suppose that DCC holds on* $\{ r_{Rf}(I) \mid {}_{eRe}I \subseteq eR \}$. *Then the following are equivalent.*

(a) eR_R *is quasi-injective with* $S(eR_R) \cong T(fR_R)$.

(b) $E(T({}_R Re))$ *is quasi-projective of the form* ${}_R Rf/r_{Rf}(eR)$.
(c) $S(eR_R) \cong T(fR_R)$ *and* $S({}_{eRe}eRf)$ *is simple.*

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Definition 7. Let M be an R-module and let e, f be primitive idempotents in R. We say that M is *colocal* if $S(M)$ is simple and essential in M. And we say that (eR, Rf) is a *colocal pair* (abbreviated *c-pair*) if both eRf_{fRf} and ${}_{eRe}eRf$ are colocal.

First we characterize $\alpha_r[e, g, f]$ using a quasi-projective right R-module $eR/l_{eR}(Rf)_R$ in case that (eR, Rf) is a c-pair.

Proposition 8. *Let e, f be primitive idempotents in R and let g be an idempotent in R. Suppose that (eR, Rf) is a c-pair.*

(1) *Consider the following two conditions:*
  (a) *R satisfies $\alpha_r[e, g, f]$.*
  (b) *$eR/l_{eR}(Rf)_R$ is $gR/r_{gR}(eRg)$-simple-injective.*
  Then (a) $\Rightarrow$ (b) holds. And, if fRf is a right or left perfect ring, then the converse also holds.

(2) *The following are equivalent.*
  (c) *$eR/l_{eR}(Rf)_R$ is $gR/l_{gR}(Rf)$-simple-injective.*
  (d) (i) *The condition (b) holds.*
      (ii) *$r_{gRf}(eRg) = 0$.*

Definition 9. Let M be an R-module. We say that M is *simple-quasi-injective* if M is M-simple-injective.

Next we give an equivalent condition of a quasi-projective module $eR/l_{eR}(Rf)$ to be simple-quasi-injective. This proposition will give more important successive results.

Theorem 10. *Let R be a left perfect ring and let e, f be primitive idempotents in R with $eRf \neq 0$. The following are equivalent.*

(a) *$eR/l_{eR}(Rf)_R$ is simple-quasi-injective.*
(b) (i) *(eR, Rf) is a c-pair.*
    (ii) *R satisfies $\alpha_r[e, e, f]$.*

As a corollary we have the following interesting result.

Corollary 11. *Let R be a semiprimary ring, let e, f be primitive idempotents in R with $eRf \neq 0$. Suppose that ACC holds on right annihilator ideals. Then the following are equivalent.*

(a) *${}_R Rf/r_{Rf}(eR)$ is quasi-injective.*

(b) $eR/l_{eR}(Rf)_R$ *is quasi-injective.*
(c) (eR, Rf) *is a c-pair.*

Next we characterize indecomposable projective simple-quasi-injective modules and indecomposable quasi-projective R-simple-injective modules, which is a generalized result of Theorem 5 (I).

Theorem 12. (1) *Let R be a right perfect ring and let f be a primitive idempotent in R. The following are equivalent.*

(a) ${}_RRf$ *is simple-quasi-injective.*
(b) *There exists a primitive idempotent e in R such that*
  (i) $S({}_RRf) \cong T({}_RRe)$,
  (ii) eRf_{fRf} *is colocal,*
  (iii) *R satisfies $\alpha_l[e, f, f]$.*

(2) *Let R be a left perfect ring and let e, f be primitive idempotents in R. The following are equivalent.*

(a) $eR/l_{eR}(Rf)_R$ *is R-simple-injective.*
(b) (i) *$S({}_RRf)$ is simple essential with $S({}_RRf) \cong T({}_RRe)$,*
  (ii) eRf_{fRf} *is colocal,*
  (iii) *R satisfies $\alpha_r[e, e, f]$.*

Further we generalize Theorem 5 (II) to c-pairs. We note that, in the following theorem, the equivalence between (c) and (d) is already given by Hoshino and Sumioka in [13].

Theorem 13. *Let e, f be primitive idempotents in R and let g be an idempotent of R. Suppose that (eR, Rf) is a c-pair and fRf is a left perfect ring. Then the following are equivalent.*

(a) (i) $eR/l_{eR}(Rf)_R$ *is $gR/r_{gR}(eRg)$-injective.*
  (ii) ${}_RRf/r_{Rf}(eR)$ *is $Rg/l_{Rg}(gRf)$-injective.*
(b) (i) $eR/l_{eR}(Rf)_R$ *is $gR/r_{gR}(eRg)$-simple-injective.*
  (ii) ${}_RRf/r_{Rf}(eR)$ *is $Rg/l_{Rg}(gRf)$-simple-injective.*
(c) $|\,(gRf/r_{gRf}(eRg))_{fRf}\,| < \infty$.
(d) $|\,{}_{eRe}(eRg/l_{eRg}(gRf))\,| < \infty$.
(e) *ACC holds on* $\{\, r_{gRf}(I) \mid {}_{eRe}I \subseteq eRg \,\}$
  ($\Leftrightarrow$ *DCC holds on* $\{\, l_{eRg}(I') \mid I'_{fRf} \subseteq gRf \,\}$).

As a corollary we obtain the following corollary. We note that, in the following theorem, the equivalence between (c) and (d) is already given by Hoshino and Sumioka in [13].

Corollary 14. *Let e, f be primitive idempotents in R. Suppose that (eR, Rf) is a c-pair and fRf is a left perfect ring. Then the following are equivalent.*

(a) *Both $eR/l_{eR}(Rf)_R$ and ${}_RRf/r_{Rf}(eR)$ are injective.*
(b) *Both $eR/l_{eR}(Rf)_R$ and ${}_RRf/r_{Rf}(eR)$ are R-simple-injective.*
(c) $|\, Rf/r_{Rf}(eR)_{fRf} \,| < \infty$.
(d) $|\, {}_{eRe}eR/l_{eR}(Rf) \,| < \infty$.

(e) *ACC holds on* $\{\, r_{Rf}(I) \mid {}_{eRe}I \subseteq eR \,\}$.

Last we give another corollary.

Corollary 15. *Let* e, f *be primitive idempotents in* R. *Suppose that* (eR, Rf) *is an i-pair and* fRf *is a left perfect ring. Then the following are equivalent.*

(a) *Both* eR_R *and* ${}_RRf$ *are injective.*
(b) *Both* eR_R *and* ${}_RRf$ *are R-simple-injective.*
(c) $|\, Rf_{fRf} \,| < \infty$.
(d) $|\, {}_{eRe}eR \,| < \infty$.
(e) *ACC holds on* $\{\, r_{Rf}(I) \mid {}_{eRe}I \subseteq eR \,\}$.

References

[1] F. W. Anderson and K. R. Fuller, "Rings and categories of modules (second edition)," Graduate Texts in Math. 13, Springer-Verlag (1991)
[2] Y. Baba and K. Oshiro, *On a Theorem of Fuller,* J. Algebra **154** (1993), no.1, 86-94.
[3] Y. Baba, *Injectivity of quasi-projective modules, projectivity of quasi-injective modules, and projective cover of injective modules,* J. Algebra **155** (1993), no.2, 415-434.
[4] Y. Baba and K. Iwase, *On quasi-Harada rings,* J. Algebra **185** (1996), 415-434.
[5] Y. Baba, *Some classes of QF-3 rings,* Comm. Alg. **28** (2000), no.6, 2639-2669.
[6] Y. Baba, *On Harada rings and quasi-Harada rings with left global dimension at most* 2*,* Comm. Alg. **28** (2000), no.6, 2671-2684.
[7] Y. Baba, *On self-duality of Auslander rings of local serial rings,* Comm. Alg. **30** (2002), no.6, 2583-2592.
[8] H. Bass, *Finitistic dimension and a homological generalization of semiprimary rings,* Trans. Amer. Math. Soc. **95** (1960), 466-486.
[9] K. R. Fuller, *On indecomposable injectives over artinian rings,* Pacific J. Math **29** (1968), 343-354.
[10] M. Harada, *Non-small modules and non-cosmall modules,* in "Ring Theory , Proceedings of 1978 Antwerp Conference" (F. Van Oystaeyen, Ed.), pp. 669-690, Dekker, New York 1979.
[11] M. Harada, "Factor categories with applications to direct decomposition of modules," Lecture Note in Pure and Appl. Math., Vol. 88, Dekker, New York, (1983).
[12] M. Hoshino and T. Sumioka, *Injective pairs in perfect rings,* Osaka J. Math. **35** (1998), no.3, 501-508.
[13] M. Hoshino and T. Sumioka, *Colocal pairs in perfect rings,* Osaka J. Math. **36** (1999), no.3, 587-603.
[14] T. Kato, *Self-injective rings,* Tohoku Math. J. **19** (1967), 485-494.
[15] M. Morimoto and T. Sumioka, *Generalizations of theorems of Fuller,* Osaka J. Math. **34** (1997), 689-701.
[16] M. Morimoto and T. Sumioka, *On dual pairs and simple-injective modules,* J. Algebra **226** (2000), no.1, 191-201.
[17] M. Morimoto and T. Sumioka, *Semicolocal pairs and finitely cogenerated injective modules,* Osaka J. Math. **37** (2000), no.4, 801-820.
[18] K. Oshiro, *Semiperfect modules and quasi-semiperfect mofules,* Osaka J. Math. **20** (1983), 337-372.
[19] K. Oshiro, *Lifting modules, extending modules and their applications to QF-rings,* Hokkaido Math. J. **13** (1984), 310-338.
[20] M. Rayer, "Small and Cosmall Modules," Ph.D. Dissertation, Indiana University, 1971.
[21] L. E. T. Wu and J. P. Jans, *On quasi-projectives,* Illinois J. Math. **11** (1967), 439-448.

Department of Mathematics
Osaka Kyoiku University
Kashiwara, Osaka 582-8582 JAPAN
E-mail address: ybaba@cc.osaka-kyoiku.ac.jp