REMARKS ON QF-2 RINGS, QF-3 RINGS AND HARADA RINGS

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1. Introduction

In the procedings of the 1978 antwerp conference, M.Harada studied those rings whose non-small left modules contains non-zero injective submodules. K.Oshiro called perfect rings with this condition "*left Harada* rings". These rings are two sided artinian, right QF-2, and right and left QF-3 rings containing *QF* rings and *Nakayama* rings, and moreover, these rings have left and also right ideal theoretic characterizations.

The purpose of this paper is to study the following well known theorems (*see Anderson-Fuller* [1]) :

Theorem I. Right or left artinian QF-2 rings are QF-3.
Theorem II. For a right or left artinian ring R, R is QF-3 if and only if its injective hull $E(R_R)$ is projective.
Theorem III. Every Nakayama ring R with a simple projective right ideal is expressed as a factor ring of an upper triangular matrix ring over a division ring.

In Theorems I, II, we are little anxious whether the assumption "right or left artinian" is natural or not. This assumption also appears in the following well known theorem due to Fuller [6] :

Let R be a right or left artinian ring and let e be a primitive idempotent in R. Then eR_R is injective if and only if there exists a primitive idempotent f in R such that $S(eR) \cong fR/fJ$ and $S(Rf) \cong Re/Je$, where $S(X)$ and J mean the socle of X and the Jacobson radical of R, respectively.

In Baba-Oshiro [2], this theorem is improved for a semiprimary ring with "ACC or DCC" for right annihilator ideals, where ACC and DCC mean the ascending chain condition and the descending chain condition, respectively. As the condition ACC or DCC for right annihilator ideals is equivalent to the condition ACC or DCC for left annihilator ideals, the replacement of "right or left artinian" with "semiprimary ring with ACC or DCC for annihilator right ideals" is quite natural.

In this paper, from this view point, we improve Theorem I as follows: Semiprimary QF-2 rings with ACC or DCC for right annihilator ideals are QF-3. For Theorem II, we show that, for a left perfect ring R with ACC or DCC for right annihilator ideals, R is QF-3 if its injective hull $E(R_R)$ is projective. For Theorem III, using the structure theorem of left Harada rings, we improve the theorem as follows: Left Harada rings with a simple projective right ideal is expressed as a factor ring of an upper triangular matrix ring over a division ring.

The detailed version of this paper will be submitted for publication elsewhere.

2. Improve versions of Theorem I and Theorem II

Recall that a right R-module M is called *uniform* if every non-zero submodule of M is essential. We note that, if R is left perfect, M_R is uniform if and only if M_R is colocal.

The *uniform dimension*of a module M is the infimum of those cardinal numbers c such that $\#I \leq c$ for every independent set $\{N_i\}_{i \in I}$ of non-zero submodules of M. We denote the uniform dimension of M by $unif.dimM$, where $\#I$ means the number of elements of I.

PROPOSITION 1. (*c.f.*[3, Proposition.3.1.2]) Let R be a ring. We consider the following four conditions.

(a) R is right *QF-3*.
(b) R contains a faithful injective right ideal.
(c) For any projective right R module P_R, $E(P_R)$ is projective.
(d) $E(R_R)$ is projective.

Then the following hold.

(1) $(a) \Rightarrow (b)$ holds. Further, if R is a left perfect ring, then $(b) \Rightarrow (a)$ also holds.
(2) If R is left perfect, then $(b) \Leftrightarrow (c)$ holds.
(3) If ACC or DCC holds on right annihilator ideals, then $(d) \Rightarrow (b)$ does.
(4) $(c) \Rightarrow (d)$ holds in general.

Remark : By Proposition 1, when R is a left perfect ring, we have $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ and $(a) \Rightarrow (d)$, but in general $(d) \Rightarrow (a)$ is not true.

For example, for the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{Z}$ of integers, we consider $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then ${}_RR$ is noetherian and has a faithful injective right ideal, so that $E(R_R)$ is projective, but R does not have minimal faithful right R module. (c.f. [18, Theorem 6.2 (Vinsinhaler)])

The following Theorem is due to K.R.Fuller.

THEOREM A. ([1, Theorem 31.3]) Let R be a right or left artinian ring and let $f \in Pi(R)$. Then ${}_RRf$ is injective if and only if there is a primitive idempotent e in R such that $S({}_RRf) \cong T({}_RRe)$ and $S(eR_R) \cong T(fR_R)$.

Using this Theorem A, he showed that every right or left artinian QF-2 ring is QF-3.

Y. Baba and K. Oshiro improved Theorem A in [2] as follows:

THEOREM B. ([2]) Let R be a semiprimary ring which satisfies ACC or DCC for right annihilator ideais and let $e, f \in Pi(R)$. Then the following conditions are equivalent:

(1) ${}_RRf$ is injective with $S({}_RRf) \cong T({}_RRe)$.
(2) eR_R is injective with $S(eR_R) \cong T(fR_R)$.

Now we show the following.

THEOREM 2. If R is a semiprimary *QF-2* ring with *ACC* or *DCC* for right annihilator ideals, then R is *QF-3*.

3. An improve version of Theorem III

For our purpose, we need the following structure theorem due to Oshiro ([15]-[17]):

THEOREM C. Let R be a basic left Harada ring. Then R can be constructed as an upper staircase factor ring of a block extension of its frame QF-subring $F(R)$.

In order to understand this structure theorem, we must review the sketch of the proof of Theorem C (for details, see Baba-Oshiro's Lecture Note).

Let F be a basic QF-ring with $Pi(F) = \{e_1, \ldots, e_y\}$. We put $A_{ij} := e_i F e_j$ for any i, j, and, in particular, put $Q_i := A_{ii}$ for any i. Then we may represent F as

$$F \quad = \quad \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1y} \\ A_{21} & A_{22} & \cdots & A_{2y} \\ \cdots & \cdots & \cdots & \cdots \\ A_{y1} & A_{y2} & \cdots & A_{yy} \end{pmatrix} \quad = \quad \begin{pmatrix} Q_1 & A_{12} & \cdots & A_{1y} \\ A_{21} & Q_2 & \cdots & A_{2y} \\ \cdots & \cdots & \cdots & \cdots \\ A_{y1} & \cdots & A_{y,y-1} & Q_y \end{pmatrix}.$$

For $k(1), \ldots, k(y) \in \mathbb{N}$, the *block extension* $F(k(1), \ldots, k(y))$ of F is defined as follows: For each $i,\ s \in \{1, \ldots, y\},\ \ j \in \{1, \ldots, k(i)\},\ \ t \in \{1, \ldots, k(s)\}$, let

$$P_{ij,st} = \begin{cases} Q_i & \text{if } \ i = s,\ j \le t, \\ J(Q_i) & \text{if } \ i = s,\ j > t, \\ A_{is} & \text{if } \ i \ne s, \end{cases}$$

and

$$P(i,s) = \begin{pmatrix} P_{i1,s1} & P_{i1,s2} & \cdots & P_{i1,sk(s)} \\ P_{i2,s1} & P_{i2,s2} & \cdots & P_{i2,sk(s)} \\ \vdots & \vdots & \ddots & \vdots \\ P_{ik(i),s1} & P_{ik(i),s2} & \cdots & P_{ik(i),sk(s)} \end{pmatrix}.$$

Consequently, when $i = s$, we have the $k(i) \times k(i)$ matrix

$$P(i,i) = \begin{pmatrix} Q_i & \cdots & \cdots & Q_i \\ J(Q_i) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J(Q_i) & \cdots & J(Q_i) & Q_i \end{pmatrix}$$

which we denote by $Q(i)$, and, when $i \ne s$, we have the $k(i) \times k(s)$ matrix

$$P(i,s) = \begin{pmatrix} A_{is} & \cdots & A_{is} \\ & \cdots & \\ A_{is} & \cdots & A_{is} \end{pmatrix}$$

Furthermore, we set

$$P = F(k(1),\ldots,k(y)) = \begin{pmatrix} P(1,1) & P(1,2) & \cdots & P(1,y) \\ P(2,1) & P(2,2) & \cdots & P(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ P(y,1) & P(y,2) & \cdots & P(y,y) \end{pmatrix}$$

$$= \begin{pmatrix} Q(1) & P(1,2) & \cdots & P(1,y) \\ P(2,1) & Q(2) & \cdots & P(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ P(y,1) & P(y,2) & \cdots & Q(y) \end{pmatrix}.$$

Since F is a basic *QF*-ring, we see that P is a basic left *Harada* ring with matrix size $k(1)+\cdots+k(y)$. We say that $F(k(1),\ldots,k(y))$ is a *block extension* of F for $\{k(1),\ldots,k(y)\}$. In more detail, this matrix representation is given by

$$P = F(k(1),\ldots,k(y)) = \begin{pmatrix} P_{11,11} & \cdots & P_{11,1k(1)} & \cdots & P_{11,y1} & \cdots & P_{11,yk(y)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{1k(1),11} & \cdots & P_{1k(1),1k(1)} & \cdots & P_{1k(1),y1} & \cdots & P_{1k(1),yk(y)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{y1,11} & \cdots & P_{y1,1k(1)} & \cdots & P_{y1,y1} & \cdots & P_{y1,yk(y)} \\ \vdots & & \vdots & & \vdots & & \vdots \\ P_{yk(y),11} & \cdots & P_{yk(y),1k(1)} & \cdots & P_{yk(y),y1} & \cdots & P_{yk(y),yk(y)} \end{pmatrix}.$$

If we set

$$p_{ij} = \langle 1 \rangle_{ij,ij}\ ,$$

where this means an element of P which the $(ij,\ ij)$-position $=1$, and another positions are 0.

For each $i=1,\ldots,y,\ j=1,\ldots,k(i)$, then $\{p_{ij}\}_{i=1,j=1}^{y\quad k(i)}$ is a well-indexed set of a complete set of orthogonal primitive idempotents of $P=F(k(1),\ldots,k(y))$.

For $Pi(P)$, we note that

$$p_{ij}P_P \cong p_{i1}J(P)_P^{j-1}$$

for any $i=1,\ldots,y$ and $j=1,\ldots,k(i)$.

Given the situation above, the following are equivalent:

(1) F is a *QF* ring with a Nakayama permutation:

$$\begin{pmatrix} e_1 & \cdots & e_y \\ e_{\sigma(1)} & \cdots & e_{\sigma(y)} \end{pmatrix}.$$

(2) $P=F(k(1),\ldots,k(y))$ is a basic left *Harada* ring of type $(*)$ with a well-indexed set $Pi(P)=\{p_{ij}\}_{i=1,j=1}^{y\quad k(i)}$.

Let R be a basic *Harada* ring. We call R a basic *Harada* ring of *type* $(*)$ if there is a permutation σ of $\{1,2,\ldots,m\}$ such that $(e_{j1}R, Re_{\sigma(j)n(\sigma(j))})$ is an *i*-pair for every $j\in\{1,2,\ldots,m\}$.

From now on, we assume that the Nakayama permutation of F is

$$\begin{pmatrix} e_1 & \cdots & e_y \\ e_{\sigma(1)} & \cdots & e_{\sigma(y)} \end{pmatrix},$$

and

we take the block extension $P = F(k(1), \ldots, k(y))$ of F. Let $i \in \{1, \ldots, y\}$ and consider the i-pair $(e_iF; Fe_{\sigma(i)})$. Put $S(A_{ij}) := S({}_{Q_i}A_{ij}) = S(A_{ij\,Q_j})$. Then we define an upper staircase left $Q(i)$- right $Q(\sigma(i))$-subbimodule $S(i, \sigma(i))$ of $P(i, \sigma(i))$ with tiles $S(A_{ij})$ as follows:

(I) Suppose that $i = \sigma(i)$: Then we see from above argument that $S(A_{ij})$ is simple as both a left and a right ideal of $Q_i = A_{ii}$. Put $Q := Q_i$, $J := J(Q_i)$ and $S := S(Q_i)$. Then, in the $k(i) \times k(i)$ matrix ring,

$$Q(i) = P(i,i) = \begin{pmatrix} Q & \cdots & \cdots & Q \\ J & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & Q \end{pmatrix},$$

we define an upper staircase left $Q(i)$- right $Q(i)$-subbimodule $S(i,i) = S(i, \sigma(i))$ of $Q(i)$ as follows:

$$S(i,i) = \begin{pmatrix} 0 \cdots 0 & & S \\ & & \\ 0 & & \end{pmatrix}. \qquad (\text{ the } (1,1)\text{-position} = 0\,),$$

where, for the form of $S(i,i)$, we assume that
(1) the $(1,1)$-position $= 0$,
(2) when Q is a division ring, that is, $Q = S$,

$$S(i,i) = \begin{pmatrix} 0 \cdots 0 & & & S \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix}$$

Then, since S is an ideal of Q, we see that $S(i,i) = S(i, \sigma(i))$ is an ideal of $Q(i)$.

We let $\overline{Q(i)} = \overline{P(i,\sigma)} = P(i,\sigma)/S(i,\sigma(i))$ for the subbimodule $S(i, \sigma(i))$. In $Q(i)$, we replace Q or J of the (p,q)-position by $\overline{Q} = Q/S$ or $\overline{J} = J/S$, respectively, when the (p,q)-position of $S(i,i)$ is S. Then we may represent $\overline{Q(i)}$ with the matrix ring which is made by these replacements.

For example,

$$Q(i) = \begin{pmatrix} Q & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} \\ J & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} \\ J & \overline{J} & \overline{Q} & \overline{Q} & \overline{Q} & \overline{Q} \\ J & J & J & Q & \overline{Q} & \overline{Q} \\ J & J & J & J & \overline{Q} & \overline{Q} \\ J & J & J & J & \overline{J} & \overline{Q} \end{pmatrix} = \begin{pmatrix} Q & Q & Q & Q & Q & Q \\ J & Q & Q & Q & Q & Q \\ J & J & Q & Q & Q & Q \\ J & J & J & Q & Q & Q \\ J & J & J & J & Q & Q \\ J & J & J & J & J & Q \end{pmatrix} \Big/ \begin{pmatrix} 0 & S & S & S & S & S \\ 0 & S & S & S & S & S \\ 0 & S & S & S & S & S \\ 0 & 0 & 0 & 0 & S & S \\ 0 & 0 & 0 & 0 & S & S \\ 0 & 0 & 0 & 0 & S & S \end{pmatrix}$$

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(II) Now suppose that $i \neq \sigma(i)$: Put $S := S_{i\sigma(i)} = S({}_{Q_i}A_{i\sigma(i)}) = S({A_{i\sigma(i)}}_{Q_{\sigma(i)}})$. Then S is a left Q_i- right $Q_{\sigma(i)}$-subbimodule of $A = A_{i\sigma(i)}$. In the left $Q(i)$- right $Q(\sigma(i))$-bimodule

$$P(i, \sigma(i)) = \begin{pmatrix} A & \cdots & A \\ & \cdots & \\ A & \cdots & A \end{pmatrix} \qquad (\, k(i) \times k(\sigma(i))\text{-matrix}\,),$$

we define an upper staircase subbimodule $S(i, \sigma(i))$ of $P(i, \sigma(i))$ with tiles S of $P(i, \sigma(i))$ as follows:

$$S(i, \sigma(i)) = \begin{pmatrix} 0 \cdots 0 & & S \\ & & \\ & & \\ 0 & & \end{pmatrix} \qquad (\text{ the } (1,1)\text{-position} = 0\,)$$

and put $\overline{P(i,\sigma)} := P(i, \sigma(i))/S(i, \sigma(i))$. We may represent $\overline{P(i,\sigma)}$ as

$$\overline{P(i,\sigma)} = \begin{pmatrix} A \cdots A & & \overline{A} \\ & & \\ & & \\ A & & \end{pmatrix}.$$

Next we define a subset X of $P = F(k(1), \ldots, k(y))$ by

$$X = \begin{pmatrix} X(1,1) & X(1,2) & \cdots & X(1,y) \\ X(2,1) & X(2,2) & \cdots & X(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ X(y,1) & X(y,2) & \cdots & X(y,y) \end{pmatrix},$$

where $X(i,j)$ $(\subseteq Q_i)$ and $X(i,j)$ $(\subseteq P(i,j))$ are defined by

$$X(i,i) = \begin{cases} 0 & \text{if } \ i \neq \sigma(i), \\ S(i,i) & \text{if } \ i = \sigma(i), \end{cases}$$

$$X(i,j) = \begin{cases} 0 & \text{if } \ j \neq \sigma(i), \\ S(i,j) & \text{if } \ j = \sigma(i). \end{cases}$$

Then we see that X is an ideal of $P = F(k(1), \ldots, k(y))$. The factor ring $F(k(1), \ldots, k(y))/X$ is then called an *upper staircase factor ring* of $P = F(k(1), \ldots, k(y))$. If, in the representation

$$P = F(k(1), \ldots, k(y)) = \begin{pmatrix} P(1,1) & P(1,2) & \cdots & P(1,y) \\ P(2,1) & P(2,2) & \cdots & P(2,y) \\ \cdots & \cdots & \cdots & \cdots \\ P(y,1) & P(y,2) & \cdots & P(y,y) \end{pmatrix},$$

we replace $P(i,\sigma(i))$ with $\overline{P(i,\sigma(i))}$ and put $\overline{P} := F(k(1), \ldots, k(y))/X$, then it is convenient to represent $\overline{P}$ as follows:

$$\overline{P} = \begin{pmatrix} P(1,1) & \cdots & \overline{P(1,\sigma(1))} & \cdots & \cdots & \cdots & P(1,y) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P(i,1) & \cdots & \cdots & \overline{P(i,\sigma(i))} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P(y,1) & \cdots & \cdots & \cdots & \overline{P(y,\sigma(y))} & \cdots & P(y,y) \end{pmatrix}.$$

From the form of $\overline{P}$ together with $k \geq 1$, where the k appears in the matrices above (**I**), (**II**), we can see that $\overline{P} = F(k(1), \ldots, k(y))/X$ is a basic left *Harada* ring. Moreover, by the upper staircase form of $S(i,\sigma(i))$, we have left *Harada* rings $P = P_1 = F(k(1), \ldots, k(y))$, P_2, P_3, $\ldots$, P_{l-1}, $P_l = \overline{P}$ and canonical surjective ring homomorphisms $\varphi_i : P_i \to P_{i+1}$ with $\ker \varphi_i$ a simple ideal of P_i as follows:

$$P_1 \xrightarrow{\varphi_1} P_2 \xrightarrow{\varphi_2} P_3 \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{l-2}} P_{l-1} \xrightarrow{\varphi_{l-1}} P_l = \overline{P} = F(k(1), \ldots, k(y))/X.$$

The following is the fundamental structure theorem (see Oshiro [17]).

THEOREM D. For a given basic *QF*-ring F, every upper staircase factor ring P/X of a block extension $P = F(k(1), \ldots, k(y))$ is a basic left *Harada* ring, and, for any basic left Harada ring R, there is a basic QF-subring $F(R)$ which is called the frame QF-subring, R is represented in this form by $F(R)$.

Using this theorem, we show the following

THEOREM 3. Let R be a basic indecomposable left *Harada* ring. If R has a simple projective right R-module, then R can be represented as an upper triangular matrix ring over a division ring as follows:

$$R \cong \begin{pmatrix} \mathrm{D} & & & & \\ & \ddots & \mathrm{D} & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \mathrm{D} \end{pmatrix}$$

By *THEOREM 3* we have the following corollary.

COROLLARY 4. (*c.f.*[1, Theorem 32.8]) Let R be a basic indecomposable Nakayama ring. If R has a simple projective right R-module, then R can be represented as a factor ring of an upper triangular matrix ring over a division ring.

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