PRIMITIVITY OF GROUP RINGS OF ASCENDING HNN EXTENSIONS OF FREE GROUPS

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Abstract. Let H be a group, and let $\varphi : H \longrightarrow H$ be a monomorphism. The ascending HNN extension corresponding to φ is the group $H_\varphi = \langle H, t | t^{-1}ht = \varphi(h)\rangle$. A ring is (right) primitive if it has a faithful irreducible (right) module. Let F be a free group and K a field. We give a necessary and sufficient condition for the group ring KF_φ to be primitive.

1. Introduction

Let H be a group, and let $\varphi : H \longrightarrow H$ be a monomorphism. The ascending HNN extension corresponding to φ is the group $H_\varphi = \langle H, t | t^{-1}ht = \varphi(h)\rangle$. A ring is right primitive if it has a faithful irreducible right module. One can analogously define left primitive and generally two propreties are not equivalent. For our purpose, the two concepts are equivalent, for the group ring possesses a nice involution. Let F be a free group and K a field. Our purpose of this paper is the study of primitivity of the group ring KF_φ.

If $H \neq 1$ is a finite group or an abelian group, then the group ring KH can never be primitive. In fact, the only primitive commutative rings are fields, and in the case of finite $H \neq 1$, the density theorem would imply that primitive KH be simple, but the augmentention ideal belies that. The first nontrivial example of primitive group ring was offered by Formanek and Snider [7] in 1972. After that, many examples which include the complete solution for primitivity of group rings of polycyclic groups settled by Domanov [3], Farkas-Passman [4] and Roseblade [14] were constructed. Perhaps one of the most interesting result is the one on free products obtained by Formanek:

Theorem 1. ([6, Theorem 5]) *Let K be a field and $G = A * B$ a free product non-trivial groups (except $G = \mathbb{Z}_2 \times \mathbb{Z}_2$). Then KG is primitive.*

As a special case of the theorem, says KF is primitive for every field K provided that F is a nonabelian free group. Moreover, in the same paper, he remarks

Theorem 2. ([6]) *Let $G = \langle t\rangle \times F$ be the direct product of a free group F and the infinite cyclic group $\langle t\rangle$. Then KG is primitive if and only if $|K| \leq |F|$ (the cardinality of K is not larger than that of F).*

It is not difficult to see the result applying to the case of the cyclic extension of F by $\langle t\rangle$ (see Theorem 3 (i) below).

Now, the ascending HNN extension H_φ of a group H, which is a generalization of the cyclic extension of H by $\langle t\rangle$, is a well-studied class of groups. For example, Feighn and

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Handel [5] described all subgroups of ascending HNN extensions of free groups and showed that ascending HNN extensions of free groups are coherent (that is, their f.g. subgroups are finitely presented). Hsu and Wise [9] have recently shown that ascending HNN extensions of polycyclic-by-finite groups are residually finite (that is, each nontrivial element of those groups can be mapped to a non-identity element in some homomorphism onto a finite group). They also study on residual finiteness for ascending HNN extensions of finitely generated free groups, which is conjectured in [8] (the condition 'finitely generated' cannot be dropped [1]). More recently, Borisov and Sapir, in their paper [2], have shown that the conjecture has a positive solution. That is, the ascending HNN extension F_φ of a finitely generated free group F is residually finite [2, Theorem 1.2]. Then by reduction method based on residual properties and on series in groups, we can see that KF_φ is semiprimitive (that is, the Jacobson radical is trivial) if the characteristic of K is zero. One might therefore hope that KF_φ is semiprimitive for any field K. In this paper, we shall show that KF_φ is semiprimitive for all K even if the rank of F is countably infinite (Corollary 10). In fact, KF_φ is often primitive, which is our main result:

Theorem 3. ([11, Theorem 1.1]) *Let F be a nonabelian free group, and F_φ the ascending HNN extension of F determined by φ.*
(i) *In case $\varphi(F) = F$, the group ring KF_φ is primitive for a field K if and only if either $|K| \leq |F|$ or F_φ is not virtually the direct product $F \times \mathbb{Z}$.*
(ii) *In case $\varphi(F) \neq F$, if the rank of F is at most countably infinite, then the group ring KF_φ is primitive for any field K.*

2. Ascending HNN Extensions of Free Groups

Throughout this paper, F denotes the nonabelian free group with the basis X, and $F_\varphi = \langle F, t | t^{-1}ft = \varphi(f)\rangle$ denotes the ascending HNN extension of F determined by φ. Let H be a group and N a subgroup of H. We denote by $[H : N]$ the index of N in H. For a group N', H is said to be virtually N' if N' is isomorphic to N and $[H : N] < \infty$. If h is an element of H, we let $C_N(h)$ denote the centralizer of h in N. Let $C(H)$ be the center of H and $\triangle(H)$ the FC center of H, that is $\triangle(H) = \{h \in H \mid [H : C_H(h)] < \infty\}$.

If f is a non-trivial element in F then $C_F(f)$ is infinite cyclic, and so $\triangle(F) = C(F)$ is trivial. On the other hand, $\triangle(F_\varphi)$ is not trivial in general. However, if $\triangle(F_\varphi)$ is non-trivial then $\triangle(F_\varphi) = C(F_\varphi)$ and F_φ is virtually the direct product $F \times \mathbb{Z}$:

Lemma 4. *Let F be a nonabelian free group.*
(i) $\triangle(F_\varphi) = C(F_\varphi)$.
(ii) $C(F_\varphi) \neq 1$ *if and only if F_φ is virtually the direct product $F \times \mathbb{Z}$. When this is the case, φ is an automorphism of F and there exist $n > 0$ and $f \in F$ such that $C(F_\varphi) = \langle t^n f\rangle$.*

Proof. Since $\triangle(F_\varphi) \supseteq C(F_\varphi)$, we may assume $\triangle(F_\varphi) \neq 1$. Let $1 \neq g \in \triangle(F_\varphi)$. We shall show $C(F_\varphi) = \langle g\rangle$. Since $[F_\varphi : C_{F_\varphi}(g)] < \infty$, we have $[F : C_F(g)] < \infty$, which implies $g \notin F$ because of $\triangle(F) = 1$. By the normal form theorem, there exist $n, l \geq 0$ and $f \in F$ such that $g = t^n f t^{-l}$, where $f \notin \varphi(F)$ if $n \neq 0$ and $l \neq 0$. Then replacing g by g^{-1} if necessary, we may assume that $n \geq l \geq 0$, and then $f \notin \varphi(F)$

unless $l = 0$. Since $[F_\varphi : C_{F_\varphi}(g)] < \infty$, there exists $m \geq 1$ such that $t^m g t^{-m} = g$, and so $t^{m+n} f t^{-l-m} = t^n f t^{-l}$; thus $f = \varphi^m(f) \in \varphi(F)$. Hence we get $l = 0$, that is $g = t^n f$ with $n > 0$. Then we may assume that n is minimal in $\{n' > 0 \mid t^{n'} f' \in \triangle(F_\varphi) \text{ with } f' \in F\}$. Again by $[F_\varphi : C_{F_\varphi}(g)] < \infty$, there exists $k > 0$ such that for each $x \in F$, $x^k g x^{-k} = g$, and so $x^k g x^{-k} = x^k t^n f x^{-k} = t^n \varphi^n(x)^k f x^{-k} = t^n f$; thus $\varphi^n(x)^k = (f x f^{-1})^k$. This implies $\varphi^n(x) = f x f^{-1}$ because F is a free group (c.f. [10]). In particular, $\varphi^n(f) = f$ and also $x \in C_{F_\varphi}(g)$. Furthermore we see that φ^n is an automorphism and so is φ.

Now, if $f = 1$, then $g = t^n \in C(F_\varphi)$, which completes the proof, and therefore we may assume $f \neq 1$. Since F is free, as is well known, $C_F(f)$ is cyclic, and thus $C_F(f) = \langle h \rangle$ for some $1 \neq h \in F$ and $f = h^m$ for some $m \neq 0$. Then $h^m = f = \varphi^n(f) = \varphi^n(h)^m$, and so $\varphi^n(h) = h$. Moreover, $\varphi(h) = \varphi^n(\varphi(h)) = f \varphi(h) f^{-1}$, which implies $\varphi(h) \in C_F(f)$ and thus $\varphi(h) = h^l$ for some $l \neq 0$. Since $h = \varphi^n(h) = h^{l^n}$, we have that $l = 1$, that is, $\varphi(h) = h$. Hence we get that $\varphi(f) = f$ which means $g \in C(F_\varphi)$. We have thus seen that the assertion of (i) holds and $C(F_\varphi) \supseteq \langle g \rangle$. Conversely, $C(F_\varphi) \subseteq \langle g \rangle$. In fact, if $g_1 \in C(F_\varphi)$, then we may assume that $g_1 = t^{n_1} f_1$ for some n_1 with $n_1 \geq n$ and for some $f_1 \in F$. It is obvious that $g_1 \in C(F_\varphi)$ if and only if $\varphi(f_1) = f_1$ and $\varphi^{n_1}(x) = f_1 x f_1^{-1}$ for every $x \in F$. Let $n_1 = mn + k$, where $m > 0$ and $0 \leq k < n$. For each $x \in F$, $f_1 x f_1^{-1} = \varphi^{n_1}(x) = \varphi^k(\varphi^{nm}(x)) = \varphi^k(f^m x f^{-m}) = f^m \varphi^k(x) f^{-m}$, and therefore, if we put $f_2 = f^{-m} f_1$, then $\varphi(f_2) = f_2$ and $\varphi^k(x) = f_2 x f_2^{-1}$ for every $x \in F$; thus $t^k f_2 \in C(F_\varphi)$. By the minimality of n, we get $k = 0$. That is, $f_2 \in C(F) = 1$, and so $f_1 = f^m$. Hence we conclude that $g_1 = t^{mn} f^m = (t^n f)^m = g^m \in \langle g \rangle$.

Since $FC(F_\varphi) = F\langle g \rangle \simeq F \times \mathbb{Z}$ and $[F_\varphi : FC(F_\varphi)] = [F\langle t \rangle : F\langle t^n \rangle] < \infty$, we see that F_φ is virtually $F \times \mathbb{Z}$. Conversely, if F_φ is virtually $F \times \mathbb{Z}$, then there exists $1 \neq g \in F_\varphi$ such that $g \in \triangle(F_\varphi)$, and so $g \in C(F_\varphi)$ by (i); thus $C(F_\varphi) \neq 1$. This completes the proof. $\square$

In what follows, for $f \in F$ and $i \geq 0$, we denote by $f^{[i]}$ the element $t^i f t^{-i}$ of $t^i F t^{-i}$. The next assertions are elementary and some of them can be found in [5].

Lemma 5. *Let N_0 be a subgroup of F with $\varphi(N_0) \subseteq N_0$. For each non-negative integer i, let $N_i = t^i N_0 t^{-i}$ and $N = \cup_{i=0}^\infty N_i$.*
(i) *$N_i \simeq N_0$ and $N_i \subseteq N_{i+1}$, where the equality holds if and only if $\varphi(N_0) = N_0$.*
(ii) *If N_0 is a normal subgroup of F, then N is a normal subgroup of F_φ.*
(iii) *If $[N_i, N_i]$ is the derived subgroup of N_i, then $[N, N] = \cup_{i=0}^\infty [N_i, N_i]$.*
(vi) *If the rank of N_0 is finite and $\varphi(N_0) \subset N_0$, then $\varphi([N_0, N_0]) \subset [N_0, N_0]$.*

3. Primitivity of Group Rings of F_φ

We will start this section with presenting the next two lemmas which are basic results on group rings (c.f. [13]).

Lemma 6. *Let K be a field, H a group and N a subgroup of H.*
(i)([16, Theorem 1]) *Suppose that N is normal. If $\triangle(H) = 1$ and $\triangle(H/N) = H/N$, then KN is primitive implies KH is primitive.*

(ii)([15, Theorem 3]) *If $\triangle(H)$ is torsion free abelian and $[H : N]$ is finite, then KN is primitive implies KH is primitive.*

Lemma 7. ([12, Theorem 2]) *Let K be a field and H a group. If $\triangle(H) = 1$ and KH is primitive, then for any field extension K' of K, $K'H$ is primitive.*

In view of Lemma 4, 6 (ii) and Theorem 2, we have immediately

Corollary 8. ([11, Corollary 2.6]) *Let K be a field, and suppose that $C(F_\varphi) \neq 1$. Then the group ring KF_φ is primitive if and only if K is any field with $|K| \leq |F|$.*

In what follows, let F be a free group with a countably infinite basis X, and $F_\varphi = \langle F, t | t^{-1}ft = \varphi(f)\rangle$ the ascending HNN extension of F determined by φ. For an element w in F, $\mathcal{R}(w)$ denotes the reduced word equivalent to w on X, and we set $\mathfrak{L}(w) = \{x^{\pm 1} \in X^{\pm 1} \mid x \text{ is a letter contained in } \mathcal{R}(w)\}$. For a non-negative integer i, let G_i be the subgroup of F_φ generated by $\{t^i f t^{-i} \mid f \in F\}$, and $G = \cup_{i=0}^{\infty} G_i$. Moreover, let K be a field with $|K| \leq |G|$, and KG denotes the group ring of G over K.

Let $\mathbb{N}$ be the set of positive integers. Since $|KG| = |\mathbb{N}|$, there exists bijection s from $\mathbb{N}$ to the elements of KG except for the zero element. Let $s(i) = s_i = \sum_{j=1}^{m_i} \alpha_{ij} f_{ij}^{[l_{ij}]}$, where $\alpha_{ij} \in K$, $f_{ij} \in F$, $m_i > 0$, $l_{ij} \geq 0$ and $f_{ij}^{[l_{ij}]} = t^{l_{ij}} f_{ij} t^{-l_{ij}} \in G_i$ satisfying

$$f_{ij}^{[l_{ij}]} \neq f_{ij'}^{[l_{ij'}]} \text{ if } j \neq j', \text{ and } f_{ij} \notin \varphi(F) \text{ if } l_{ij} \neq 0. \tag{3.1}$$

For s_i above, we set $q_1 = max\{l_{1j} \mid 1 \leq j \leq m_1\}$, $S_1 = \mathfrak{L}(\varphi^{q_1 - l_{1j}}(f_{1j}) \mid 1 \leq j \leq m_1)$, and for $i > 1$, inductively $q_i = max\{q_{i-1} + 1,\ l_{ij} \mid 1 \leq j \leq m_i\}$ and $S_i = \mathfrak{L}(\varphi^{q_i - l_{ij}}(f_{ij}),\ \varphi^{q_i - q_{i-1}}(x) \mid 1 \leq j \leq m_i,\ x \in S_{i-1})$. We choose three elements x_{11}, x_{12} and x_{13} in $X \setminus S_1$ which are different from each other, and set $B_1 = \widehat{B_1} = \{x_{11}, x_{12}, x_{13}\}$ and $S_{B_1} = \mathfrak{L}(\varphi^{q_2 - q_1}(x) \mid x \in B_1)$. Moreover, for $i > 1$, we set inductively $B_i = \{x_{i1}, x_{i2}, x_{i3}\}$, $\widehat{B_i} = \widehat{B_{i-1}} \cup B_i$, where $x_{i1}, x_{i2}, x_{i3} \in X \setminus (S_i \cup S_{B_{i-1}} \cup \widehat{B_{i-1}})$ with $x_{ik} \neq x_{ik'}$ ($k \neq k'$), and $S_{B_i} = \mathfrak{L}(\varphi^{q_{i+1} - q_i}(x) \mid x \in S_{B_{i-1}} \cup B_i)$. Because $|X|$ is countably infinite, $X \setminus (S_i \cup S_{B_{i-1}} \cup \widehat{B_{i-1}})$ is non-empty for every $i > 0$, in fact, it is an infinite set, and thereby the above argument is valid. Then we have that

$$i \neq i' \Longrightarrow B_i \cap B_{i'} = \emptyset, \tag{3.2}$$

$$i' \geq i \Longrightarrow \{x_{i'1}, x_{i'2}, x_{i'3}\} \cap \mathfrak{L}(\varphi^{q_{i'} - l_{ij}}(f_{ij}) \mid 1 \leq j \leq m_i) = \emptyset, \tag{3.3}$$

$$i' > i \Longrightarrow \{x_{i'1}, x_{i'2}, x_{i'3}\} \cap \mathfrak{L}(\varphi^{q_{i'} - q_i}(x_{ik}) \mid 1 \leq k \leq 3) = \emptyset. \tag{3.4}$$

Here we define the element $\varepsilon(s_i)$ in KG for each s_i as follows;

$$\varepsilon(s_i) = z_i^{[q_i]} s_i z_i^{[q_i]^{-1}} + x_{i1}^{[q_i]} z_i^{[q_i]} s_i z_i^{[q_i]^{-1}} + \alpha_{i1} x_{i2}^{[q_i]} z_i^{[q_i]} f_{i1}^{[q_i]} z_i^{[q_i]^{-1}}, \tag{3.5}$$

where $z_i = x_{i2}^{-1} x_{i3}$ and $\{x_{i1}, x_{i2}, x_{i3}\} = B_i$.

The next lemma plays an essential role in the proof of our main result Theorem 3.

Lemma 9. ([11, Lemma 3.3]) *Let $\varepsilon(s_i)$ be as defined by (3.5) and let $\rho = \sum_{i=1}^{\infty} \varepsilon(s_i)KG$ be the right ideal of KG. Then ρ is a proper right ideal of KG.*

The proof of the above lemma is not short and so we omit it. The reader should refer to the paper [11]. By making use of the above lemma, we can prove Theorem 3:

Proof of Theorem 3 (i): If $\varphi(F) = F$ then F_φ/F is isomorphic to $\langle t\rangle$, and so $\triangle(F_\varphi/F) = F_\varphi/F$. In addition, if $C(F_\varphi) = 1$ then $\triangle(F_\varphi) = 1$ by Lemma 4 (i). By [6, Theorem 2], KF is primitive for any field K, and therefore it follows from Lemma 6 (i) that KF_φ is primitive. By virtue of Lemma 4, $C(F_\varphi) \neq 1$ if and only if F_φ is virtually $F \times \mathbb{Z}$, and hence the result follows from Corollary 8.

(ii): If $\varphi(F) \neq F$, then $\triangle(F_\varphi) = C(F_\varphi) = 1$ by Lemma 4. By virtue of Lemma 7, we may assume that K is a prime field. For each non-negative integer i, let $G_i = t^i F t^{-i}$, and $G = \cup_{i=0}^{\infty} G_i$. Moreover, let $D_i = [G_i, G_i] = t^i[F,F]t^{-i}$, the derived subgroup of G_i, and $D = \cup_{i=0}^{\infty} D_i$. If we put $N_0 = F$ in Lemma 5 (ii), then the lemma asserts that G is a normal subgroup of F_φ. It is obvious that F_φ/G is isomorphic to $\langle t\rangle$, and thereby, by virtue of Lemma 6 (i), it suffices to show that KG is primitive. If the rank of F is finite, then $D_0 = [F,F]$ is a free group of countably infinite rank. If we put $N_0 = D_0$ in Lemma 5 (i), then the lemma asserts that D_i is isomorphic to D_0 and $D_i \subseteq D_{i+1}$ (in fact, $\varphi(D_0) \neq D_0$; thus $D_i \subset D_{i+1}$ by (iv)) for every $i \geq 0$. Since G is locally free by lemma 5 (i), we see that the finite conjugate center of G is trivial. Moreover, G/D is abelian by Lemma 5 (iii), and therefore, again by Lemma 6 (i), it suffices to show that KD is primitive. In other words, we may further assume that the rank of F is countably infinite. Then KG satisfies all of the conditions supposed in Lemma 9.

Let $\varepsilon(s_i)$ be the element in KG defined by (3.5), and let $\rho = \sum_{i=1}^{\infty} \varepsilon(s_i)KG$ be the right ideal of KG. By Lemma 9, ρ is a proper right ideal of KG, and therefore, ρ is extended to a maximal right ideal ρ_m of KG. To complete the proof, we shall show that KG acts faithfully on the irreducible module KG/ρ_m. Let κ be the kernel of the action of KG on KG/ρ_m so that, certainly, $\kappa \subseteq \rho_m$. Now, if $\kappa \neq 0$, then κ contains the element s_i for some $i \in \mathbb{N}$, and therefore, by (3.5) the definition of $\varepsilon(s_i)$, we see that $\varepsilon(s_i) - g_i \in \kappa \subseteq \rho_m$, where $g_i = \alpha_{i1} x_{i2}^{[q_i]} z_i^{[q_i]} f_{i1}^{[q_i]} {z_i^{[q_i]}}^{-1}$ is a trivial unit in KG. On the other hand, $\varepsilon(s_i)$ is also contained in ρ_m; thus we conclude that $g_i \in \rho_m$, a contradiction. Hence the action is faithful, and KG is primitive. □

As a corollary of Theorem 3, we finally state the semiprimitivity of F_φ.

Corollary 10. ([11, Corollary 3.7]) *Let F be a nonabelian free group of at most countably infinite rank, and F_φ the ascending HNN extension of F determined by φ. If K is any field then the group ring KF_φ is semiprimitive.*

Proof. Let K_0 be the prime field of K. Since $|K_0| \leq |F|$, by virtue of Theorem 3, K_0F_φ is primitive and so semiprimitive. As is well known, semiprimitive group rings are separable algebras, thus semiprimitivity of group rings close under extensions of coefficient fields, and therefore KF_φ is semiprimitive. □

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