ON RINGS ALL OF WHOSE IDEALS ARE *N*-PRIMARY

Yasuyuki Hirano and Hisaya Tsutsui

Abstract: *Let k be a positive integer. The structure of rings all of whose ideals are n-primary for some positive integer $n \le k$ is studied and several examples of such rings are constructed. Rings all of whose nonzero ideals are n-primary for some positive integer $n \le k$ is also considered.*

Throughout this paper, we assume that a ring R is associative with an identity element but not necessarily commutative.

Definition. An ideal P of a ring R will be called right k-primary if there exists an integer $k \ge 1$ minimum with respect to the following condition: for any ideals I, J of R, $IJ \subseteq P$ implies $I \subseteq P$, or $J^k \subseteq P$. An ideal P of a ring R will be called left k-primary if there exists an integer $k \ge 1$ minimum with respect to the following condition: for any ideals I, J of R, $IJ \subseteq P$ implies $J \subseteq P$, or $I^k \subseteq P$. A ring R will be called right (left) k-primary if 0 is a right (left) k-primary ideal. A ring R will be called fully right (left) k-primary if there exists an integer $k \ge 1$ minimum with respect to the following property: every ideal of R is right (left) n-primary for some positive integer $n \le k$. A fully right and left k-primary ring will be called a fully k-primary ring.

The properties of commutative rings in which all ideals are primary were studied by Satyanarayana [9] and Chaudhuri [3]. Let R be a commutative Noetherian ring. An ideal A of R is called irreducible if there are no ideals B, C properly containing A such that $A = B \cap C$. It is well-known that an irreducible ideal is primary. Hence if the set of ideals is linearly ordered, then every ideal I of R is primary. However, we should not that a ring in which every ideal is primary is not necessarily a fully k-primary ring. The formal power series ring $R = Z_2[[x]]$ is an example of a ring in which every ideal is primary but not a fully k-primary ring.

A commutative fully 1-primary ring is a field. The ring Z_n of integer modulo n is fully k-primary if and only if $n = p^k$ for some prime p. If a ring R has a unique maximal ideal M and $M^t = 0$ for some integer $t \ge 1$, then it is clear that R is fully k-primary for some integer $k \le t$, and M is the only prime ideal in R. A fully right(left) k-primary ring has a unique maximal ideal but in general the maximal ideal of a fully

[1] The detailed version of this paper has been submitted elsewhere.

right(left) k-primary ring does not have to be nilpotent (Example 1). However, if the center of a ring R is not a field, then R is fully right (left) k-primary if and only if $M^k = 0$ where M is the unique maximal ideal of R (Theorem 2). A commutative ring R is fully k-primary if and only if R has a unique maximal ideal M and $M^k = 0$. (Theorem 3). This result can be extended to PI- rings (Theorem 4) and FBN rings (Theorem 5). We will show a necessary and sufficient condition for a ring to be fully right (left) k-primary (Theorem1). Using this condition, one can show that if a ring R is a fully right (left) k-primary ring, so are any n by n matrix rings over R, and eRe for any idempotent element e in R. Hence, fully right (left) primary is a Morita invariant property. Among other observations, we will point out that if a ring R is not n-primary but all nonzero ideals of R are n-primary for some positive integer $n \le k$, then R has either one or two minimal nonzero ideals.

Recently, Gorton-Heatherly [7] investigated some characteristics of a k-primary rings and ideals. They started their paper by showing any powers of a maximal ideal of a ring are right and left k-primary for some integer k, while not all right k- primary ideal is left primary. We begin our paper by a few examples that show a fully left (right) k-primary ring is not necessarily a fully right (left) k-primary ring. These examples also show that the maximal ideal of a fully left (right) k-primary ring is not necessarily nilpotent.

Example 1. Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in F \right\}$, a subring (without identity) of $M_{2\times 2}(F)$ where F is a field. Let $P = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \middle| a \in F \right\}$, the only nonzero proper ideal of S. Consider $R = \{(s,t) \mid s \in S, t \in F\}$ with component-wise addition and the multiplication defined by $(s_1,t_1)(s_2,t_2) = (s_1 s_2 + s_1 t_2 + s_2 t_1,\ t_1 t_2)$. $\left(S \times F \to S \text{ is defined by} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \cdot t = \begin{bmatrix} at & bt \\ 0 & 0 \end{bmatrix}. \right)$
Then R is a ring (with identity) and it has two nonzero proper ideals $M = \{(s,0) \mid s \in S\}$ and $I = \{(p,0) \mid p \in P\}$. Since $M^2 = M$, $I^2 = 0$, $MI = I$, and $IM = 0$, R is a fully left 2-primary ring but not a fully right 2-primary ring. If we use $S' = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \middle| a, b \in F \right\}$ for the same construction, we can obtain a right fully 2-primary ring that is not fully left 2-primary. □

If we slightly modify the example above, we can obtain other examples of our interest. The following is an example of a ring that is not fully 2-primary, not fully right k-primary for any k, but it is fully left 3-primary.

Example 2. Let $S=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b\in T\right\}$, a subring (without identity) of $M_{2\times 2}(T)$ where T is a ring with unique nonzero proper ideal P and $P^2=0$ (e.g., Z_4).

Consider $R=\{(s,r)\mid s\in S, r\in Z_2\}$ with component-wise addition and the multiplication defined by $(s_1,t_1)(s_2,t_2)=(s_1s_2+s_1t_2+s_2t_1,\ t_1t_2)$. (Give Z_2 -module structure on S by the obvious manner.)

Let $I_0=\{(s,0)\mid s\in S\}$, $I_1=\{(p,0)\mid p\in P_1\}$ where $P_1=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a\in P,\ b\in T\right\}$, $I_2=\{(p,0)\mid p\in P_2\}$ where $P_2=\left\{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a,b\in P\right\}$, $I_3=\{(p,0)\mid p\in P_3\}$ where $P_3=\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| b\in T\right\}$, and $I_4=\{(p,0)\mid p\in P_4\}$ where $P_4=\left\{\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| b\in P\right\}$.

Since $I_4I_1=0$ and I_1 is idempotent, R is not a right fully k-primary ring for any integer k. Since $I_1I_t=I_t$ for $t=1,2,3,4$, $I_2I_4=0$, $I_2{}^2=I_3\neq 0$, and $I_2{}^3=I_3{}^2=I_4{}^2=0$, R is not fully left 2-primary but a fully left 3-primary ring. □

A right ideal I of a ring R is called eventually idempotent if there is a natural number n, in general depending on I such that $I^n=I^{n+1}$. Clark [4] studied the structure of rings in which every right ideal is eventually idempotent (called eventually idempotent rings.) If, for a ring R, there is a natural number n such that $I^n=I^{n+1}$ for all ideals I of R, then the least such n is called the idempotent bound for R. By Proposition 1 of Clark [4], Proposition 1 below, and Example 3 below, the class of eventually idempotent rings with idempotent bound k strictly contains the class of fully right k-primary rings.

Proposition 1. If a ring R is fully right k-primary then $I^k=I^{k+1}$ for any ideal I.

Example 3.

(A) Let $R=\{(a,b)\mid a,b\in F\}$ where F is a field, with component-wise addition and multiplication. Then R is an eventually idempotent ring with idempotent bound 1, known in literature as a fully idempotent, but R is not a fully right (left) k-primary ring for any positive integer k. □

(B) Let $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in F \right\}$, where F is a field. Then R has three nonzero proper ideals $I_1 = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \middle| a \in F \right\}$, $I_2 = \left\{ \begin{bmatrix} b & a \\ 0 & 0 \end{bmatrix} \middle| a, b \in F \right\}$, and $I_3 = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \middle| a, b \in F \right\}$.
Since I_2 and I_3 are idempotent and $I_1^2 = 0$, R is eventually idempotent with idempotent bound 2. However, since I_2 and I_3 are idempotent and $I_3 I_2 = 0$, R is not fully right *k*-primary for any positive integer *k*. □

Notice that the ring in Example 3 has two maximal ideals. It is an immediate consequence of the next proposition that a fully right *k*-primary ring has a unique maximal ideal.

Proposition 2. If a ring *R* is fully right *k*-primary then for any ideals I, J of *R*, one of the following conditions must hold:

1. $I \subseteq J$
2. $J \subseteq I$
3. $I^k = J^k$

Corollary 1. The set of prime ideals in a fully right *k*-primary ring is linearly ordered.

By Example 2 and 3, we see that the conditions given in Proposition 2 are not sufficient for a ring to be fully right (left) *k*-primary.

Theorem 1. A ring R is fully right *k*-primary if and only if R is eventually idempotent with idempotent bound *k*, and for any ideals I and *J* of *R*, $I = IJ$, $J = JI$, or $I^k = J^k$. A ring S is fully left *k*-primary if and only S is eventually idempotent with idempotent bound *k*, and for any ideals I and *J* of S, $J = IJ$, $I = JI$, or $I^k = J^k$.

Theorem 2. Let R be a ring whose center $Z(R)$ is not a field. Then R is a fully right k-primary ring if and only if R has a unique maximal ideal M and $M^k = 0$.

Theorem 3. Let R be a commutative-ring. Then R is fully k-primary ring if and only if R has a unique maximal ideal M and $M^k = 0$.

As a natural generalization of commutative rings, we consider rings that satisfy a polynomial identity.

Theorem 4. Let R be a PI-ring. Then the following are equivalent:

(1) R is a fully right k-primary ring.
(2) R is a fully left k-primary ring.
(3) R has a unique maximal ideal M and $M^k = 0$.

Recall that a PI-ring is fully right and left fully bounded.

Theorem 5: Let R be a FBN (fully right bounded right Noetherian) ring. Then the following are equivalent:

(1) R is a fully right k-primary ring.
(2) R is a fully left k-primary ring.
(3) R has a unique maximal ideal M and $M^k = 0$.

We now consider a subring S of a fully k-primary ring R that might help in studying the structure of R.

Lemma 1: Let R be a fully k-primary ring with idempotent maximal ideal M. Let L be an ideal of M when we consider M as a ring without identity. Then L is an ideal of R.

An ideal of a maximal ideal of a fully k-primary ring R (when I is considered as a ring without identity) is in general, not an ideal of R as the following examples shows.

Example 4. Let $R = \{(r_1, r_2) \mid r_1 \in P,\ r_2 \in T\}$ where T is a ring with unique nonzero proper ideal P and $P^2 = 0$; with component-wise addition and multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$. Then R is a fully 2- primary ring whose only nonzero proper ideals is the maximal ideal $M = 0 \oplus \mathbb{R}$. Let $I = 0 \oplus \mathbb{Q}\left(\sqrt{2}\right)$. Then I is an ideal of M when M is considered as a ring without identity but not an ideal of R. □

Lemma 1 above yields the following Theorem.

Theorem 6. Let R be a fully k- primary ring whose maximal ideal M is idempotent. Let $Z(R)$ be the center of R. Then $S = M + Z(R)$ is a fully k- primary ring. Further R and S have the same set of proper ideals.

Theorem 7. Let R and $S = M + Z(R)$ be as stated in Theorem 4. Then

(1) R is semiprime if and only if S is semiprime
(2) R is prime if and only if S is prime
(3) S is semiprimitive if and only if R is semiprimitive.
(4) S is right primitive if and only if R is right primitive.
(5) S is right Artinian if and only if R is right Artinian..

Definition. A ring R will be called almost fully right (left) k-primary if there exists an integer $k \geq 1$ minimum with respect to the following property: every <u>nonzero</u> ideal of R is right (left) n-primary for some positive integer $n \leq k$. A almost fully right and left k-primary ring will be called an almost k-primary ring.

Theorem 8. An almost fully right k-primary ring R has one or two maximal ideals. Further, R has two maximal ideals if and only if R is a direct sum of two simple rings.

Theorem 9. An almost fully right k-primary ring R that is not fully right k-primary has one or two minimal non-zero ideals. If R has exactly one minimal nonzero ideal, then every nonzero ideal contains the minimal ideal. Further, if R has two minimal ideals, then R is a direct sum of two fully right k-primary rings.

Note that a direct sum of two fully right k-primary rings is not necessarily almost fully right k-primary. For example, Z_4 is a fully 2-primary ring but $Z_4 \oplus Z_4$ contains nonzero ideals that are not 2-primary.

Theorem 10. If a ring R has a unique minimal nonzero ideal L and $J(R) = 0$, then R is almost fully right k-primary if and only if R is fully right k-primary R.

If an almost fully right k-primary ring R has two maximal ideals M_1 and M_2, then $M_1 \cap M_2 = 0$. Thus, if R has a unique minimal nonzero ideal, then since every nonzero ideal contains the minimal ideal, R must have a unique maximal ideal. The example below shows that the converse of the statement is false.

Example 5. Let R be a simple domain but not a division ring, and let $0 \neq a \in R$. Consider $S = \{(r_1, r_2) \mid r_1 \in aR, r_2 \in aR + Z(R)\}$ with component-wise addition and multiplication defined by $(a, b)(c, d) = (ac, ad + bc + bd)$. Let $I_1 = \{(r,s) \mid r, s \in aR\}, I_2 = \{(r,0) \mid r \in aR\}$, and $I_3 = \{(r,-r) \mid r \in aR\}$. Then since

I_1, I_2, I_3 are all idempotent but $I_2 \cdot I_3 = 0$, S is an almost 2-fully primary but not a 2-primary ring, with unique maximal ideal I_1 and two minimal ideals I_2 and I_3. $\square$

Theorem 11. A PI ring R is almost fully right k-primary if and only if

(1) R has a unique maximal ideal M and $M^k = 0$,
(2) R is a direct sum of two simple Artinian rings,
(3) R has a unique maximal ideal M and M^k is the unique minimal nonzero ideal of R and every nonzero ideal contains M^k, or
(4) R has a unique maximal ideal M and $M^k = M^{k+1}$ is a minimal nonzero ideal of R and there exists exactly one nonzero ideal that does not contain M^k.

Theorem 12. A FBN ring R is almost fully right k-primary if and only if

(1) R has a unique maximal ideal M and $M^k = 0$,
(2) R is a direct sum of two simple Artinian rings,
(3) R has a unique maximal ideal M and M^k is the unique minimal nonzero ideal of R and every nonzero ideal contains M^k, or
(4) R has a unique maximal ideal M and $M^k = M^{k+1}$ is a minimal nonzero ideal of R and there exists exactly one nonzero ideal that does not contain M^k.

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Department of Mathematics
Naruto University
Tokushima, JAPAN
(yhirano@naruto-u.ac.jp)

Department of Mathematics
ERAU
Prescott, AZ, USA
(hisaya.tsutsui@erau.edu)