A TILED ORDER OF FINITE GLOBAL DIMENSION WITH NO NEAT PRIMITIVE IDEMPOTENT

HISAAKI FUJITA AND AKIRA OSHIMA

Let R be a discrete valuation ring with a unique maximal ideal πR and a quotient field K, and let $F = R/\pi R$ be the residue class field. Let $n \geq 2$ be an integer and $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ a set of n^2 integers satisfying

$$\lambda_{ii} = 0, \quad \lambda_{ik} + \lambda_{kj} \geq \lambda_{ij}, \quad \lambda_{ij} + \lambda_{ji} > 0 \quad (\text{if } \ i \neq j)$$

for all $1 \leq i, j, k \leq n$. Then $\Lambda = (\pi^{\lambda_{ij}} R)$ is a basic semiperfect Noetherian R-subalgebra of the full $n \times n$ matrix algebra $\mathrm{M}_n(K)$. We call such Λ a *tiled R-order* in $\mathrm{M}_n(K)$.

Let S be a semiperfect Noetherian ring and e a primitive idempotent of S. Following Ágoston, Dlab and Wakamatsu [1], we call e a *neat* primitive idempotent if $\mathrm{Ext}^i_S(V, V) = 0$ for all $i \geq 1$, where V is a simple right S-module with $Ve \neq 0$ (see [5], too).

It was proved by Jategaonkar [7] that for a fixed integer $n \geq 2$, there are, up to isomorphism, only finitely many tiled R-orders of finite global dimension in $\mathrm{M}_n(K)$. The literature contains a number of papers concerned with determining tiled R-orders of finite global dimension. Tiled R-orders of global dimension two were studied by Roggenkamp and Wiedemann in connection with the interest of orders of finite lattice type (see [2], [11], [12], [20]). As for the problem to determine the maximum finite global dimension among tiled R-orders in $\mathrm{M}_n(K)$ for a fixed n, some authors studied tiled R-orders having large global dimension, but it is not known what is the maximum (see [4], [5], [6], [7], [8], [9], [14], [17], [18]). In such examples, neat primitive idempotents play an essential role when we compute global dimension inductively. Then in [5], we posed a question "Does any tiled R-order of finite global dimension have a neat primitive idempotent?", which can be considered as an improved version of Jategaonkar's conjecture disproved by Kirkman and Kuzmanovich [9] and [4] for all $n \geq 6$.

We notice that in those studies, almost all known results hold if R is an arbitrary discrete valuation ring. However, among other things, Rump [14] proved that global dimension gl.dim Λ of a tiled R-order $\Lambda = (\pi^{\lambda_{ij}} R)$ is determined by the set $\{\lambda_{ij} \mid 1 \leq i, j \leq n\}$ and $\mathrm{char}\, F$ (characteristic of F), and that if gl.dim $\Lambda \leq 2$ then gl.dim Λ does not depend on $\mathrm{char}\, F$, by using matroid theory (see Tutte [19]). Moreover, he provided an example of a tiled R-order Λ in $\mathrm{M}_n(K)$ such that gl.dim $\Lambda = 3$ if $\mathrm{char}\, F \neq 2$, and gl.dim $\Lambda = 4$ if $\mathrm{char}\, F = 2$, where $n = 14$. In accordance with matroid theory, Rump calls a tiled R-order *regular* if its global dimension does not depend on $\mathrm{char}\, F$, and he added the following sentence: "For the present, at least, we have demonstrated that the problem to determine the tiled orders of finite global dimension can hardly be solved without a careful inspection of regularity."

In this report, we announce a new example of non-regular tiled R-orders. Namely, for an arbitrary prime p, we construct a tiled R-order Λ in $\mathrm{M}_n(K)$ such that gl.dim $\Lambda = 5$ if $\mathrm{char}\, F \neq p$ and gl.dim $\Lambda = \infty$ if $\mathrm{char}\, F = p$, where $n = 4p + 5$. Moreover, in the

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computation of gl.dim Λ, we see that Λ has no neat primitive idempotent. Thus, if char $F \neq p$, Λ is a counterexample to the question mentioned above.

1. Example

Let $(\mathcal{P}, \leq)$ be a finite poset. We can consider $\mathcal{P}$ a finite quiver $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)$ as follows. $\mathcal{P}_0$ is the set of vertices in $\mathcal{P}$, that is, the set $\mathcal{P}$ itself. $\mathcal{P}_1$ is the set of arrows of $\mathcal{P}$ defined by $a \to b \in \mathcal{P}_1$ provided $a < b$ and there is no $x \in \mathcal{P}_0$ with $a < x < b$. Note that the order of $\mathcal{P}$ is generated by $\mathcal{P}_1$. That is, for $a, b \in \Omega$, $a < b$ if and only if there is a path from a to b in the quiver $\mathcal{P}$.

From a given finite poset $\mathcal{P}$ with n vertices, we can construct a tiled R-order $\Lambda = (\pi^{\lambda_{xy}} R)$ in $\mathrm{M}_n(K)$ by defining $\lambda_{xy} = 0$ if $x \leq y$, and $\lambda_{xy} = 1$ otherwise.

The following is the example of our tiled R-order.

Example. Let p be an arbitrary prime, and put $l := p + 1$. Then we define a finite quiver $\mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1)$ as follows. The set $\mathcal{P}_0$ has the following $4l + 1\ (= 4p + 5)$ vertices.

$$\mathcal{P}_0 := \{a_i,\ b_i,\ c_i,\ d_i \mid 1 \leq i \leq l\} \cup \{d\}$$

The set $\mathcal{P}_1$ has the following $5l + l^2\ (= p^2 + 7p + 6)$ arrows.

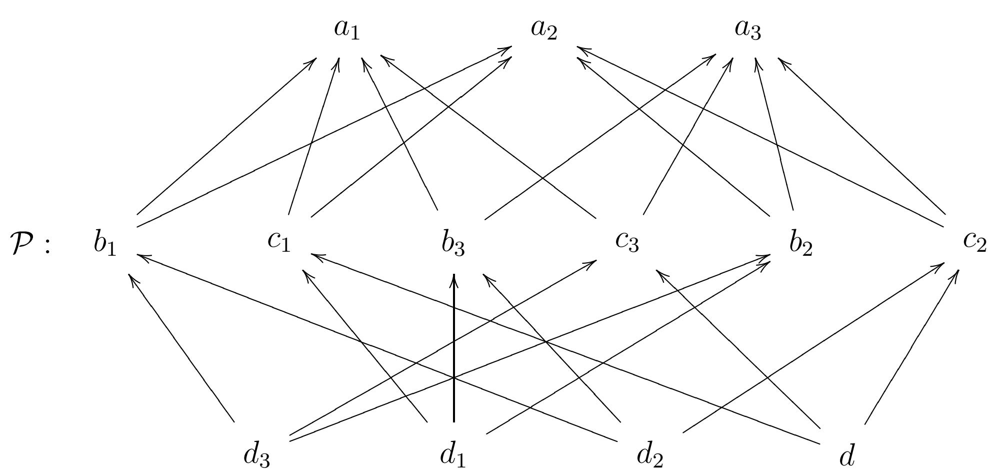
$$\begin{cases} b_i \to a_i \quad (1 \leq i \leq l) & b_i \to a_{i+1} \quad (1 \leq i \leq l) \\ c_i \to a_i \quad (1 \leq i \leq l) & c_i \to a_{i+1} \quad (1 \leq i \leq l) \\ d_i \to c_i \quad (1 \leq i \leq l) & d_i \to b_{i+k} \quad (1 \leq i \leq l,\ 1 \leq k \leq p) \\ d \to c_i \quad (1 \leq i \leq l) & \end{cases}$$

where we consider the indices i of a_i, b_i modulo l. Let Λ be the tiled R-order in $\mathrm{M}_n(K)$ corresponding to $\mathcal{P}$, where $n = 4p + 5$. Then

$$\mathrm{gl.dim}\,\Lambda = \begin{cases} 5 & \text{if} \quad \mathrm{char}\, F \neq p \\ \infty & \text{if} \quad \mathrm{char}\, F = p. \end{cases}$$

Moreover, all primitive idempotents e_i $(1 \leq i \leq n)$ of Λ are not neat.

In the case of $p = 2$, the quiver $\mathcal{P}$ and its tiled R-order Λ in $\mathrm{M}_{13}(K)$ are as follows.



$$\Lambda := \begin{pmatrix}
R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
\boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & R & \boldsymbol{\pi} & R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} \\
R & R & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & R & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R & \boldsymbol{\pi} \\
R & R & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & R & \boldsymbol{\pi} & \boldsymbol{\pi} & \boldsymbol{\pi} & R
\end{pmatrix}$$

where $\boldsymbol{\pi} = \pi R$.

Let $J(\Lambda)$ be the Jacobson radical of Λ. We compute minimal projective resolutions of $J(\Lambda)e_i$ $(1 \leq i \leq n)$ by using Rump's theory [14], which is slightly modified in the detailed version of this paper.

The exponent matrix (λ_{ij}) of a tiled R-order $\Lambda = (\pi^{\lambda_{ij}} R)$ defines an infinite poset Ω_Λ (called *σ-poset* in [14] with an automorphism σ of Ω_Λ). If R is the formal power series ring $F[[t]]$ in the indeterminate t, then there is a correspondence between left Λ-lattices and bounded finite dimensional Ω_Λ-representations over F (see Zavadskij and Kirichenko [21], [22], Roggenkamp and Wiedemann [13], de la Peña and Raggi-Cárdenas [3], and Simson [15]). Using this correspondence, in [14], Rump develops an axiomatic theory to compute global dimension of arbitrary tiled R-orders.

Remark. In [14], Rump provided an example of a non-regular tiled R-order in $\mathrm{M}_{14}(K)$ with $\operatorname{char} F = 2$, which is constructed from a finite poset. A similar finite poset can be found in [16]. Oshima [10] extended Rump's example to the case of an arbitrary prime p, that is, he constructed a tiled R-order Λ in $\mathrm{M}_n(K)$ such that $\mathrm{gl.dim}\,\Lambda = 3$ if $\operatorname{char} F \neq p$, and $\mathrm{gl.dim}\,\Lambda = 4$ if $\operatorname{char} F = p$, where $n = 8p - 2$.

As suggested in [14], it may be an interesting problem to find smaller size n which admits non-regular tiled R-orders. When $p = 2$, our example provides a non-regular tiled R-order in $\mathrm{M}_n(K)$ such that $n = 13 < 14$, at present, that is the minimum among known examples.

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Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571 JAPAN
E-mail: fujita@math.tsukuba.ac.jp