ON HOCHSCHILD COHOMOLOGY RING OF AN ORDER OF A QUATERNION ALGEBRA

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Abstract. We will give an efficient bimodule projective resolution of an order Γ, where Γ is an order of a simple component of the rational group ring $\mathbb{Q}Q_{2^r}$ of the generalized quaternion 2-group Q_{2^r} of order 2^{r+2}. Moreover we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution.

1. Introduction

The cohomology theory of associative algebras was initiated by Hochschild [6], Cartan and Eilenberg [1] and MacLane [7]. Let R be a commutative ring with identity and Λ an R-algebra which is a finitely generated projective R-module. If M is a Λ-bimodule (i.e., a $\Lambda^{\mathrm{e}} = \Lambda \otimes_R \Lambda^{\mathrm{op}}$-module), then the nth Hochschild cohomology of Λ with coefficients in M is defined by $H^n(\Lambda, M) := \mathrm{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda, M)$. We set $HH^n(\Lambda) = H^n(\Lambda, \Lambda)$. The Yoneda product gives $HH^*(\Lambda) := \bigoplus_{n \geq 0} HH^n(\Lambda)$ a graded ring structure with $1 \in Z\Lambda \simeq HH^0(\Lambda)$ where $Z\Lambda$ denotes the center of Λ. $HH^*(\Lambda)$ is called the Hochschild cohomology ring of Λ. The Hochschild cohomology ring $HH^*(\Lambda)$ is graded-commutative, that is, for $\alpha \in HH^p(\Lambda)$ and $\beta \in HH^q(\Lambda)$ we have $\alpha\beta = (-1)^{pq}\beta\alpha$. The Hochschild cohomology is an important invariant of algebras. However the Hochschild cohomology ring is difficult to compute in general.

Let G be a finite group and e a centrally primitive idempotent of the rational group ring $\mathbb{Q}G$. Then $\mathbb{Q}Ge$ is a central simple algebra over the center K. We set $\Gamma = \mathbb{Z}Ge$. Then Γ is an R-order of $\mathbb{Q}Ge$, where R denotes the ring of integers of K. The author is interested in the Hochschild cohomology ring $HH^*(\Gamma)$ of an R-algebra Γ, which is an invariant of the finite group G and the central idempotent e. On the other hand, a ring homomorphism $\phi : \mathbb{Z}G \to \Gamma; x \mapsto xe$ induces a ring homomorphism $HH^*(\Gamma) \to H^*(G, {}_{\psi}\Gamma)$, where ${}_{\psi}\Gamma$ denotes Γ regarded as a G-module by conjugation and $H^*(G, {}_{\psi}\Gamma)$ denotes the ordinary cohomology ring of G with coefficients in ${}_{\psi}\Gamma$. In fact, we consider that the study of the ring structure of $H^*(G, {}_{\psi}\Gamma)$ and the ring homomorphism gives us much helpful information about $HH^*(\Gamma)$. So there are some examples of the ring structure of $H^*(G, {}_{\psi}\Gamma)$ and the ring homomorphism $HH^*(\Gamma) \to H^*(G, {}_{\psi}\Gamma)$ ([4], [5]). The Hochschild cohomology ring $HH^*(\Gamma)$ is in general hard to compute, however it is theoretically possible to calculate if an efficient Γ^{e}-projective resolution is given. In this paper, as an example of it, we will give the ring structure of the Hochschild cohomology $HH^*(\Gamma)$, where Γ is an order of a simple component of the rational group ring of the generalized quaternion 2-group of order 2^{r+2}.

The detailed version of this paper will be submitted for publication elsewhere.

Let G be the generalized quaternion 2-group of order 2^{r+2} for $r \geq 1$:

$$Q_{2^r} = \langle x, y \mid x^{2^{r+1}} = 1, x^{2^r} = y^2, yxy^{-1} = x^{-1} \rangle.$$

We set $e = (1 - x^{2^r})/2 \in \mathbb{Q}G$ and denote xe by ζ, a primitive 2^{r+1}-th root of e. Then e is a centrally primitive idempotent of $\mathbb{Q}G$ and $\mathbb{Q}Ge$ is the (ordinary) quaternion algebra over the field $K := \mathbb{Q}(\zeta + \zeta^{-1})$ with identity e, that is, $\mathbb{Q}Ge = K \oplus Ki \oplus Kj \oplus Kij$ where we set $i = x^{2^{r-1}}e$ and $j = ye$ (see [2, (7.40)]). Note that $i^2 = j^2 = -e, ij = -ji$ hold. In the following we set $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, the ring of integers of K, and we set $\Gamma = \mathbb{Z}Ge = R \oplus R\zeta \oplus Rj \oplus R\zeta j$. Note that R is a commuting parameter ring, because y commutes with $x + x^{-1}$. Then Γ is an R-order of $\mathbb{Q}Ge$. In particular if $r = 1$, $\Gamma = \mathbb{Z}e \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij$ is just the (ordinary) quaternion algebra over $\mathbb{Z}$ with identity e.

We will give an efficient bimodule projective resolution of Γ, and we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ by calculating the Yoneda products using this bimodule projective resolution. This paper is a summary of [3].

2. A bimodule projective resolution of Γ

In this section, we state a Γ^{e}-projective resolution of Γ. For each $q \geq 0$, let Y_q be a direct sum of $q+1$ copies of $\Gamma \quad \Gamma$. As elements of Y_q, we set

$$c_q^s = \begin{cases} (0, \dots, 0, \underbrace{e \quad e}_{s}, 0, \dots, 0) & \text{(if } 1 \leq s \leq q+1\text{)}, \\ 0 & \text{(otherwise)}. \end{cases}$$

Then we have $Y_q = \bigoplus_{k=1}^{q+1} \Gamma c_q^k \Gamma$. Let $t = 2^r$. Define left Γ^{e}-homomorphisms $\pi : Y_0 \to \Gamma; c_0^1 \mapsto e$ and $\delta_q : Y_q \to Y_{q-1} \ \ (q > 0)$ given by

$$\delta_q(c_q^s) = \begin{cases} -\zeta c_{q-1}^s + c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta - c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ even,} \\ \displaystyle\sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j + c_{q-1}^{s-1} & \text{for } q \text{ even, } s \text{ odd,} \\ \displaystyle -\sum_{l=0}^{t-1} \zeta^{t-1-l} c_{q-1}^s \zeta^l + (-1)^{(q-s-1)/2} j c_{q-1}^{s-1} j - c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ even,} \\ \zeta c_{q-1}^s - c_{q-1}^s \zeta + (-1)^{(q-s)/2} \zeta j c_{q-1}^{s-1} j \zeta + c_{q-1}^{s-1} & \text{for } q \text{ odd, } s \text{ odd.} \end{cases}$$

Theorem 1. *The above (Y, π, δ) is a Γ^{e}-projective resolution of Γ.*

Proof. By the direct calculations, we have $\pi \cdot \delta_1 = 0$ and $\delta_q \cdot \delta_{q+1} = 0 \ \ (q \geq 1)$.

To see that the complex (Y, π, δ) is acyclic, we state a contracting homotopy. In general, it suffices to define the homotopy as an abelian group homomorphism. However, we can see that there exists a homotopy as a right Γ-module, which permits us to cut down the number of cases. We define right Γ-homomorphisms $T_{-1} : \Gamma \to Y_0$ and $T_q : Y_q \to Y_{q+1} \ \ (q \geq 0)$ as follows:

$$T_{-1}(\gamma) = c_0^1 \gamma \quad \text{(for } \gamma \in \Gamma\text{)}.$$

If $q(\geq 0)$ is even, then

$$T_q(\zeta^k c_q^s) = \begin{cases} 0 & (k=0,\ s=1), \\ \displaystyle\sum_{l=0}^{k-1} \zeta^{k-1-l} c_{q+1}^1 \zeta^l & (1 \leq k < t,\ s=1), \\ 0 & (s(\geq 2) \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s(\geq 3) \text{ odd}), \end{cases}$$

$$T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{q/2} c_{q+1}^2 j & (k=0,\ s=1), \\ (-1)^{q/2} \left(\displaystyle\sum_{l=0}^{k-1} \zeta^{k-1-l} c_{q+1}^1 \zeta^l j + \zeta^k c_{q+1}^2 j \right) & (1 \leq k < t,\ s=1), \\ \zeta^k j c_{q+1}^{s+1} & (s(\geq 2) \text{ even}), \\ 0 & (s(\geq 3) \text{ odd}). \end{cases}$$

If $q(\geq 1)$ is odd, then

$$T_q(\zeta^k c_q^s) = \begin{cases} 0 & (0 \leq k \leq t-2,\ s=1), \\ c_{q+1}^1 & (k=t-1,\ s=1), \\ 0 & (s(\geq 2) \text{ even}), \\ -\zeta^k c_{q+1}^{s+1} & (s(\geq 3) \text{ odd}), \end{cases}$$

$$T_q(\zeta^k j c_q^s) = \begin{cases} (-1)^{(q-1)/2} \left(c_{q+1}^1 j \zeta + \zeta^{t-1} c_{q+1}^2 j \zeta \right) & (k=0,\ s=1), \\ (-1)^{(q+1)/2} \zeta^{k-1} c_{q+1}^2 j \zeta & (1 \leq k < t,\ s=1), \\ \zeta^k j c_{q+1}^{s+1} & (s(\geq 2) \text{ even}), \\ 0 & (s(\geq 3) \text{ odd}). \end{cases}$$

Then by the direct calculations, we have

$$\delta_{q+1} T_q + T_{q-1} \delta_q = \mathrm{id}_{Y_q}$$

for $q \geq 0$. Hence (Y, π, δ) is a Γ^{e}-projective resolution of Γ. $\square$

3. Hochschild cohomology $HH^*(\Gamma)$

In this section, we will determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$. This is obtained by using the Γ^{e}-projective resolution (Y, π, δ) of Γ stated in Theorem 1. In the following we denote a direct sum of q copies of a module M by M^q.

3.1. Module structure. In this subsection, we give the module structure of $HH^*(\Gamma)$.

As elements of Γ^{q+1}, we set

$$\iota_q^s = \begin{cases} (0, \dots, 0, \overset{s}{\check{e}}, 0, \dots, 0) & (\text{if } 1 \leq s \leq q+1), \\ 0 & (\text{otherwise}). \end{cases}$$

Then we have $\Gamma^{q+1} = \bigoplus_{k=1}^{q+1} \Gamma \iota_q^k$.

Applying the functor $\mathrm{Hom}_{\Gamma^e}(-,\Gamma)$ to the resolution (Y,π,δ), we have the following complex, where we identify $\mathrm{Hom}_{\Gamma^e}(Y_q,\Gamma)$ with Γ^{q+1} using an isomorphism $\mathrm{Hom}_{\Gamma^e}(Y_q,\Gamma) \to \Gamma^{q+1}; f \mapsto \sum_{k=1}^{q+1} f(c_q^k)\iota_q^k$:

$$\big(\mathrm{Hom}_{\Gamma^e}(Y,\Gamma),\delta^{\#}\big):\quad 0\to\Gamma \xrightarrow{\delta_1^{\#}} \Gamma^2 \xrightarrow{\delta_2^{\#}} \Gamma^3 \xrightarrow{\delta_3^{\#}} \Gamma^4 \xrightarrow{\delta_4^{\#}} \Gamma^5 \to \cdots,$$

$$\delta_{q+1}^{\#}(\gamma\iota_q^s)=\begin{cases} -\sum_{l=0}^{t-1}\zeta^{t-1-l}\gamma\zeta^l\iota_{q+1}^s+((-1)^{(q-s)/2}\zeta j\gamma j\zeta+\gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ even,}\\ (\zeta\gamma-\gamma\zeta)\iota_{q+1}^s+((-1)^{(q-s-1)/2}j\gamma j-\gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ even, } s \text{ odd,}\\ -(\zeta\gamma-\gamma\zeta)\iota_{q+1}^s+((-1)^{(q-s-1)/2}j\gamma j+\gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ even,}\\ \sum_{l=0}^{t-1}\zeta^{t-1-l}\gamma\zeta^l\iota_{q+1}^s+((-1)^{(q-s)/2}\zeta j\gamma j\zeta-\gamma)\iota_{q+1}^{s+1} & \text{for } q \text{ odd, } s \text{ odd.}\end{cases}$$

In the above, note that

$$\gamma\iota_q^s=\begin{cases}(0,\dots,0,\overset{s}{\check{\gamma}},0,\dots,0) & (\text{if } 1\le s\le q+1),\\ 0 & (\text{otherwise}),\end{cases}$$

for $\gamma\in\Gamma$, and so on. By the direct calculations, we have the following theorem:

Theorem 2. (1) *If $r=1$, then we have*

$$HH^n(\Gamma)=\begin{cases}\mathbb{Z} & (n=0),\\ (\mathbb{Z}/2\mathbb{Z})^{2n+1} & (n\ge 1).\end{cases}$$

(2) *If $r\ge 2$, then we have*

$$HH^n(\Gamma)=\begin{cases}R & (n=0),\\ (R/(\zeta+\zeta^{-1})R)^{2n+1} & (n\ \textit{odd}),\\ R/2^rR\oplus(R/(\zeta+\zeta^{-1})R)^{2n} & (n(\neq 0)\ \textit{even}).\end{cases}$$

3.2. Ring structure. Recall the Yoneda product in $HH^*(\Gamma)$. Let $\alpha\in HH^n(\Gamma)$ and $\beta\in HH^m(\Gamma)$, where α and β are represented by cocycles $f_\alpha: Y_n\to\Gamma$ and $f_\beta: Y_m\to\Gamma$, respectively. There exists the commutative diagram of Γ^e-modules:

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\delta_{n+m+1}} & Y_{n+m} & \xrightarrow{\delta_{n+m}} & \cdots & \xrightarrow{\delta_{m+2}} & Y_{m+1} & \xrightarrow{\delta_{m+1}} & Y_m & \xrightarrow{f_\beta} & \Gamma & & \\
 & & \downarrow{\scriptstyle\mu_n} & & & & \downarrow{\scriptstyle\mu_1} & & \downarrow{\scriptstyle\mu_0} & & \| & & \\
\cdots & \xrightarrow[\delta_{n+1}]{} & Y_n & \xrightarrow[\delta_n]{} & \cdots & \xrightarrow[\delta_2]{} & Y_1 & \xrightarrow[\delta_1]{} & Y_0 & \xrightarrow[\pi]{} & \Gamma & \longrightarrow & 0,
\end{array}$$

where μ_l $(0\le l\le n)$ are liftings of f_β. We define the product $\alpha\cdot\beta\in HH^{n+m}(\Gamma)$ by the cohomology class of $f_\alpha\mu_n$. This product is independent of the choice of representatives f_α and f_β, and liftings μ_l $(0\le l\le n)$.

First, we consider the case $r=1$. Note the Hochschild cohomology ring $HH^*(\Gamma)$ is graded-commutative. From Theorem 2 (1), $HH^*(\Gamma)$ is a commutative ring in this case.

We take generators of $HH^1(\Gamma)$ as follows (see [3, Theorem 2 (1)]):

$$A = \zeta\iota_1^2,\ B = \zeta j\iota_1^1,\ C = j\iota_1^1 + \zeta j\iota_1^2.$$

Then we have $2A = 2B = 2C = 0$. We calculate the Yoneda products. Then $HH^n(\Gamma)$ ($n \geq 2$) is multiplicatively generated by A, B and C, and the equation $A^2 + B^2 + C^2 = 0$ holds. Moreover the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r = 1$ (see [3, Section 3.1] for details).

Next, we consider the case $r \geq 2$. The computation is similar to the case where $r = 1$, however it is more complicated. By [3, Theorem 2 (2)], we take generators of $HH^1(\Gamma)$ as follows:

$$A = (e - \eta\zeta)\iota_1^2,\ B = (j - \eta\zeta j)\iota_1^1,\ C = (\zeta j - \eta j)\iota_1^1 + (j - \eta\zeta j)\iota_1^2.$$

In the above η denotes $2e/(\zeta + \zeta^{-1}) \in R$ (see also [3, Lemma 2.1]). Then we have $(\zeta + \zeta^{-1})A = (\zeta + \zeta^{-1})B = (\zeta + \zeta^{-1})C = 0$.

Note that products of A, B, C and $X \in HH^n(\Gamma)$ ($n \geq 0$) are commutative, because $HH^*(\Gamma)$ is graded-commutative and the equations $2A = 2B = 2C = 0$ hold. We calculate the Yoneda products. Then the following equations hold in $HH^2(\Gamma)$:

$$A^2 = \iota_2^3,\ AB = j\iota_2^2,\ AC = \zeta j\iota_2^2 - j\iota_2^3,\ B^2 = 2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2,$$
$$BC = 2^{r-1}\eta(e - \eta\zeta)\iota_2^1,\ C^2 = 2^{r-1}\eta\zeta\iota_2^1 + \zeta\iota_2^2 + \iota_2^3.$$

In particular, generators of $HH^2(\Gamma)$ except $(e - \eta\zeta)\iota_2^1$ are generated by the products of A, B and C, and the equation $A^2 + B^2 + C^2 = 0$ holds.

In the following, we put $D = (e - \eta\zeta)\iota_2^1$ which is a generator of $HH^2(\Gamma)$, and then we have $2^r D = 0$ and $BC = 2^{r-1}\eta D$. Similarly, we calculate the Yoneda products. Then $HH^n(\Gamma)$ ($n \geq 3$) is multiplicatively generated by A, B, C and D, and the relations are enough. Thus we can determine the ring structure of $HH^*(\Gamma)$ in the case $r \geq 2$ (see [3, Section 3.2] for details).

Finally we state the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$:

Theorem 3. (1) *If $r = 1$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to*

$$\mathbb{Z}[A, B, C]/(2A, 2B, 2C, A^2 + B^2 + C^2),$$

where $\deg A = \deg B = \deg C = 1$.

(2) *If $r \geq 2$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is isomorphic to*

$$R[A, B, C, D]/((\zeta + \zeta^{-1})A, (\zeta + \zeta^{-1})B, (\zeta + \zeta^{-1})C, 2^r D,$$
$$A^2 + B^2 + C^2, BC - 2^{r-1}\eta D),$$

where $R = \mathbb{Z}[\zeta + \zeta^{-1}]$, $\deg A = \deg B = \deg C = 1$ *and* $\deg D = 2$.

Remark 4. In the case $r = 1$, this cohomology ring is already known by Sanada [8, Section 3.4]. In [8], he treats the Hochschild cohomology of crossed products over a commutative ring and its product structure using a spectral sequence of a double complex. As a special case, he determines the Hochschild cohomology ring of the quaternion algebra over $\mathbb{Z}$.

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