ON HOCHSCHILD COHOMOLOGY RING OF AN ORDER OF A QUATERNION ALGEBRA

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ABSTRACT. We will give an efficient bimodule projective resolution of an order \( \Gamma \), where \( \Gamma \) is an order of a simple component of the rational group ring \( \mathbb{Q}Q_{2^r} \) of the generalized quaternion 2-group of order \( 2^{r+2} \). Moreover we will determine the ring structure of the Hochschild cohomology \( HH^*(\Gamma) \) by calculating the Yoneda products using this bimodule projective resolution.

1. INTRODUCTION

The cohomology theory of associative algebras was initiated by Hochschild [6], Cartan and Eilenberg [1] and MacLane [7]. Let \( R \) be a commutative ring with identity and \( \Lambda \) an \( R \)-algebra which is a finitely generated projective \( R \)-module. If \( M \) is a \( \Lambda \)-bimodule (i.e., a \( \Lambda \)-\( \Lambda \)-\( \text{op} \)-module), then the \( n \)th Hochschild cohomology of \( \Lambda \) with coefficients in \( M \) is defined by \( HH^n(\Lambda, M) := \text{Ext}^n_{\Lambda \otimes \Lambda \text{op}}(\Lambda, M) \). We set \( HH^n(\Lambda) = HH^n(\Lambda, \Lambda) \). The Yoneda product gives \( HH^*(\Lambda) \) a graded ring structure with 1 graded-commutative, that is, for \( \alpha \in HH^p(\Lambda) \) and \( \beta \in HH^q(\Lambda) \) we have \( \alpha \beta = (-1)^{pq} \beta \alpha \). The Hochschild cohomology is an important invariant of algebras. However the Hochschild cohomology ring is difficult to compute in general.

Let \( G \) be a finite group and \( e \) a centrally primitive idempotent of the rational group ring \( \mathbb{Q}G \). Then \( \mathbb{Q}Ge \) is a central simple algebra over the center \( K \). We set \( \Gamma = ZGe \). Then \( \Gamma \) is an \( R \)-order of \( \mathbb{Q}Ge \), where \( R \) denotes the ring of integers of \( K \). The author is interested in the Hochschild cohomology ring \( HH^*(\Gamma) \) of an \( R \)-algebra \( \Gamma \), which is an invariant of the finite group \( G \) and the central idempotent \( e \). On the other hand, a ring homomorphism \( \phi : ZG \to \Gamma; x \mapsto xe \) induces a ring homomorphism \( HH^*(\Gamma) \to HH^*(G, \psi \Gamma) \), where \( \psi \Gamma \) denotes \( \Gamma \) regarded as a \( G \)-module by conjugation and \( HH^*(G, \psi \Gamma) \) denotes the ordinary cohomology ring of \( G \) with coefficients in \( \psi \Gamma \). In fact, we consider that the study of the ring structure of \( HH^*(G, \psi \Gamma) \) and the ring homomorphism gives us much helpful information about \( HH^*(\Gamma) \). So there are some examples of the ring structure of \( HH^*(G, \psi \Gamma) \) and the ring homomorphism \( HH^*(\Gamma) \to HH^*(G, \psi \Gamma) \) ([4], [5]). The Hochschild cohomology ring \( HH^*(\Gamma) \) is in general hard to compute, however it is theoretically possible to calculate if an efficient \( \Gamma \)-projective resolution is given. In this paper, as an example of it, we will give the ring structure of the Hochschild cohomology \( HH^*(\Gamma) \), where \( \Gamma \) is an order of a simple component of the rational group ring of the generalized quaternion 2-group of order \( 2^{r+2} \).

The detailed version of this paper will be submitted for publication elsewhere.
Let $G$ be the generalized quaternion 2-group of order $2^{r+2}$ for $r \geq 1$:

$$Q_{2^r} = \langle x, y \mid x^{2^{r+1}} = 1, x^2 = y^2, yxy^{-1} = x^{-1} \rangle.$$ 

We set $e = (1 - x^{2^r})/2 \in QG$ and denote $xe$ by $\zeta$, a primitive $2^{r+1}$-th root of $e$. Then $e$ is a centrally primitive idempotent of $QG$ and $QGe$ is the (ordinary) quaternion algebra over the field $K := \mathbb{Q}(\zeta + \zeta^{-1})$, with identity $e$, that is, $QGe = K \oplus Ki \oplus Kj \oplus Kij$ where we set $i = x^{2^{r-1}} e$ and $j = ye$ (see [2, (7.40)]). Note that $i^2 = j^2 = -e, ij = -ji$ hold. In the following we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of $K$, and we set $\Gamma = ZGe = R \oplus R\zeta \oplus Rj \oplus R\zeta j$. Note that $R$ is a commuting parameter ring, because $y$ commutes with $x + x^{-1}$. Then $\Gamma$ is an $R$-order of $QGe$. In particular if $r = 1$, $\Gamma = Z \oplus Zi \oplus Zj \oplus Zij$ is just the (ordinary) quaternion algebra over $\mathbb{Z}$ with identity $e$.

We will give an efficient bimodule projective resolution of $\Gamma$, and we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution. This paper is a summary of [3].

2. A BIMODULE PROJECTIVE RESOLUTION OF $\Gamma$

In this section, we state a $\Gamma^e$-projective resolution of $\Gamma$. For each $q \geq 0$, let $Y_q$ be a direct sum of $q + 1$ copies of $\Gamma$. As elements of $Y_q$, we set

$$e^s_q = \begin{cases} 
(0, \ldots, 0, e, 0, \ldots, 0) & \text{if } 1 \leq s \leq q + 1, \\
0 & \text{(otherwise)}.
\end{cases}$$

Then we have $Y_q = \bigoplus_{k=1}^{q+1} \Gamma^e_q \Gamma$. Let $t = 2^r$. Define left $\Gamma^e$-homo morphisms $\pi : Y_0 \to \Gamma; c_0^1 \mapsto e$ and $\delta_q : Y_q \to Y_{q-1}$ ($q > 0$) given by

$$\delta_q(e_q^s) = \begin{cases} 
-\zeta c_{q-1}^s + c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta - c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ even,} \\
\sum_{l=0}^{t-1} \zeta^{t-l-1} c_{q-1}^{s-l} \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j + c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ odd,} \\
-\sum_{l=0}^{t-1} \zeta^{t-l-1} c_{q-1}^{s-l} \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j - c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ even,} \\
(\zeta c_{q-1}^s - c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta c_{q-1}^{s-1} j \zeta + c_{q-1}^{s-1}) & \text{for } q \text{ odd, } s \text{ odd.}
\end{cases}$$

**Theorem 1.** The above $(Y, \pi, \delta)$ is a $\Gamma^e$-projective resolution of $\Gamma$.

**Proof.** By the direct calculations, we have $\pi \cdot \delta_1 = 0$ and $\delta_q \cdot \delta_{q+1} = 0$ ($q \geq 1$).

To see that the complex $(Y, \pi, \delta)$ is acyclic, we state a contracting homotopy. In general, it suffices to define the homotopy as an abelian group homomorphism. However, we can see that there exists a homotopy as a right $\Gamma$-module, which permits us to cut down the number of cases. We define right $\Gamma$-homomorphisms $T_{-1} : \Gamma \to Y_0$ and $T_q : Y_q \to Y_{q+1}$ ($q \geq 0$) as follows:

$$T_{-1}(\gamma) = c_0^1 \gamma - 2^{-}$$ (for $\gamma \in \Gamma$).
If \( q(\geq 0) \) is even, then

\[
T_q(\zeta^k \iota_q^s) = \begin{cases} 
0 & (k = 0, \ s = 1), \\
\sum_{l=0}^{k-1} \zeta^{k-1-l} \iota_{q+1}^l \zeta^l & (1 \leq k < t, \ s = 1), \\
0 & (s(\geq 2) \ even), \\
-\zeta^k \iota_{q+1}^s & (s(\geq 3) \ odd), 
\end{cases}
\]

\[
T_q(\zeta^k \iota_{q+1}^*) = \begin{cases} 
0 & (k = 0, \ s = 1), \\
\left(\begin{array}{c} (-1)^{q/2} c_{q+1}^2 \zeta^l \\ -(-1)^{q/2} \sum_{l=0}^{k-1} \zeta^{k-1-l} \iota_{q+1}^l \zeta^l + \zeta^k c_{q+1}^2 \zeta^l \end{array}\right) & (1 \leq k < t, \ s = 1), \\
\zeta^k \iota_{q+1}^* & (s(\geq 2) \ even), \\
0 & (s(\geq 3) \ odd).
\end{cases}
\]

If \( q(\geq 1) \) is odd, then

\[
T_q(\zeta^k \iota_q^s) = \begin{cases} 
0 & (0 \leq k \leq t - 2, \ s = 1), \\
c_{q+1}^1 & (k = t - 1, \ s = 1), \\
0 & (s(\geq 2) \ even), \\
-\zeta^k \iota_{q+1}^s & (s(\geq 3) \ odd), 
\end{cases}
\]

\[
T_q(\zeta^k \iota_{q+1}^*) = \begin{cases} 
(-1)^{(q-1)/2} c_{q+1}^1 \zeta^l + \zeta^{t-1} c_{q+1}^2 \zeta^l & (k = 0, \ s = 1), \\
(-1)^{(q+1)/2} \zeta^{k-1} c_{q+1}^2 \zeta^l & (1 \leq k < t, \ s = 1), \\
\zeta^k \iota_{q+1}^* & (s(\geq 2) \ even), \\
0 & (s(\geq 3) \ odd).
\end{cases}
\]

Then by the direct calculations, we have

\[
\delta_{q+1} T_q + T_{q-1} \delta_q = \text{id}_{Y_q}
\]

for \( q \geq 0 \). Hence \((Y, \pi, \delta)\) is a \( \Gamma^e\)-projective resolution of \( \Gamma \). \( \square \)

3. HOCHSCHILD COHOMOLOGY \( HH^*(\Gamma) \)

In this section, we will determine the ring structure of the Hochschild cohomology \( HH^*(\Gamma) \). This is obtained by using the \( \Gamma^e\)-projective resolution \((Y, \pi, \delta)\) of \( \Gamma \) stated in Theorem 1. In the following we denote a direct sum of \( q \) copies of a module \( M \) by \( M^q \).

3.1. Module structure. In this subsection, we give the module structure of \( HH^*(\Gamma) \).

As elements of \( \Gamma^{q+1} \), we set

\[
\iota_q^s = \begin{cases} 
(0, \ldots, 0, \hat{s}, 0, \ldots, 0) & (if \ 1 \leq s \leq q + 1), \\
0 & (otherwise).
\end{cases}
\]

Then we have \( \Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma \iota_q^k \).
Applying the functor $\text{Hom}_{\Gamma^e}(-, \Gamma)$ to the resolution $(Y, \pi, \delta)$, we have the following complex, where we identify $\text{Hom}_{\Gamma^e}(Y_q, \Gamma)$ with $\Gamma^{q+1}$ using an isomorphism $\text{Hom}_{\Gamma^e}(Y_q, \Gamma) \rightarrow \Gamma^{q+1}$; $f \mapsto \sum_{k=1}^{q+1} f(c_{q,k}^e) t_q$:

$$(\text{Hom}_{\Gamma^e}(Y, \Gamma), \delta^\#) : 0 \rightarrow \Gamma \xrightarrow{\delta^e_0} \Gamma^2 \xrightarrow{\delta^e_1} \Gamma^3 \xrightarrow{\delta^e_2} \Gamma^4 \xrightarrow{\delta^e_3} \Gamma^5 \rightarrow \cdots$$

$$\delta^e_{q+1}(\gamma t_q^s) = \begin{cases} -\sum_{l=0}^{t-1} \zeta^{t-1-l} \gamma \zeta l t_{q+1}^s + ((-1)^{(q-s)/2} \gamma \zeta j \gamma j \zeta + \gamma) t_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ even}, \\ (\zeta \gamma - \gamma \zeta) t_{q+1}^s + ((-1)^{(q-s-1)/2} \gamma j \gamma j \gamma - \gamma) t_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ odd}, \\ -(\zeta \gamma - \gamma \zeta) t_{q+1}^s + ((-1)^{(q-s-1)/2} \gamma j \gamma j \gamma - \gamma) t_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ even}, \\ \sum_{l=0}^{t-1} \zeta^{t-1-l} \gamma \zeta l t_{q+1}^s + ((-1)^{(q-s)/2} \gamma \zeta j \gamma j \zeta - \gamma) t_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ odd}. \end{cases}$$

In the above, note that

$$\gamma t_q^s = \begin{cases} (0, \ldots, 0, \gamma, 0, \ldots, 0) & \text{if } 1 \leq s \leq q+1, \\ 0 & \text{(otherwise)}, \end{cases}$$

for $\gamma \in \Gamma$, and so on. By the direct calculations, we have the following theorem:

**Theorem 2.** (1) If $r = 1$, then we have

$$HH^n(\Gamma) = \begin{cases} \mathbb{Z} & (n = 0), \\ (\mathbb{Z}/2\mathbb{Z})^{2n+1} & (n \geq 1). \end{cases}$$

(2) If $r \geq 2$, then we have

$$HH^n(\Gamma) = \begin{cases} R & (n = 0), \\ R/(\zeta + \zeta^{-1})R^{2n+1} & (n \text{ odd}), \\ R/2^c R \oplus (R/(\zeta + \zeta^{-1})R)^2 \mathbb{Z} & (n \neq 0) \text{ even}. \end{cases}$$

### 3.2. Ring structure

Recall the Yoneda product in $HH^*(\Gamma)$. Let $\alpha \in HH^n(\Gamma)$ and $\beta \in HH^m(\Gamma)$, where $\alpha$ and $\beta$ are represented by cocycles $f_\alpha : Y_n \rightarrow \Gamma$ and $f_\beta : Y_m \rightarrow \Gamma$, respectively. There exists the commutative diagram of $\Gamma^e$-modules:

$$\cdots \xrightarrow{\delta_{n+1}} Y_{n+1} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_{n+2}} Y_{n+2} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_{n+1}} \cdots \xrightarrow{\mu_n} \cdots \xrightarrow{\delta_{n+2}} Y_{n+1} \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_{n+1}} \cdots \xrightarrow{\mu_0} Y_0 \xrightarrow{\pi} \Gamma \xrightarrow{0},$$

where $\mu_l (0 \leq l \leq n)$ are liftings of $f_\beta$. We define the product $\alpha \cdot \beta \in HH^{n+m}(\Gamma)$ by the cohomology class of $f_\alpha \mu_n$. This product is independent of the choice of representatives $f_\alpha$ and $f_\beta$, and liftings $\mu_l (0 \leq l \leq n)$.

First, we consider the case $r = 1$. Note the Hochschild cohomology ring $HH^*(\Gamma)$ is graded-commutative. From Theorem 2 (1), $HH^*(\Gamma)$ is a commutative ring in this case.
We take generators of $HH^1(\Gamma)$ as follows (see [3, Theorem 2 (1)]):
\[ A = \zeta i_1^2, \ B = \zeta j i_1, \ C = j i_1 + \zeta j i_2. \]

Then we have $2A = 2B = 2C = 0$. We calculate the Yoneda products. Then $HH^n(\Gamma) \ (n \geq 2)$ is multiplicatively generated by $A$, $B$ and $C$, and the equation $A^2 + B^2 + C^2 = 0$ holds. Moreover the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r = 1$ (see [3, Section 3.1] for details).

Next, we consider the case $r \geq 2$. The computation is similar to the case where $r = 1$, however it is more complicated. By [3, Theorem 2 (2)], we take generators of $HH^1(\Gamma)$ as follows:
\[ A = (e - \eta \zeta)i_1^2, \ B = (j - \eta \zeta j)i_1, \ C = (\zeta j - \eta j)i_1 + (j - \eta \zeta j)i_2. \]

In the above $\eta$ denotes $2e/(\zeta + \zeta^{-1}) \in R$ (see also [3, Lemma 2.1]). Then we have $(\zeta + \zeta^{-1})A = (\zeta + \zeta^{-1})B = (\zeta + \zeta^{-1})C = 0$.

Note that products of $A$, $B$, $C$ and $X \in HH^n(\Gamma) \ (n \geq 0)$ are commutative, because $HH^*(\Gamma)$ is graded-commutative and the equations $2A = 2B = 2C = 0$ hold. We calculate the Yoneda products. Then the following equations hold in $HH^2(\Gamma)$:
\[ A^2 = i_2^3, \ AB = ji_2^2, \ AC = \zeta ji_2 - ji_2^3, \ B^2 = 2r^{-1} \eta \zeta i_1 + i_2^3, \]
\[ BC = 2r^{-1} \eta (e - \eta \zeta) i_2, \ C^2 = 2r^{-1} \eta \zeta i_2 + i_2^3. \]

In particular, generators of $HH^2(\Gamma)$ except $(e - \eta \zeta)i_2^3$ are generated by the products of $A$, $B$ and $C$, and the equation $A^2 + B^2 + C^2 = 0$ holds.

In the following, we put $D = (e - \eta \zeta)i_2^3$ which is a generator of $HH^2(\Gamma)$, and then we have $2rD = 0$ and $BC = 2r^{-1} \eta D$. Similarly, we calculate the Yoneda products. Then $HH^n(\Gamma) \ (n \geq 3)$ is multiplicatively generated by $A$, $B$, $C$ and $D$, and the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r \geq 2$ (see [3, Section 3.2] for details).

Finally we state the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$:

**Theorem 3.** (1) If $r = 1$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to
\[ \mathbb{Z}[A, B, C]/(2A, 2B, 2C, A^2 + B^2 + C^2), \]
where $\deg A = \deg B = \deg C = 1$.

(2) If $r \geq 2$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to
\[ R[A, B, C, D]/((\zeta + \zeta^{-1})A, (\zeta + \zeta^{-1})B, (\zeta + \zeta^{-1})C, 2rD, \]
\[ A^2 + B^2 + C^2, \ BC - 2r^{-1} \eta D), \]
where $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, $\deg A = \deg B = \deg C = 1$ and $\deg D = 2$.

**Remark 4.** In the case $r = 1$, this cohomology ring is already known by Sanada [8, Section 3.4]. In [8], he treats the Hochschild cohomology of crossed products over a commutative ring and its product structure using a spectral sequence of a double complex. As a special case, he determines the Hochschild cohomology ring of the quaternion algebra over $\mathbb{Z}$. 

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