AN INTRODUCTION TO NONCOMMUTATIVE ALGEBRAIC GEOMETRY

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Abstract. Since classification of low dimensional projective varieties has been active and successful in algebraic geometry for many years, one of the major projects in noncommutative algebraic geometry is to classify low dimensional noncommutative projective varieties defined by Artin and Zhang. In this note, we will survey this project. Classification of noncommutative projective curves were completed by Artin and Stafford (1995). For classification of noncommutative projective surfaces, we have the following conjecture due to Artin (1997); every noncommutative projective surface is birationally equivalent to either (1) a quantum projective plane, (2) a quantum ruled surface, or (3) a surface finite over its center. Classification of quantum projective planes were completed by Artin, Tate and Van den Bergh (1990), however, classification of the other types of surfaces together with the above conjecture are still open.

1. Quasi-schemes

Throughout, let k be an algebraically closed field. In this paper, we assume that all rings, schemes and abelian categories are noetherian. First, we define the basic object of study in noncommutative algebraic geometry.

Definition 1 (Artin-Zhang 1994 [6], Van den Bergh 2001 [18])**.** A quasi-scheme (over k) is a pair $X = (\operatorname{mod} X, \mathcal{O}_X)$ where $\operatorname{mod} X$ is a (k-linear) abelian category, and $\mathcal{O}_X \in \operatorname{mod} X$ is an object. Two quasi-schemes X and Y are isomorphic (over k) if there exists a (k-linear) equivalence functor $F : \operatorname{mod} X \to \operatorname{mod} Y$ such that $F(\mathcal{O}_X) \cong \mathcal{O}_Y$.

The above definition was modeled by the following example.

Example 2. A (usual) scheme X is a quasi-scheme $X = (\operatorname{mod} X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is the structure sheaf on X, and $\operatorname{mod} X$ is the category of coherent $\mathcal{O}_X$-modules.

Notion of quasi-scheme includes noncommutative schemes.

Example 3. For a ring R, the noncommutative affine scheme associated to R is a quasi-scheme $\operatorname{Spec} R := (\operatorname{mod} R, R)$ where $\operatorname{mod} R$ is the category of finitely generated right R-modules. In fact, if R is commutative and $X = \operatorname{Spec} R$ in the usual sense, then the global section functor $\Gamma(X, -) : \operatorname{mod} X \to \operatorname{mod} R$ induces an isomorphism of quasi-schemes $(\operatorname{mod} X, \mathcal{O}_X) \to (\operatorname{mod} R, R)$.

Example 4. For a graded ring A, the noncommutative homogeneous affine scheme associated to A is a quasi-scheme $\operatorname{GrSpec} A := (\operatorname{grmod} A, A)$ where $\operatorname{grmod} A$ is the category of finitely generated graded right A-modules.

This is an expository paper. The detailed version of this paper will be submitted for publication elsewhere.

The most important example of a quasi-scheme in noncommutative algebraic geometry is the following.

Example 5 (Artin-Zhang 1994 [6])**.** For a graded ring A, the noncommutative projective scheme associated to A is a quasi-scheme $\operatorname{Proj} A := (\operatorname{tails} A, \mathcal{A})$ where

- $\operatorname{tors} A = \{M \in \operatorname{grmod} A \mid M_n = 0 \text{ for all } n \gg 0\}$ is the full subcategory consisting of torsion modules,
- $\operatorname{tails} A = \operatorname{grmod} A / \operatorname{tors} A$ is the quotient category,
- $\pi : \operatorname{grmod} A \to \operatorname{tails} A$ is the quotient functor, and
- $\mathcal{A} = \pi A \in \operatorname{tails} A$.

Note that $\mathcal{M} \cong \mathcal{N}$ in $\operatorname{tails} A$ if and only if $M_{\geq n} \cong N_{\geq n}$ in $\operatorname{grmod} A$ for some n.

The above definition was inspired by the following classical result.

Theorem 6 (Serre 1955 [15])**.** *If A is a commutative graded algebra finitely generated in degree 1 over k and $X = \operatorname{Proj} A$ in the usual sense, then the composition of functors*

$$\begin{array}{ccccc} \operatorname{mod} X & \longrightarrow & \operatorname{grmod} A & \xrightarrow{\pi} & \operatorname{tails} A \\ \mathcal{F} & \longrightarrow & \Gamma_*(X, \mathcal{F}) := \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)) & & \end{array}$$

induces an isomorphism of quasi-schemes $(\operatorname{mod} X, \mathcal{O}_X) \to (\operatorname{tails} A, \mathcal{A})$.

If A is a graded domain finitely generated in degree 1 over k of $\operatorname{GKdim} A = d + 1$, then it is reasonable to call $\operatorname{Proj} A$ a noncommutative projective variety of dimension d. In particular, we call $\operatorname{Proj} A$ a noncommutative projective curve (resp. surface) if $\operatorname{GKdim} A = 2$ (resp. $\operatorname{GKdim} A = 3$). Since noncommutative projective curves were classified by Artin and Stafford (1995) [3], the next project is to classify noncommutative projective surfaces. This project is still wide open. We only have the conjecture below.

If A is a graded domain over k and $X = \operatorname{Proj} A$, then we define the function field of X by

$$k(X) := \{a/b \mid a, b \in A \text{ are homogeneous elements of the same degree}\}.$$

We say that two noncommutative projective varieties X and Y are birationally equivalent if $k(X) \cong k(Y)$ as k-algebras.

Conjecture (Artin 1997 [1]) Every noncommutative projective surface is birationally equivalent to one of the following:

(1) a quantum projective plane.
(2) a quantum ruled surface.
(3) a surface finite over its center.

Classification of quantum projective planes were completed by Artin, Tate and Van den Bergh (1990)[4], however, classification of the other types of surfaces together with the above conjecture are still open.

2. Weak Divisors

Definition 7. [8] Let X be a quasi-scheme over k. A weak divisor on X is a k-linear autoequivalence $D : \operatorname{mod} X \to \operatorname{mod} X$.

We denote by $\mathrm{WPic}\, X$ the group of weak divisors on X. For $D \in \mathrm{WPic}\, X$ and $n \in \mathbb{Z}$, we denote the n-fold composition of D by

$$\begin{aligned} D^n : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{M} &\mapsto \mathcal{M}(nD). \end{aligned}$$

If X is a (usual) scheme over k and D is a Cartier divisor on X, then

$$\begin{aligned} - \otimes_X \mathcal{O}_X(D) : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{F} &\mapsto \mathcal{F}(D) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \end{aligned}$$

is a weak divisor. More generally, the pair $(\sigma, \mathcal{L})$ where $\sigma \in \mathrm{Aut}\, X$ is an automorphism of X and $\mathcal{L} \in \mathrm{Pic}\, X$ is an invertible sheaf on X defines a weak divisor by

$$\begin{aligned} D = (\sigma, \mathcal{L}) : \operatorname{mod} X &\to \operatorname{mod} X \\ \mathcal{F} &\mapsto \mathcal{F}(D) := \sigma_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}). \end{aligned}$$

In fact, if X is a smooth projective variety with an ample or anti-ample canonical divisor, then every weak divisor is given by the pair as above [7].

If X is a quasi-scheme over k and $D \in \mathrm{WPic}\, X$, then we can construct a graded algebra over k by

$$B(X, D) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(nD))$$

with the multiplication defined as follows:

$$\begin{array}{ccc} \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(mD)) \times \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X(nD)) & & (a, b) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_X(\mathcal{O}_X, \mathcal{O}_X((m+n)D)) & & ab := a(nD) \circ b \end{array}$$

$$\begin{tikzcd} & \mathcal{O}_X \arrow[r, "a"] \arrow[d, "nD"'] & \mathcal{O}_X(mD) \arrow[d, "nD"] & \\ \mathcal{O}_X \arrow[r, "b"] & \mathcal{O}_X(nD) \arrow[r, "a(nD)"] & \mathcal{O}_X(mD)(nD) = \mathcal{O}_X((m+n)D). \end{tikzcd}$$

Example 8. If A is a graded algebra and $X = \mathrm{GrSpec}\, A$, then

$$\begin{aligned} (1) : \operatorname{grmod} A &\to \operatorname{grmod} A \\ M &\mapsto M(1) \end{aligned}$$

where $M(1)_i = M_{i+1}$ is a weak divisor on X, and

$$B(X, (1)) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\operatorname{grmod} A}(A, A(n)) \cong A.$$

Example 9. Let X be a projective scheme over k. If D is a very ample divisor on X, then $B(X, D)$ is a homogeneous coordinate ring of X so that $X \cong \mathrm{Proj}\, B(X, D)$.

3. Noncommutative Projective Curves

We have a nice characterization of a quasi-scheme to be a noncommutative projective scheme as in the commutative case.

Definition 10 (Artin-Zhang 1994 [6])**.** Let X be a quasi-scheme over k and $D \in \operatorname{WPic} X$ a weak divisor. We say that D is ample if

- $\{\mathcal{O}_X(-nD)\}_{n\in\mathbb{N}}$ is a set of generators for $\operatorname{mod} X$, and
- for every epimorphism $\mathcal{M} \to \mathcal{N}$ in $\operatorname{mod} X$,

$$\operatorname{Hom}_X(\mathcal{O}_X(-nD), \mathcal{M}) \to \operatorname{Hom}_X(\mathcal{O}_X(-nD), \mathcal{N})$$

  is surjective for all $n \gg 0$.

Roughly speaking, $D \in \operatorname{WPic} X$ is ample if and only if $\mathcal{O}_X(-nD)$ is a projective generator for $\operatorname{mod} X$ for $n \gg 0$.

Definition 11 (Artin-Zhang 1994 [6])**.** We say that a graded algebra A satisfies χ_1 if $\dim_k \operatorname{Ext}^1_A(A/A_{\geq 1}, M) < \infty$ for all $M \in \operatorname{grmod} A$.

The following result, analogous to the commutative case, is a characterization of a quasi-scheme to be projective.

Theorem 12 (Artin-Zhang 1994 [6])**.** *Let X be a* Hom*-finite quasi-scheme over k. Then $X \cong \operatorname{Proj} A$ for some graded algebra A satisfying χ_1 if and only if X has an ample weak divisor. In fact, if D is an ample weak divisor on X, then $X \cong \operatorname{Proj} B(X, D)$.*

Let X be a (usual) scheme over k. Recall that the pair $(\sigma, \mathcal{L}) \in \operatorname{Aut} X \times \operatorname{Pic} X$ defines a weak divisor $D = (\sigma, \mathcal{L}) \in \operatorname{WPic} X$. We denote $B(X, \sigma, \mathcal{L}) := B(X, D)$. If D is ample, then $X \cong \operatorname{Proj} B(X, \sigma, \mathcal{L})$, so we call $B(X, \sigma, \mathcal{L})$ a twisted homogeneous coordinate ring of X. Note that if $\sigma \neq \mathrm{id}$, then $B(X, \sigma, \mathcal{L})$ is typically a noncommutative graded algebra over k.

The following results says that every noncommutative projective curve is isomorphic to a commutative one, which completes the classification of noncommutative projective curves.

Theorem 13 (Artin-Stafford 1995 [3])**.** *If A is a graded domain finitely generated in degree 1 over k of* $\operatorname{GKdim} A = 2$*, so that* $\operatorname{Proj} A$ *is a noncommutative projective curve, then there exist a commutative projective curve X and an ample weak divisor $D = (\sigma, \mathcal{L})$ on X such that $A_{\geq n} \cong B(X, \sigma, \mathcal{L})_{\geq n}$ for some n. In particular,* $\operatorname{Proj} A \cong \operatorname{Proj} B(X, \sigma, \mathcal{L}) \cong X$.

4. Quantum Projective Planes

Next, we will define quantum projective planes and explain their classification.

Definition 14 (Artin-Schelter 1987 [2])**.** A graded algebra A is called a quantum polynomial algebra if

- $\operatorname{gldim} A = d < \infty$,
- $H_A(t) := \sum_{i\in\mathbb{N}} (\dim_k A_i) t^i = (1-t)^{-d}$, and
- $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d \\ 0 & \text{if } i \neq d. \end{cases}$

Since the only commutative quantum polynomial algebra is a commutative polynomial algebra generated in degree 1 over k, if A is a quantum polynomial algebra of $\operatorname{gldim} A = d+1$, then it is reasonable to call $\operatorname{Proj} A$ a quantum projective space of dimension d. In particular, we call $\operatorname{Proj} A$ a quantum projective plane if $\operatorname{gldim} A = 3$.

Let X be a (usual) scheme over k. Recall that every very ample invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$ on X defines an embedding into a projective space $X \to \mathbb{P}(V^*)$ where $V = \Gamma(X, \mathcal{L})$ and V^* is the vector space dual of V. If $\sigma \in \operatorname{Aut} X$ is an automorphism of X, then we can construct a quadratic algebra

$$A(X, \sigma, \mathcal{L}) := T(V)/(\{f \in V \otimes_k V \mid f|_{\Gamma_\sigma} = 0\}$$

where

$$\Gamma_\sigma := \{(p, \sigma(p)) \mid p \in X\} \subset \mathbb{P}(V^*) \times \mathbb{P}(V^*)$$

is the graph of X under σ.

There is a natural graded algebra homomorphism (often surjective) $A(X, \sigma, \mathcal{L}) \to B(X, \sigma, \mathcal{L})$, which induces a map of quasi-schemes (often an embedding) $\operatorname{Proj} B(X, \sigma, \mathcal{L}) \to \operatorname{Proj} A(X, \sigma, \mathcal{L})$.

The following result completes the classification of quantum projective planes.

Theorem 15 (Artin-Tate-Van den Bergh 1990 [4])**.** *A graded algebra A is a quantum polynomial algebra of* $\operatorname{gldim} A = 3$*, so that* $\operatorname{Proj} A$ *is a quantum projective plane, if and only if $A \cong A(X, \sigma, \mathcal{L})$ where either*

(1) $X = \mathbb{P}^2$, $\mathcal{L} = \mathcal{O}_X(1)$, *and* $\sigma \in \operatorname{Aut} \mathbb{P}^2$, *or*

(2) $X \subset \mathbb{P}^2$ *is a cubic divisor,* $\mathcal{L} = \mathcal{O}_X(1)$, *and* $\sigma \in \operatorname{Aut} X$ *such that*

$$\sigma^*(\mathcal{L}) \not\cong \mathcal{L}, \ \text{but } (\sigma^2)^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L} \cong \sigma^*(\mathcal{L}) \otimes_{\mathcal{O}_X} \sigma^*(\mathcal{L}).$$

Example 16. For a generic choice of $(a, b, c) \in \mathbb{P}^2$,

$$A := k\langle x, y, z\rangle/(cx^2 + bzy + ayz, azx + cy^2 + bxz, byx + axy + cz^2) \cong A(X, \sigma, \mathcal{O}_X(1))$$

is a quantum polynomial algebra of $\operatorname{gldim} A = 3$ where

$$X = \mathcal{V}((a^3 + b^3 + c^3)xyz - abc(x^3 + y^3 + z^3)) \subset \mathbb{P}^2$$

is a smooth elliptic curve and $\sigma \in \operatorname{Aut} X$ is the translation by the point $(a, b, c) \in X$ in the group law of X. The above algebra A is called a 3-dimensional Sklyanin algebra.

5. Quantum Ruled Surfaces

Let X be a smooth projective curve over k. We will define a quantum ruled surface over X. First, we recall a commutative ruled surface over X.

One of the characterizations of a ruled surface over X is a scheme defined by $\mathbb{P}(\mathcal{E}) := \operatorname{Proj} S(\mathcal{E})$ where

- $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of rank 2, and
- $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_X$.

Note that $S(\mathcal{E}) \cong T(\mathcal{E})/(\mathcal{Q})$ where

- $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ over $\mathcal{O}_X$, and
- $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is an invertible $\mathcal{O}_X$-subbimodule locally generated by the sections of the form $xy - yx$.

We will extend this construction.

Recall that if R is a commutative ring, then R-R bimodules can be identified with $R \otimes R$-modules. If $X = \operatorname{Spec} R$, then $\operatorname{Spec}(R \otimes R) = X \times X$, so X-X bimodules can be identified with $X \times X$-modules.

Definition 17. Let X be a smooth projective variety over k. A coherent $\mathcal{O}_X$-bimodule is a coherent sheaf $\mathcal{M}$ on $X \times X$ such that

$$pr_i : \operatorname{Supp} \mathcal{M} \subset X \times X \to X$$

are finite for $i = 1, 2$ where $pr_i(x_1, x_2) = x_i$ are projection maps.

We say that a coherent $\mathcal{O}_X$-bimodule $\mathcal{E}$ is locally free of rank r if $pr_{i*}\mathcal{E}$ are locally free of rank r on X for $i = 1, 2$.

If X is a smooth projective variety over k, then every coherent locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$ of rank r has a right adjoint $\mathcal{E}^*$ which is also a locally free $\mathcal{O}_X$-bimodule of rank r, that is,

$$\operatorname{Hom}_X(- \otimes_{\mathcal{O}_X} \mathcal{E}, -) \cong \operatorname{Hom}_X(-, - \otimes_{\mathcal{O}_X} \mathcal{E}^*).$$

We say that an invertible $\mathcal{O}_X$-subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is non-degenerate if the composition

$$\mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{Q} \to \mathcal{E}^* \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E}$$

is an isomorphism.

For the rest of this section, let X be a smooth projective curve over k.

Definition 18 (Van den Bergh 1996 [17])**.** A quantum ruled surface over X is a quasi-scheme $\mathbb{P}(\mathcal{E}) := (\operatorname{mod} \mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})})$ where

- $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2,
- $\mathcal{Q} \subset \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}$ is a non-degenerate invertible $\mathcal{O}_X$-subbimodule,
- $\mathcal{A} = T(\mathcal{E})/(\mathcal{Q})$ is the graded $\mathcal{O}_X$-algebra,
- $\operatorname{mod} \mathbb{P}(\mathcal{E}) = \operatorname{tails} \mathcal{A}$, and
- $\mathcal{O}_{\mathbb{P}(\mathcal{E})} = \pi(\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{A}) \in \operatorname{mod} \mathbb{P}(\mathcal{E})$, called the structure sheaf on $\mathbb{P}(\mathcal{E})$.

It is known that $\mathbb{P}(\mathcal{E})$ is independent of the choice of a non-degenerate $\mathcal{Q}$. In fact, $\mathcal{Q}$ is not even needed to define $\mathbb{P}(\mathcal{E})$ [19].

Although quantum ruled surfaces have been studied intensively (e.g. [9], [11], [12], [13], [14], [19]), classification of them is still wide open. We will end this paper by showing a recent progress on it.

Theorem 19. [10] *If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, and $\mathcal{L}, \mathcal{M}$ are invertible $\mathcal{O}_X$-bimodules, then*

$$\mathbb{P}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{M}) \cong \mathbb{P}(\mathcal{E}).$$

Corollary 20. [10] *If $\mathcal{E}$ is a decomposable locally free $\mathcal{O}_X$-bimodule of rank 2, then $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L})$ for some invertible $\mathcal{O}_X$-bimodule $\mathcal{L}$.*

Every invertible $\mathcal{O}_X$-bimodule is isomorphic to

$$\mathcal{L}_\sigma := pr_2^* \mathcal{L} \otimes_{\mathcal{O}_{X \times X}} \mathcal{O}_{\Gamma_\sigma}$$

where $(\sigma, \mathcal{L}) \in \operatorname{Aut} X \times \operatorname{Pic} X$ [5], so quantum ruled surfaces $\mathbb{P}(\mathcal{E})$ such that $\mathcal{E}$ are decomposable are also classified by the triples $(X, \sigma, \mathcal{L})$.

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