IWASAWA ALGEBRAS, CROSSED PRODUCTS AND FILTERED RINGS

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Abstract. We apply the theory of crossed product to Iwasawa algebra $\Lambda(G) = \Lambda(H) * (G/H)$. A J-adic filtration of $\Lambda(H)$ can be extended to that of $\Lambda(G)$. We study Gorenstein dimension of a graded module over $\Lambda(G)$.

1. Introduction: Iwasawa Algebras

Let p be a prime integer and $\mathbb{Z}_p$ denote the ring of p-adic integers. A topological group G is a compact p-adic analytic group if and only if G has an open normal uniform pro-p subgroup H of finite index [6]. The *Iwasawa algebra* of G is defined by

$$\Lambda(G) := \varprojlim \mathbb{Z}_p[G/N],$$

where N ranges over the open normal subgroups of G.

The ring theoretical survey of Iwasawa algebras is given by K. Ardakov and K.A. Brown [1]. In this paper, we address crossed products and filtered rings arising from Iwasawa algebras. Therefore, we direct our attension to the fact that a ring $\Lambda(G)$ is a crossed product of a finite group G/H over a ring $\Lambda(H)$ (Iwasawa algebra of H): $\Lambda(G) \cong \Lambda(H) * (G/H)$. Since the topological group H has good conditions, a ring $\Lambda(H)$ has good properties among them, we need:

(1) local with the radical $J := \mathrm{rad}\Lambda(H)$ and $\Lambda(H)/J \cong \mathbb{F}_p$, a field of p-elements,

(2) a filtered ring with the J-adic filtration whose associated graded ring is isomorphic to a polynomial ring $\mathbb{F}_p[x_0, \cdots, x_d]$, where $d = \mathrm{dim}G$ is a minimal number of generators of G as a topological group.

(3) a left and right Noetherian domain,

(4) Auslander regular with $\mathrm{gldim}\Lambda(H) = d + 1$.

(cf. [1], [4], [5], [6], [12])

2. A crossed product and a filtered ring

2.1. Let R be a ring and A a finite group. A *crossed product* S ([8], [11]) of a group A over a ring R, denoted by $S := R * A$, is a ring such that:

1) R is a subring of $R * A$

2) $\bar{A} = \{\bar{a} : a \in A\}$ is a subset of $R * A$ consisting of units of $R * A$

3) $R * A$ is a free right R-module with basis $\bar{A}$

The detailed version of this paper will be submitted for publication elsewhere.

4) For all $a, b \in A$, the equalities $\bar{a}R = R\bar{a}$ and $\bar{a}\bar{b}R = \overline{ab}R$ hold.

Remarks. (see [11]) (1) We may assume $\bar{1}_A = 1_S$. A left R-module $R * A$ is also free with basis $\bar{A}$. Usually, we write

$$R * A = \bigoplus_{a \in A} \bar{a}R.$$

(2) There exists a map $\sigma : A \to \mathrm{Aut}R$ such that $r\bar{a} = \bar{a}r^{\sigma(a)}$, $r \in R$, $a \in A$. In what follows, we shortly write $r\bar{a} = \bar{a}r^a$. There exists a map $\tau : A \times A \to U(R)$ such that $\bar{a}\bar{b} = \overline{ab}\tau(a, b)$. In order to assure the associativity of $R * A$, maps σ, τ satisfy some conditions (see [11]).

We start with the theorem which implies that the Iwasawa algebra is Auslander Gorenstein.

A ring R is said to satisfy *Auslander condition*, if, for all finitely generated left R-module M, for all $i \geq 0$ and for all right R-submodules N of $\mathrm{Ext}^i_R(M, R)$, grade of N is greater than or equal to i, where grade of an R-module X is $\inf\{j \geq 0 : \mathrm{Ext}^j_R(X, R) \neq 0\}$.

Theorem 1. *Let $S = R * A$ be a crossed product. Then* $\mathrm{id}R = \mathrm{id}S$ *holds, where* id *stands for injective dimension. Moreover, if R satisfies Auslander condition, then S satisfies it, too.*

Proof. It follows from [2] that, for all finitely generated left S-modules M and for all $i \geq 0$,

$$\mathrm{Ext}^i_S(M, S) \cong \mathrm{Ext}^i_R(M, R).$$

The statement is an easy consequence of this formula. □

Since $\mathrm{gldim}\Lambda(H) = d + 1$, we see $\mathrm{id}\Lambda(H) = d + 1$. Hence $\Lambda(G)$ is Auslander Gorenstein of $\mathrm{id}\Lambda(G) = d + 1$.

2.2. A ring R is called a *filtered ring* with a filtration $\mathcal{F} = \{\mathcal{F}_i R\}_{i \in \mathbb{Z}}$ if

i) $\mathcal{F}_i R$ is an additive subgroup of R for all $i \in \mathbb{Z}$ and $1 \in \mathcal{F}_0 R$,

ii) $\mathcal{F}_i R \subset \mathcal{F}_{i+1} R$ $(i \in \mathbb{Z})$,

iii) $(\mathcal{F}_i R)(\mathcal{F}_j R) \subset \mathcal{F}_{i+j} R$ $(i, j \in \mathbb{Z})$,

iv) $\cup_{i \in \mathbb{Z}} \mathcal{F}_i R = R$.

([7])

Let $S = R * A$ be a crossed product and further, assume that R is a filtered ring. Then a filtration $\mathcal{F}$ is called *A-stable*, if

v) $(\mathcal{F}_i R)^a \subset \mathcal{F}_i R$ for all $a \in A$ and $i \in \mathbb{Z}$.

Let $\mathrm{gr}R = \oplus_{p \in \mathbb{Z}} \mathcal{F}_p R / \mathcal{F}_{p-1} R$ an associated graded ring of R. Then forming a crossed product '$*$' and an associated graded ring 'gr' commutes each other.

Theorem 2. *Let R, A, S be as above. Assume that R is a filtered ring with an A-stable filtration such that every unit of R sits in $\mathcal{F}_0R \setminus \mathcal{F}_{-1}R$. Then $\mathcal{F}' := \{\mathcal{F}'_iS\}_{i\in\mathbb{Z}}$, $\mathcal{F}'_iS = \oplus_{a\in A}\bar{a}(\mathcal{F}_iR)$ $(i \in \mathbb{Z})$, is a filtration of S and there is a ring isomorphism*

$$\mathrm{gr}_{\mathcal{F}'}S \cong (\mathrm{gr}_{\mathcal{F}}R) * A.$$

We put the J-adic filtration $\mathcal{F} = \{\mathcal{F}_i\Lambda(H)\}_{i\in\mathbb{Z}}$ of $\Lambda(H)$ by

$$\mathcal{F}_i\Lambda(H) = \begin{cases} J^{-i} \ (i < 0) \\ \Lambda(H) \ (i \geq 0) \end{cases}$$

It follows from [1],[4],[6],[12] that the associated graded ring satisfies $\mathrm{gr}_{\mathcal{F}}\Lambda(H) \cong \mathbb{F}_p[x_0, \cdots, x_d]$.

Since $J^\alpha \subset J$ for all $\alpha \in \mathrm{Aut}\Lambda(H)$, the J-adic filtration $\mathcal{F}$ of $\Lambda(H)$ is G/H-stable. Since $\Lambda(H)$ is a local ring, all units of $\Lambda(H)$ sit in $\Lambda(H) \setminus J$, i.e., in $\mathcal{F}_0\Lambda(H) \setminus \mathcal{F}_{-1}\Lambda(H)$. Therefore, we see $\mathrm{gr}\Lambda(G) \cong \mathrm{gr}\Lambda(H) * (G/H)$.

3. Graded modules over a crossed product

Let $S = R * A$ be a crossed product of a finite group A over a ring R. We assume that R is Noetherian, so that S is , too. A left S-module with a decomposition $M = \oplus_{a\in A}M_a$ as an abelian group is called a (*strongly*) *A-graded* module, if $\bar{a}RM_b \subset M_{ab}$ $(\bar{a}RM_b = M_{ab})$ for all $a, b \in A$. By the decomposition $S = \oplus_{a\in A}\bar{a}R$, S itself is an A-graded module with $S_a = \bar{a}R$, so S is an A-graded ring ([9], [10]). Since $\bar{a}R\bar{b}R = \overline{ab}R$, S is a strongly graded ring, therefore, every graded module over S is strongly graded ([9]).

Let $f \in \mathrm{Hom}_S(M, N)$ for M, N graded S-modules. We call f a graded homomorphism of degree $a \in A$, whenever $f(M_b) \subset N_{ba}$ for all $b \in A$. We put, for $a \in A$, $\mathrm{Hom}_a(M, N) := \{f \in \mathrm{Hom}_R(M, N) : f \text{ is graded of degree } a\}$. Then $\mathrm{Hom}_R(M, N) = \oplus_{a\in A}\mathrm{Hom}_a(M, N)$ holds.

Proposition 3. *Let N be a left R-module and M a left graded S-module. Then there exists an isomorphism* $\mathrm{Hom}_R(M_1, N) \cong \mathrm{Hom}_1(M, \mathrm{Hom}_R(S, N))$.

Proof. Note that $\mathrm{Hom}_R(S, N)$ is graded by the grading $\mathrm{Hom}_R(S, N)_a = \mathrm{Hom}_R(\overline{a^{-1}}R, N)$, $a \in A$ □

Lemma 4. [2] *Define* $\alpha : \mathrm{Hom}_R(S, R) \to S$, *by* $\alpha(f) = \sum_{a\in A}(\bar{a})^{-1}f(\bar{a})$ *for* $f \in \mathrm{Hom}_R(S, R)$. *Then α is an S-R-bimodule isomorphism.*

Combining Proposition 3 and Lemma 4, we get

Corollary 5. *Let M be a graded S-module. Then there is an isomorphism of right R-module:* $\mathrm{Hom}_R(M_1, R) \cong \mathrm{Hom}_1(M, S)$.

We study Gorenstein dimension(cf. [3]), one of the important homological invariants of a Noetherian ring. An R-module M is said to have *Gorenstein dimension zero*, denoted by $\text{G-dim}_R M = 0$, if $M^{**} \cong M$ and $\mathrm{Ext}^k_R(M, R) = \mathrm{Ext}^k_{R^{\mathrm{op}}}(M^*, R) = 0$ for $k > 0$, where $M^* = \mathrm{Hom}_R(M, R)$. For a positive integer k, M is said to have *Gorenstein dimension less than or equal to k*, denoted by $\text{G-dim } M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with $\text{G-dim } G_i = 0$ for $0 \leq i \leq k$. We have that G-dim

$M \leq k$ if and only if G-dim $\Omega^k M = 0$. It is also proved that if G-dim $M < \infty$ then G-dim $M = \sup\{k : \mathrm{Ext}_R^k(M, R) \neq 0\}$.

For a graded S-module, G-dimension is controlled by an R-module.

Theorem 6. *Let M be a graded S-module, then* $\text{G-dim}_S M = \text{G-dim}_R M_1$

We will prove this theorem in the following.

Let $M = \oplus_{a\in A} M_a = \oplus_{a\in A} \bar{a} M_1$ be a graded S-module. Note that $M \cong S \otimes_R M_1$. Then a right S-module $M^* = \mathrm{Hom}_S(M, S) = \oplus_{a\in A}\mathrm{Hom}_a(M, S)$. We see $\mathrm{Hom}_a(M, S) = \mathrm{Hom}_1(M, S)\bar{a}$ and M^* is a graded right S-module of grading $\mathrm{Hom}_S(M, S)_a = \mathrm{Hom}_1(M, S)\bar{a}$. By Corollary 6, it holds that $\mathrm{Hom}_1(M, S) \cong \mathrm{Hom}_R(M_1, R)$, hence $M^* \cong \oplus_a M_1^* \bar{a}$, whare $M_1^* = \mathrm{Hom}_R(M_1, R)$. Similary, there is an isomorphism $M^{**} \cong \oplus_a \bar{a} M_1^{**}$.

Let $\theta : M \to M^{**}$ be a canonical evaluation map. Then θ is a graded homomorphism of degree 1. Therefore, the following holds.

Lemma 7. *M is reflexive as an S-module if and only if M_1 is reflexive as an R-module.*

Concerning extension groups, the following holds.

Lemma 8. $\mathrm{Ext}_S^i(M, S) = 0$ *if and only if* $\mathrm{Ext}_R^i(M_1, R) = 0$ *for all* $i \geq 0$.

Proof. The combination of isomorphisms:

$$\mathrm{Ext}_S^i(M, S) \cong \mathrm{Ext}_R^i(M, R) \quad ([2])$$

$$\mathrm{Ext}_R^i(S \otimes_R M_1, R) \cong \mathrm{Ext}_R^i(M_1, \mathrm{Hom}_R(S, R))$$

$$\mathrm{Hom}_R(S, R) \cong S \quad (\text{Lemma } 4)$$

induces an isomorphism

$$\mathrm{Ext}_S^i(M, S) \cong \oplus_{a\in A}\mathrm{Ext}_R^i(M_1, \bar{a}R).$$

Consequently, the assertion holds. □

We deal with the case of G-dimension zero. Note that $(M^*)_1 = M_1^*$ for a graded S-module M.

Theorem 9. *Let M be a graded S-module. Then G-$\dim_S M = 0$ if and only if G-$\dim_R M_1 = 0$.*

Let $\cdots \to P_1 \to P_0 \to M_1 \to 0$ be a projective resolution of an R-module M_1, for a graded S-module. Then $\cdots \to S \otimes_R P_1 \to S \otimes_R P_0 \to S \otimes_R M_1 \to 0$ is a projective resolution of an S-module $S \otimes_R M_1 = M$. Hence $\Omega^i M \cong S \otimes_R \Omega^i M_1$, and then $(\Omega^i M)_1 \cong \Omega^i(M_1)$. Hence $\text{G-dim}_S \Omega^i M = 0$ if and only if $\text{G-dim}_R \Omega^i M_1 = 0$ by Theorem 9. This proves Theorem 6.

3.1. **Concluding Remarks.** Let M be a graded $\Lambda(G)$-module and take a good filtration of M_1 ([7]). Then the following (in)equalities hold:

$$\text{G-dim}_{\Lambda(G)} M + \mathfrak{m}\text{-depth}(\text{gr} M_1) \leq d+1$$
$$\text{grade}_{\Lambda(G)} M + \dim_{\text{gr}\Lambda(H)}(\text{gr} M_1) = d+1,$$

where $\mathfrak{m}$ is the *maximal ideal of $\text{gr}\Lambda(H) = \mathbb{F}[x_0, \cdots, x_d]$.

These formulae will be able to apply to homological theory of modules over the Iwasawa algebra. For example:
Suppose that $\text{gr} M_1$ is Cohen-Macaulay, then M is perfect, i.e., $\text{grade}_{\Lambda(G)} M = \text{G-dim}_{\Lambda(G)} M$.

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