

# SUPPORT VARIETIES AND THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE

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ABSTRACT. This paper is based on my talks given at the ‘41st Symposium on Ring Theory and Representation Theory’ held at Shizuoka University, Japan, 5-7 September 2008. It begins with a brief introduction to the use of Hochschild cohomology in developing the theory of support varieties of [50] for a module over an artin algebra. I then describe the current status of research concerning the structure of the Hochschild cohomology ring modulo nilpotence.

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## INTRODUCTION

This survey article is based on talks given at the ‘41st Symposium on Ring Theory and Representation Theory’, Shizuoka University in September 2008, and is organised as follows. Section 1 gives a brief introduction to the use of Hochschild cohomology in developing the theory of support varieties of [50]. Section 2 considers the Hochschild cohomology ring of  $\Omega$ -periodic algebras. In [50] it had been conjectured that the Hochschild cohomology ring modulo nilpotence of a finite-dimensional algebra is always finitely generated as an algebra. Section 3 describes many classes of algebras where this holds, that is, that the Hochschild cohomology ring modulo nilpotence is finitely generated as an algebra. The final section is devoted to studying the recent counterexample of Xu ([57]) to this conjecture.

Throughout this paper, let  $\Lambda$  be an indecomposable finite-dimensional algebra over an algebraically closed field  $K$ , with Jacobson radical  $\mathfrak{r}$ . Denote by  $\Lambda^e$  the enveloping algebra  $\Lambda^{\text{op}} \otimes_K \Lambda$  of  $\Lambda$ , so that right  $\Lambda^e$ -modules correspond to  $\Lambda, \Lambda$ -bimodules. The Hochschild cohomology ring  $\text{HH}^*(\Lambda)$  of  $\Lambda$  is given by  $\text{HH}^*(\Lambda) = \text{Ext}_{\Lambda^e}^*(\Lambda, \Lambda) = \bigoplus_{i \geq 0} \text{Ext}_{\Lambda^e}^i(\Lambda, \Lambda)$  with the Yoneda product. We may consider an element of  $\text{Ext}_{\Lambda^e}^n(\Lambda, \Lambda)$  as an exact sequence of

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$\Lambda$ ,  $\Lambda$ -bimodules  $0 \rightarrow \Lambda \rightarrow E^n \rightarrow E^{n-1} \rightarrow \dots \rightarrow E^1 \rightarrow \Lambda \rightarrow 0$  where the Yoneda product is the ‘splicing together’ of exact sequences.

The low-dimensional Hochschild cohomology groups are well-understood via the bar resolution ([41] and see [3, 39]), and may be described as follows:

- $\mathrm{HH}^0(\Lambda) = Z(\Lambda)$ , the centre of  $\Lambda$ .
- $\mathrm{HH}^1(\Lambda)$  is the space of derivations modulo the inner derivations. A derivation is a  $K$ -linear map  $f : \Lambda \rightarrow \Lambda$  such that  $f(ab) = af(b) + f(a)b$  for all  $a, b \in \Lambda$ . A derivation  $f : \Lambda \rightarrow \Lambda$  is an inner derivation if there is some  $x \in \Lambda$  such that  $f(a) = ax - xa$  for all  $a \in \Lambda$ .
- $\mathrm{HH}^2(\Lambda)$  measures the infinitesimal deformations of the algebra  $\Lambda$ ; in particular, if  $\mathrm{HH}^2(\Lambda) = 0$  then  $\Lambda$  is rigid, that is,  $\Lambda$  has no non-trivial deformations.

Recently there has been much work on the structure of the entire Hochschild cohomology ring  $\mathrm{HH}^*(\Lambda)$  and its connections and applications to the representation theory of  $\Lambda$ . One important property of Hochschild cohomology in this situation is its invariance under derived equivalence, proved by Rickard in [48, Proposition 2.5] (see also [39, Theorem 4.2] for a special case). It is also well-known that  $\mathrm{HH}^*(\Lambda)$  is a graded commutative ring, that is, for homogeneous elements  $\eta \in \mathrm{HH}^n(\Lambda)$  and  $\theta \in \mathrm{HH}^m(\Lambda)$ , we have  $\eta\theta = (-1)^{mn}\theta\eta$ . Thus, when the characteristic of  $K$  is different from two, then every homogeneous element of odd degree squares to zero. Let  $\mathcal{N}$  denote the ideal of  $\mathrm{HH}^*(\Lambda)$  which is generated by the homogeneous nilpotent elements. Then, for  $\mathrm{char} K \neq 2$ , we have  $\mathrm{HH}^{2k+1}(\Lambda) \subseteq \mathcal{N}$  for all  $k \geq 0$ . Hence (in all characteristics) the Hochschild cohomology ring modulo nilpotence,  $\mathrm{HH}^*(\Lambda)/\mathcal{N}$ , is a commutative  $K$ -algebra.

Support varieties for finitely generated modules over a finite-dimensional algebra  $\Lambda$  were introduced using Hochschild cohomology by Snashall and Solberg in [50], where it was also conjectured that the Hochschild cohomology ring modulo nilpotence is itself a finitely generated algebra. We remark that the graded commutativity of  $\mathrm{HH}^*(\Lambda)$  implies that  $\mathcal{N}$  is contained in every maximal ideal of  $\mathrm{HH}^*(\Lambda)$  and so  $\mathrm{MaxSpec} \mathrm{HH}^*(\Lambda) = \mathrm{MaxSpec} \mathrm{HH}^*(\Lambda)/\mathcal{N}$ . Although the recent paper [57] provides a counterexample to the conjecture of [50], nevertheless finiteness conditions play an key role in the structure of these support varieties (see [15]), so it remains of particular importance to determine the structure of the Hochschild cohomology ring modulo nilpotence.

## 1. SUPPORT VARIETIES

One of the motivations for introducing support varieties for finitely generated modules over a finite-dimensional algebra came from the rich theory of support varieties for finitely generated modules over group algebras of finite groups. For a finite group  $G$  and finitely generated  $KG$ -module  $M$ , the variety of  $M$ ,  $V_G(M)$ , was defined by Carlson [10] to be the variety of the kernel of the homomorphism

$$- \otimes_K M : H^{\mathrm{ev}}(G, K) \rightarrow \mathrm{Ext}_{KG}^*(M, M).$$

This map factors through the Hochschild cohomology ring of  $KG$ , so that we have the commutative diagram

$$\begin{array}{ccc} H^{\text{ev}}(G, K) & \xrightarrow{-\otimes_K M} & \text{Ext}_{KG}^*(M, M) \\ & \searrow & \nearrow^{-\otimes_{KG} M} \\ & & \text{HH}^*(KG) \end{array}$$

Linckelmann considered the map  $-\otimes_{KG} M: \text{HH}^{\text{ev}}(KG) \rightarrow \text{Ext}_{KG}^*(M, M)$  when studying varieties for modules for non-principal blocks ([43]).

Now, for any finite-dimensional algebra  $\Lambda$  and finitely generated  $\Lambda$ -module  $M$ , there is a ring homomorphism  $\text{HH}^*(\Lambda) \xrightarrow{-\otimes_{\Lambda} M} \text{Ext}_{\Lambda}^*(M, M)$ . This ring homomorphism turns out to provide a similarly fruitful theory of support varieties for finitely generated modules over an arbitrary finite-dimensional algebra. As usual, let  $\text{mod } \Lambda$  denote the category of all finitely generated left  $\Lambda$ -modules.

For  $M \in \text{mod } \Lambda$ , the support variety of  $M$ ,  $V_{\text{HH}^*(\Lambda)}(M)$ , was defined by Snashall and Solberg in [50, Definition 3.3] by

$$V_{\text{HH}^*(\Lambda)}(M) = \{\mathfrak{m} \in \text{MaxSpec } \text{HH}^*(\Lambda)/\mathcal{N} \mid \text{Ann}_{\text{HH}^*(\Lambda)} \text{Ext}_{\Lambda}^*(M, M) \subseteq \mathfrak{m}'\}$$

where  $\mathfrak{m}'$  is the preimage in  $\text{HH}^*(\Lambda)$  of the ideal  $\mathfrak{m}$  in  $\text{HH}^*(\Lambda)/\mathcal{N}$ . We recall from above that  $\text{MaxSpec } \text{HH}^*(\Lambda) = \text{MaxSpec } \text{HH}^*(\Lambda)/\mathcal{N}$ .

Since we assumed that  $\Lambda$  is indecomposable, we know that  $\text{HH}^0(\Lambda)$  is a local ring. Thus  $\text{HH}^*(\Lambda)/\mathcal{N}$  has a unique maximal graded ideal which we denote by  $\mathfrak{m}_{\text{gr}}$  so that  $\mathfrak{m}_{\text{gr}} = \langle \text{rad } \text{HH}^0(\Lambda), \text{HH}^{\geq 1}(\Lambda) \rangle / \mathcal{N}$ . From [50, Proposition 3.4(a)], we have  $\mathfrak{m}_{\text{gr}} \in V_{\text{HH}^*(\Lambda)}(M)$  for all  $M \in \text{mod } \Lambda$ . We say that the variety of  $M$  is *trivial* if  $V_{\text{HH}^*(\Lambda)}(M) = \{\mathfrak{m}_{\text{gr}}\}$ .

The following result collects some of the properties of varieties from [50]. For ease of notation, we write  $V(M)$  for  $V_{\text{HH}^*(\Lambda)}(M)$ . We also denote the kernel of the projective cover of  $M \in \text{mod } \Lambda$  by  $\Omega_{\Lambda}(M)$ .

Recall that we assume throughout this paper that  $K$  is an algebraically closed field. This assumption is a necessary assumption in many of the results in this article. However, it is not needed in all of [50], and the interested reader may refer back to [50] to see precisely what assumptions are required there at each stage.

**Theorem 1.1.** ([50, Propositions 3.4, 3.7]) *Let  $M \in \text{mod } \Lambda$ .*

- (1)  $V(M) = V(\Omega_{\Lambda}(M))$  if  $\Omega_{\Lambda}(M) \neq (0)$ ,
- (2)  $V(M_1 \oplus M_2) = V(M_1) \cup V(M_2)$ ,
- (3) If  $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$  is an exact sequence, then  $V(M_{i_1}) \subseteq V(M_{i_2}) \cup V(M_{i_3})$  whenever  $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ ,
- (4) If  $\text{Ext}_{\Lambda}^i(M, M) = (0)$  for  $i \gg 0$ , or the projective or the injective dimension of  $M$  is finite, then the variety of  $M$  is trivial.
- (5) If  $\Lambda$  is selfinjective then  $V(M) = V(\tau M)$ , where  $\tau$  is the Auslander-Reiten translate. Hence all modules in a connected stable component of the Auslander-Reiten quiver have the same variety.

For a finitely generated module  $M$  over a group algebra of a finite group  $G$ , it is well-known ([10]) that the variety of  $M$  is trivial if and only if  $M$  is a projective module. In

contrast, it is still an open question as to what are the appropriate necessary and sufficient conditions on a module for it to have trivial variety in the more general case where  $\Lambda$  is an arbitrary finite-dimensional algebra. There are some partial results for a particular class of monomial algebras in [25] (see Section 3). Nevertheless, the converse to Theorem 1.1(4) does not hold in general, and a counterexample may be found in [52, Example 4.7].

However, this question was successfully answered by Erdmann, Holloway, Snashall, Solberg and Taillefer in [15], by placing some (reasonable) additional assumptions on  $\Lambda$ . (Recall that we are already assuming that the field  $K$  is algebraically closed.) Specifically, the following two finiteness conditions were introduced.

**(Fg1)**  $H$  is a commutative Noetherian graded subalgebra of  $\mathrm{HH}^*(\Lambda)$  with  $H^0 = \mathrm{HH}^0(\Lambda)$ .

**(Fg2)**  $\mathrm{Ext}_\Lambda^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$  is a finitely generated  $H$ -module.

As remarked in [15], these two conditions together imply that both  $\mathrm{HH}^*(\Lambda)$  and  $\mathrm{Ext}_\Lambda^*(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$  are finitely generated  $K$ -algebras. In particular, the properties **(Fg1)** and **(Fg2)** hold where  $\Lambda = KG$ ,  $G$  is a finite group, and  $H = \mathrm{HH}^{\mathrm{ev}}(\Lambda)$  ([21, 54]). With conditions **(Fg1)** and **(Fg2)**, we have the following results from [15], where we define the variety using the subalgebra  $H$  of  $\mathrm{HH}^*(\Lambda)$ , so that  $V_H(M) = \mathrm{MaxSpec}(H/\mathrm{Ann}_H \mathrm{Ext}_\Lambda^*(M, M))$ .

**Theorem 1.2.** ([15, Theorem 2.5]) *Suppose that  $\Lambda$  and  $H$  satisfy **(Fg1)** and **(Fg2)**. Then  $\Lambda$  is Gorenstein. Moreover the following are equivalent for  $M \in \mathrm{mod} \Lambda$ :*

- (i) *The variety of  $M$  is trivial;*
- (ii)  *$M$  has finite projective dimension;*
- (iii)  *$M$  has finite injective dimension.*

**Theorem 1.3.** ([15, Theorem 4.4]) *Suppose that  $\Lambda$  and  $H$  satisfy **(Fg1)** and **(Fg2)**. Given a homogeneous ideal  $\mathfrak{a}$  in  $H$ , there is a module  $M \in \mathrm{mod} \Lambda$  such that  $V_H(M) = V_H(\mathfrak{a})$ .*

**Theorem 1.4.** ([15, Theorem 2.5 and Propositions 5.2, 5.3]) *Suppose that  $\Lambda$  and  $H$  satisfy **(Fg1)** and **(Fg2)** and that  $\Lambda$  is selfinjective. Let  $M \in \mathrm{mod} \Lambda$  be indecomposable.*

- (1)  *$V_H(M)$  is trivial  $\Leftrightarrow M$  is projective.*
- (2)  *$V_H(M)$  is a line  $\Leftrightarrow M$  is  $\Omega$ -periodic.*

Our final results in this section concern the representation type of  $\Lambda$  and the structure of the Auslander-Reiten quiver; for more details see [15, 52]. First we recall that Heller showed that if  $\Lambda$  is of finite representation type then the complexity of a finitely generated module is at most 1 ([40]), and that Rickard showed that if  $\Lambda$  is of tame representation type then the complexity of a finitely generated module is at most 2 ([47]). However there are selfinjective preprojective algebras of wild representation type where all indecomposable modules are either projective or periodic and so have complexity at most 1. Nevertheless, the next result uses the Hochschild cohomology ring to give some information on the representation type of an algebra.

**Theorem 1.5.** ([15, Proposition 6.1]) *Suppose that  $\Lambda$  and  $H$  satisfy **(Fg1)** and **(Fg2)** and that  $\Lambda$  is selfinjective. Suppose also that  $\dim H \geq 2$ . Then  $\Lambda$  is of infinite representation type, and  $\Lambda$  has an infinite number of indecomposable periodic modules lying in infinitely many different components of the stable Auslander-Reiten quiver.*

We end this section with the statement of Webb's theorem ([55]) for group algebras of finite groups and a generalisation of this theorem from [15].

**Theorem 1.6.** ([55]) *Let  $G$  be a finite group and suppose that  $\text{char } K$  divides  $|G|$ . Then the orbit graph of a connected component of the stable Auslander-Reiten quiver of  $KG$  is one of the following:*

- (a) *a finite Dynkin diagram  $(\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8})$ ,*
- (b) *a Euclidean diagram  $(\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_{6,7,8}, \tilde{\mathbb{A}}_{12})$ , or*
- (c) *an infinite Dynkin diagram of type  $\mathbb{A}_\infty, \mathbb{D}_\infty$  or  $\mathbb{A}_\infty^\infty$ .*

**Theorem 1.7.** ([15, Theorem 5.6]) *Suppose that  $\Lambda$  and  $H$  satisfy **(Fg1)** and **(Fg2)** and that  $\Lambda$  is selfinjective. Suppose that the Nakayama functor is of finite order on any indecomposable module in  $\text{mod } \Lambda$ . Then the tree class of a component of the stable Auslander-Reiten quiver of  $\Lambda$  is one of the following:*

- (a) *a finite Dynkin diagram  $(\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_{6,7,8})$ ,*
- (b) *a Euclidean diagram  $(\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_{6,7,8}, \tilde{\mathbb{A}}_{12})$ , or*
- (c) *an infinite Dynkin diagram of type  $\mathbb{A}_\infty, \mathbb{D}_\infty$  or  $\mathbb{A}_\infty^\infty$ .*

We remark that the hypotheses of Theorem 1.7 are satisfied for all finite-dimensional cocommutative Hopf algebras ([15, Corollary 5.7]).

For more information, the reader should also see the survey paper on support varieties for modules and complexes by Solberg [52]. In addition, the paper by Bergh [5] introduces the concept of a twisted support variety for a finitely generated module over an artin algebra, where the twist is induced by an automorphism of the algebra, and, in [6], Bergh and Solberg study relative support varieties for finitely generated modules over a finite-dimensional algebra over a field.

## 2. $\Omega$ -PERIODIC ALGEBRAS

We now turn our attention to the structure of the Hochschild cohomology ring. One class of algebras where it is relatively straightforward to determine the structure of the Hochschild cohomology ring explicitly is the class of  $\Omega$ -periodic algebras. We recall that  $\Lambda$  is said to be an  $\Omega$ -periodic algebra if there exists some  $n \geq 1$  such that  $\Omega_{\Lambda^e}^n(\Lambda) \cong \Lambda$  as bimodules. Such an algebra  $\Lambda$  has a periodic minimal projective bimodule resolution, so that  $\text{HH}^i(\Lambda) \cong \text{HH}^{n+i}(\Lambda)$  for  $i \geq 1$ , and is necessarily self-injective (Butler; see [34]).

There is an extensive survey of periodic algebras by Erdmann and Skowroński in [18]. Examples of such algebras include the preprojective algebras of Dynkin type where  $\Omega_{\Lambda^e}^6(\Lambda) \cong \Lambda$  as bimodules ([49]; see also [19]), and the deformed mesh algebras of generalized Dynkin type of Białkowski, Erdmann and Skowroński [7, 18]. For the selfinjective algebras of finite representation type over an algebraically closed field, it is known from [34] that there is some  $n \geq 1$  and automorphism  $\sigma$  of  $\Lambda$  such that  $\Omega_{\Lambda^e}^n(\Lambda)$  is isomorphic as a bimodule to the twisted bimodule  ${}_1\Lambda_\sigma$ . It has now been shown that all selfinjective algebras of finite representation type over an algebraically closed field are  $\Omega$ -periodic ([13, 16, 17, 18]).

The structure of the Hochschild cohomology ring modulo nilpotence of these algebras was determined by Green, Snashall and Solberg in [34].

**Theorem 2.1.** ([34, Theorem 1.6]) *Let  $K$  be an algebraically closed field. Let  $\Lambda$  be a finite-dimensional indecomposable  $K$ -algebra such that there is some  $n \geq 1$  and some automorphism  $\sigma$  of  $\Lambda$  such that  $\Omega_{\Lambda^e}^n(\Lambda)$  is isomorphic to the twisted bimodule  ${}_1\Lambda_\sigma$ . Then*

$$\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong \begin{cases} K[x] & \text{or} \\ K. \end{cases}$$

*If there is some  $m \geq 1$  such that  $\Omega_{\Lambda^e}^m(\Lambda) \cong \Lambda$  as bimodules, then  $\mathrm{HH}^*(\Lambda)/\mathcal{N} \cong K[x]$ , where  $x$  is in degree  $m$  and  $m$  is minimal.*

Additional information on the ring structure of the Hochschild cohomology ring of the preprojective algebras of Dynkin type  $A_n$  was determined in [19], of the preprojective algebras of Dynkin types  $D_n, E_6, E_7, E_8$  in [20], and of the selfinjective algebras of finite representation type  $A_n$  over an algebraically closed field in [16, 17].

Given that the Hochschild cohomology ring of these algebras is understood, this naturally leads to the study of situations where the Hochschild cohomology rings of two algebras  $A$  and  $B$  can be related. This enables us to transfer information about the Hochschild cohomology ring of, say, an  $\Omega$ -periodic algebra, to other algebras. Apart from periodic algebras, there are other algebras where the Hochschild cohomology ring is known, and these provide additional examples where the transfer of properties between Hochschild cohomology rings may also be studied. One such class of examples is the class of truncated quiver algebras, which has been extensively studied in the literature by many authors.

Happel showed in [39, Theorem 5.3] that if  $B$  is a one-point extension of a finite-dimensional  $K$ -algebra  $A$  by a finitely generated  $A$ -module  $M$ , then there is a long exact sequence connecting the Hochschild cohomology rings of  $A$  and  $B$ :

$$\begin{aligned} 0 \rightarrow \mathrm{HH}^0(B) \rightarrow \mathrm{HH}^0(A) \rightarrow \mathrm{Hom}_A(M, M)/K \rightarrow \\ \mathrm{HH}^1(B) \rightarrow \mathrm{HH}^1(A) \rightarrow \mathrm{Ext}_A^1(M, M) \rightarrow \dots \\ \dots \rightarrow \mathrm{Ext}_A^i(M, M) \rightarrow \mathrm{HH}^{i+1}(B) \rightarrow \mathrm{HH}^{i+1}(A) \rightarrow \mathrm{Ext}_A^{i+1}(M, M) \rightarrow \dots \end{aligned}$$

It was subsequently shown by Green, Marcos and Snashall in [29, Theorem 5.1] that there is a graded ring homomorphism

$$\mathrm{HH}^*(B) \rightarrow \mathrm{HH}^*(A) \oplus \mathcal{K}$$

which induces this long exact sequence, where  $\mathcal{K}$  is the graded  $K$ -module with  $\mathcal{K}_0 = K$  and  $\mathcal{K}_n = 0$  for all  $n \neq 0$ .

These results were generalized independently to arbitrary triangular matrix algebras by Cibils [12], by Green and Solberg [36], and by Michelena and Platzeck [44].

A recent result of König and Nagase, ([42, 45]), has related the Hochschild cohomology ring of  $B$  to that of  $B/BeB$  in the case where  $B$  is an algebra with idempotent  $e$ , such that  $BeB$  is a stratifying ideal of  $B$ .

**Theorem 2.2.** ([42]) *Let  $B$  be an algebra with idempotent  $e$  such that  $BeB$  is a stratifying ideal of  $B$  and let  $A$  be the factor algebra  $B/BeB$ . Then there are long exact sequences as follows:*

1.  $\cdots \rightarrow \text{Ext}_{Be}^n(B, BeB) \rightarrow \text{HH}^n(B) \rightarrow \text{HH}^n(A) \rightarrow \cdots$ ;
2.  $\cdots \rightarrow \text{Ext}_{Be}^n(A, B) \rightarrow \text{HH}^n(B) \rightarrow \text{HH}^n(eBe) \rightarrow \cdots$ ; and
3.  $\cdots \rightarrow \text{Ext}_{Be}^n(A, BeB) \rightarrow \text{HH}^n(B) \rightarrow \text{HH}^n(A) \oplus \text{HH}^n(eBe) \rightarrow \cdots$ .

### 3. THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE

The definition of a support variety in [50] led us to consider the structure of  $\text{HH}^*(\Lambda)/\mathcal{N}$  and to conjecture that  $\text{HH}^*(\Lambda)/\mathcal{N}$  is always finitely generated as an algebra. A counterexample to this conjecture was recently given by Xu in [57]; nevertheless the Hochschild cohomology ring modulo nilpotence is finitely generated as an algebra for many diverse classes of algebras.

The Hochschild cohomology ring modulo nilpotence is known to be finitely generated as an algebra in the following cases.

- any block of a group ring of a finite group ([21, 54]);
- any block of a finite-dimensional cocommutative Hopf algebra ([24]);
- finite-dimensional selfinjective algebras of finite representation type over an algebraically closed field ([34]);
- finite-dimensional monomial algebras ([35] and see [32]);
- finite-dimensional algebras of finite global dimension (see [39]).

For the last class of examples, if  $\Lambda$  is an algebra of finite global dimension  $N$ , then  $\text{HH}^i(\Lambda) = \text{Ext}_{\Lambda^e}^i(\Lambda, \Lambda) = (0)$  for all  $i > N$ . Hence  $\text{HH}^*(\Lambda)/\mathcal{N} \cong K$ . In [39], Happel asked whether or not it was true, for a finite-dimensional algebra  $\Gamma$  over a field  $K$ , that if  $\text{HH}^n(\Gamma) = (0)$  for  $n \gg 0$  then the global dimension of  $\Gamma$  is finite. This question has now been answered in the negative by Buchweitz, Green, Madsen and Solberg in [8] by the following example.

**Example 3.1.** ([8]) Let

$$\Lambda_q = K\langle x, y \rangle / (x^2, xy + qyx, y^2)$$

with  $q \in K \setminus \{0\}$ . If  $q$  is not a root of unity then  $\dim \text{HH}^i(\Lambda_q) = 0$  for  $i \geq 3$ . Moreover,  $\Lambda_q$  is a selfinjective algebra so has infinite global dimension.

We also note from [8] that  $\dim \Lambda_q = 4$ ,  $\dim \text{HH}^*(\Lambda_q) = 5$  and  $\text{HH}^*(\Lambda)/\mathcal{N} \cong K$ .

However, the situation for commutative algebras is very different, as Avramov and Iyengar have shown.

**Theorem 3.2.** ([1]) *Let  $R$  be a commutative finite-dimensional  $K$ -algebra over a field  $K$ . If  $\text{HH}^n(R) = (0)$  for  $n \gg 0$  then  $R$  is a (finite) product of (finite) separable field extensions of  $K$ . In particular, the global dimension of  $R$  is finite.*

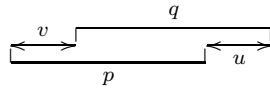
We now turn to a brief discussion of the Hochschild cohomology ring modulo nilpotence for a monomial algebra, which was studied by Green, Snashall and Solberg. Let  $\Lambda$  be a quotient of a path algebra so that  $\Lambda = K\mathcal{Q}/I$  for some quiver  $\mathcal{Q}$  and admissible ideal  $I$  of  $K\mathcal{Q}$ . Then  $\Lambda = K\mathcal{Q}/I$  is a monomial algebra if the ideal  $I$  is generated by monomials of length at least two. It should be noted that monomial algebras are very rarely selfinjective and so do not usually exhibit the same properties as group algebras. However, the

Hochschild cohomology ring modulo nilpotence of a monomial algebra turns out to have a particularly nice structure.

**Theorem 3.3.** ([35, Theorem 7.1]) *Let  $\Lambda = K\mathcal{Q}/I$  be a finite-dimensional indecomposable monomial algebra. Then  $\mathrm{HH}^*(\Lambda)/\mathcal{N}$  is a commutative finitely generated  $K$ -algebra of Krull dimension at most one.*

For some specific subclasses of monomial algebras, the structure of the Hochschild cohomology ring modulo nilpotence was explicitly determined in [32]. One of the main tools used was the minimal projective bimodule resolution of a monomial algebra of Bardzell [2]. In order to define the particular class of  $(D, A)$ -stacked monomial algebras, we require the concept of overlaps of [27, 38]; the definitions here use the notation of [32].

**Definition 3.4.** A path  $q$  in  $K\mathcal{Q}$  overlaps a path  $p$  in  $K\mathcal{Q}$  with overlap  $pu$  if there are paths  $u$  and  $v$  such that  $pu = vq$  and  $1 \leq \ell(u) < \ell(q)$ , where  $\ell(x)$  denotes the length of the path  $x \in K\mathcal{Q}$ . We illustrate the definition with the following diagram. (Note that we allow  $\ell(v) = 0$  here.)



A path  $q$  properly overlaps a path  $p$  with overlap  $pu$  if  $q$  overlaps  $p$  and  $\ell(v) \geq 1$ .

Let  $\Lambda = K\mathcal{Q}/I$  be a finite-dimensional monomial algebra where  $I$  has a minimal set of generators  $\rho$  of paths of length at least 2. We fix this set  $\rho$  and now recursively define sets  $\mathcal{R}^n$  (contained in  $K\mathcal{Q}$ ). Let

$$\begin{aligned} \mathcal{R}^0 &= \text{the set of vertices of } \mathcal{Q}, \\ \mathcal{R}^1 &= \text{the set of arrows of } \mathcal{Q}, \\ \mathcal{R}^2 &= \rho. \end{aligned}$$

For  $n \geq 3$ , we say  $R^2 \in \mathcal{R}^2$  maximally overlaps  $R^{n-1} \in \mathcal{R}^{n-1}$  with overlap  $R^n = R^{n-1}u$  if

1.  $R^{n-1} = R^{n-2}p$  for some path  $p$ ;
2.  $R^2$  overlaps  $p$  with overlap  $pu$ ;
3. there is no element of  $\mathcal{R}^2$  which overlaps  $p$  with overlap being a proper prefix of  $pu$ .

The set  $\mathcal{R}^n$  is defined to be the set of all overlaps  $R^n$  formed in this way.

Each of the elements in  $\mathcal{R}^n$  is a path in the quiver  $\mathcal{Q}$ . We follow the convention that paths are written from left to right. For an arrow  $\alpha$  in the quiver  $\mathcal{Q}$ , we write  $\mathfrak{o}(\alpha)$  for the idempotent corresponding to the origin of  $\alpha$  and  $\mathfrak{t}(\alpha)$  for the idempotent corresponding to the tail of  $\alpha$  so that  $\alpha = \mathfrak{o}(\alpha)\alpha\mathfrak{t}(\alpha)$ . For a path  $p = \alpha_1\alpha_2 \cdots \alpha_m$ , we write  $\mathfrak{o}(p) = \mathfrak{o}(\alpha_1)$  and  $\mathfrak{t}(p) = \mathfrak{t}(\alpha_m)$ .

The importance of these sets  $\mathcal{R}^n$  lies in the fact that, for a finite-dimensional monomial algebra  $\Lambda = K\mathcal{Q}/I$ , [2] uses them to give an explicit construction of a minimal projective bimodule resolution  $(P^*, \delta^*)$  of  $\Lambda$ , showing that

$$P^n = \bigoplus_{R^n \in \mathcal{R}^n} \Lambda \mathfrak{o}(R^n) \otimes_K \mathfrak{t}(R^n) \Lambda.$$

We now use these same sets  $\mathcal{R}^n$  to define a  $(D, A)$ -stacked monomial algebra.



**Definition 3.5.** ([32, Definition 3.1]) Let  $\Lambda = K\mathcal{Q}/I$  be a finite-dimensional monomial algebra, where  $I$  is an admissible ideal with minimal set of generators  $\rho$ . Then  $\Lambda$  is said to be a  $(D, A)$ -stacked monomial algebra if there is some  $D \geq 2$  and  $A \geq 1$  such that, for all  $n \geq 2$  and  $R^n \in \mathcal{R}^n$ ,

$$\ell(R^n) = \begin{cases} \frac{n}{2}D & \text{if } n \text{ even,} \\ \frac{(n-1)}{2}D + A & \text{if } n \text{ odd.} \end{cases}$$

In particular all relations in  $\rho$  are of length  $D$ .

The class of  $(D, A)$ -stacked monomial algebras includes the Koszul monomial algebras (equivalently, the quadratic monomial algebras) and the  $D$ -Koszul monomial algebras of Berger ([4]). Recall that the Ext algebra  $E(\Lambda)$  of  $\Lambda$  is defined by  $E(\Lambda) = \text{Ext}_\Lambda^*(\Lambda/\mathfrak{t}, \Lambda/\mathfrak{t})$ . It is well-known that the Ext algebra of a Koszul algebra is generated in degrees 0 and 1; moreover the Ext algebra of a  $D$ -Koszul algebra is generated in degrees 0, 1 and 2 ([28]). It was shown by Green and Snashall in [33, Theorem 3.6] that, for algebras of infinite global dimension, the class of  $(D, A)$ -stacked monomial algebras is precisely the class of monomial algebras  $\Lambda$  where each projective module in the minimal projective resolution of  $\Lambda/\mathfrak{t}$  as a right  $\Lambda$ -module is generated in a single degree and where the Ext algebra of  $\Lambda$  is finitely generated as a  $K$ -algebra. It was also shown, for a  $(D, A)$ -stacked monomial algebra of infinite global dimension, that the Ext algebra is generated in degrees 0, 1, 2 and 3.

**Theorem 3.6.** [32] *Let  $\Lambda = K\mathcal{Q}/I$  be a finite-dimensional  $(D, A)$ -stacked monomial algebra, where  $I$  is an admissible ideal with minimal set of generators  $\rho$ . Suppose  $\text{char } K \neq 2$  and  $\text{gldim } \Lambda \geq 4$ . Then there is some integer  $r \geq 0$  such that*

$$\text{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \dots, x_r]/\langle x_i x_j \text{ for } i \neq j \rangle.$$

*Moreover the degrees of the  $x_i$  and the value of the parameter  $r$  may be explicitly and easily calculated.*

We do not give the full details of the  $x_i$  and the parameter  $r$  here; they may be found in [32]. However, it is worth remarking that, given any integer  $r \geq 0$  and even integers  $n_1, \dots, n_r$ , there is a finite-dimensional  $(D, A)$ -stacked monomial algebra  $\Lambda$  with

$$\text{HH}^*(\Lambda)/\mathcal{N} \cong K[x_1, \dots, x_r]/\langle x_i x_j \text{ for } i \neq j \rangle$$

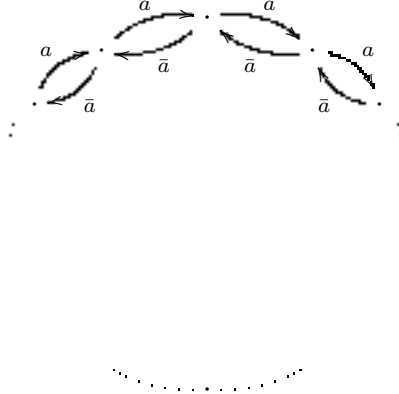
where the degree of  $x_i$  is  $n_i$ , for all  $i = 1, \dots, r$ .

In [25], necessary and sufficient conditions are given for a simple module over a  $(D, A)$ -stacked monomial algebra to have trivial variety. Referring back to Theorem 1.1(4), this goes part way to determining necessary and sufficient conditions on any finitely generated module for it to have trivial variety for this class of algebras.

We end this section with a class of selfinjective special biserial algebras  $\Lambda_N$ , for  $N \geq 1$ , studied by Snashall and Taillefer in [51]. The study of these algebras was motivated by the results of [14] where the algebras  $\Lambda_1$  arose in the presentation by quiver and relations of the Drinfeld double  $\mathcal{D}(\Lambda_{n,d})$  of the Hopf algebra  $\Lambda_{n,d}$  where  $d|n$ . The algebra  $\Lambda_{n,d}$  is given by an oriented cycle with  $n$  vertices such that all paths of length  $d$  are zero. These

algebras also occur in the study of the representation theory of  $U_q(\mathfrak{sl}_2)$ ; see work of Patra ([46]), Suter ([53]), Xiao ([56]), and also of Chin and Krop ([11]). The more general algebras  $\Lambda_N$  occur in work of Farnsteiner and Skowroński [22, 23], where they determine the Hopf algebras associated to infinitesimal groups whose principal blocks are tame when  $K$  is an algebraically closed field with  $\text{char } K \geq 3$ .

Our class of selfinjective special biserial algebras  $\Lambda_N$  is described as follows. Firstly, for  $m \geq 1$ , let  $\mathcal{Q}$  be the quiver with  $m$  vertices, labelled  $0, 1, \dots, m-1$ , and  $2m$  arrows as follows:



Let  $a_i$  denote the arrow that goes from vertex  $i$  to vertex  $i+1$ , and let  $\bar{a}_i$  denote the arrow that goes from vertex  $i+1$  to vertex  $i$ , for each  $i = 0, \dots, m-1$  (with the obvious conventions modulo  $m$ ). Then, for  $N \geq 1$ , we define  $\Lambda_N$  to be the algebra given by  $\Lambda_N = K\mathcal{Q}/I_N$  where  $I_N$  is the ideal of  $K\mathcal{Q}$  generated by

$$a_i a_{i+1}, \quad \bar{a}_{i-1} \bar{a}_{i-2}, \quad (a_i \bar{a}_i)^N - (\bar{a}_{i-1} a_{i-1})^N, \quad \text{for } i = 0, 1, \dots, m-1,$$

and where the subscripts are taken modulo  $m$ . We note that, if  $N = 1$ , then the algebra  $\Lambda_1$  is a Koszul algebra. (We continue to write paths from left to right.)

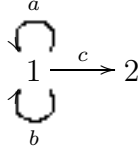
**Theorem 3.7.** ([51, Theorem 8.1]) *For  $m \geq 1$  and  $N \geq 1$ , let  $\Lambda_N$  be as defined above. Then  $\text{HH}^*(\Lambda_N)$  is a finitely generated  $K$ -algebra. Moreover  $\text{HH}^*(\Lambda_N)/\mathcal{N}$  is a commutative finitely generated  $K$ -algebra of Krull dimension two.*

Furthermore, if  $N = 1$  then [51] also showed that the conditions **(Fg1)** and **(Fg2)** hold with  $H = \text{HH}^{\text{ev}}(\Lambda_1)$ .

#### 4. COUNTEREXAMPLE TO THE CONJECTURE OF [50]

The previous section concerned algebras where the conjecture of [50] concerning the finite generation of the Hochschild cohomology ring modulo nilpotence has been shown to hold. In this section we present a counterexample to the conjecture of [50]. In [57], Xu gave a counterexample in the case where the field  $K$  has characteristic 2. It can easily be seen, for  $\text{char } K = 2$ , that the category algebra he presented in [57] is isomorphic to the following algebra  $\mathcal{A}$  given as a quotient of a path algebra. Moreover, we will show that this algebra  $\mathcal{A}$  provides a counterexample to the conjecture irrespective of the characteristic of the field.

**Example 4.1.** Let  $K$  be any field and let  $\mathcal{A} = K\mathcal{Q}/I$  where  $\mathcal{Q}$  is the quiver



and  $I = \langle a^2, b^2, ab - ba, ac \rangle$ .

The rest of this section is devoted to studying this algebra  $\mathcal{A}$  and to showing that  $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$  is not finitely generated as an algebra.

We begin by giving an explicit minimal projective resolution  $(P^*, d^*)$  for  $\mathcal{A}$  as an  $\mathcal{A}, \mathcal{A}$ -bimodule. The description of the resolution given here is motivated by [31] where the first terms of a minimal projective bimodule resolution of a finite-dimensional quotient of a path algebra were determined explicitly from the minimal projective resolution of  $\Lambda/\mathfrak{r}$  as a right  $\Lambda$ -module of Green, Solberg and Zacharia in [37]. This same technique for constructing a minimal projective bimodule resolution was used in [26] for any Koszul algebra, and in [51] for the algebras  $\Lambda_N$  which were discussed at the end of Section 3.

From Happel [39], we know that the multiplicity of  $\Lambda e_i \otimes_K e_j \Lambda$  as a direct summand of  $P^n$  is equal to  $\dim \mathrm{Ext}_\Lambda^n(S_i, S_j)$ , where  $S_i$  is the simple  $\mathcal{A}$ -module corresponding to the vertex  $i$  of  $\mathcal{Q}$ . Following [31, 37], we start by defining sets  $g^n$  in  $K\mathcal{Q}$  inductively, and then labelling the summands of  $P^n$  by the elements of  $g^n$ . The set  $g^0$  is determined by the vertices of  $\mathcal{Q}$ , the set  $g^1$  by the arrows of  $\mathcal{Q}$ , and the set  $g^2$  by a minimal generating set of the ideal  $I$ .

Let

$$\begin{aligned} g^0 &= \{g_0^0 = e_1, g_1^0 = e_2\}, \\ g^1 &= \{g_0^1 = a, g_1^1 = -b, g_2^1 = c\}, \\ g^2 &= \{g_0^2 = a^2, g_1^2 = ab - ba, g_2^2 = -b^2, g_3^2 = ac\}. \end{aligned}$$

For  $n \geq 3$  and  $r = 0, 1, \dots, n$ , let

$$g_r^n = \sum_p (-1)^s p$$

where the sum is over all paths  $p$  of length  $n$ , written  $p = \alpha_1 \alpha_2 \cdots \alpha_n$  where the  $\alpha_i$  are arrows in  $\mathcal{Q}$ , such that

- (i)  $p$  contains  $n - r$  arrows equal to  $a$  and  $r$  arrows equal to  $b$ , and
- (ii)  $s = \sum_{\alpha_j = b} j$ .

In addition, for  $n \geq 3$ , define

$$g_{n+1}^n = a^{n-1}c.$$

For  $r = 0, 1, \dots, n$ , we have that  $g_r^n = e_1 g_r^n e_1$  so we define  $\mathfrak{o}(g_r^n) = e_1 = \mathfrak{t}(g_r^n)$ . Moreover  $\mathfrak{o}(g_{n+1}^n) = e_1$  and  $\mathfrak{t}(g_{n+1}^n) = e_2$ . Thus

$$P^n = \bigoplus_{r=0}^{n+1} \mathcal{A} \mathfrak{o}(g_r^n) \otimes_K \mathfrak{t}(g_r^n) \mathcal{A}.$$

To describe the map  $d^n: P^n \rightarrow P^{n-1}$ , we first need to write each of the elements  $g_r^n$  in terms of the elements of the set  $g^{n-1}$ , that is, in terms of  $g_0^{n-1}, \dots, g_n^{n-1}$ . The following result is straightforward to verify.

**Proposition 4.2.** *Suppose  $n \geq 2$ . Then, keeping the above notation,*

$$\begin{aligned} g_0^n &= g_0^{n-1}a &&= ag_0^{n-1} \\ g_r^n &= g_r^{n-1}a + (-1)^n g_{r-1}^{n-1}b &&= (-1)^r (ag_r^{n-1} + bg_{r-1}^{n-1}) \quad \text{for } 1 \leq r \leq n-1 \\ g_n^n &= (-1)^n g_{n-1}^{n-1}b &&= (-1)^n bg_{n-1}^{n-1} \\ g_{n+1}^n &= g_0^{n-1}c &&= ag_n^{n-1}. \end{aligned}$$

We define the map  $d^0: P^0 \rightarrow \mathcal{A}$  to be the multiplication map. To define  $d^n$  for  $n \geq 1$ , we need one further piece of notation. In describing the image  $d^n(\mathfrak{o}(g_r^n) \otimes \mathfrak{t}(g_r^n))$  in the projective module  $P^{n-1}$ , we use a subscript under  $\otimes$  to indicate the appropriate summand of the projective module  $P^{n-1}$ . Specifically, let  $- \otimes_r -$  denote a term in the summand of  $P^{n-1}$  corresponding to  $g_r^{n-1}$ . Nonetheless all tensors are over  $K$ ; however, to simplify notation, we omit the subscript  $K$ . The maps  $d^n: P^n \rightarrow P^{n-1}$  for  $n \geq 1$  may now be defined. The proof of the following result is omitted and is similar to those in [31, Proposition 2.8] and [51, Theorem 1.6].

**Theorem 4.3.** *For the algebra  $\mathcal{A}$  of Example 4.1, the sequence  $(P^*, d^*)$  is a minimal projective resolution of  $\mathcal{A}$  as an  $\mathcal{A}, \mathcal{A}$ -bimodule, where, for  $n \geq 0$ ,*

$$P^n = \bigoplus_{r=0}^{n+1} \mathcal{A} \mathfrak{o}(g_r^n) \otimes \mathfrak{t}(g_r^n) \mathcal{A},$$

the map  $d^0: P^0 \rightarrow \mathcal{A}$  is the multiplication map, the map  $d^1: P^1 \rightarrow P^0$  is given by  $d^1(\mathfrak{o}(g_r^1) \otimes \mathfrak{t}(g_r^1)) =$

$$\begin{cases} \mathfrak{o}(g_0^1) \otimes_0 a - a \otimes_0 \mathfrak{t}(g_0^1) & \text{for } r = 0; \\ -\mathfrak{o}(g_1^1) \otimes_0 b + b \otimes_0 \mathfrak{t}(g_1^1) & \text{for } r = 1; \\ \mathfrak{o}(g_2^1) \otimes_0 c - c \otimes_1 \mathfrak{t}(g_2^1) & \text{for } r = 2; \end{cases}$$

and, for  $n \geq 2$ , the map  $d^n: P^n \rightarrow P^{n-1}$  is given by  $d^n(\mathfrak{o}(g_r^n) \otimes \mathfrak{t}(g_r^n)) =$

$$\begin{cases} \mathfrak{o}(g_0^n) \otimes_0 a + (-1)^n a \otimes_0 \mathfrak{t}(g_0^n) & \text{for } r = 0; \\ \mathfrak{o}(g_r^n) \otimes_r a + (-1)^n \mathfrak{o}(g_r^n) \otimes_{r-1} b \\ \quad + (-1)^{r+n} (a \otimes_r \mathfrak{t}(g_r^n) + b \otimes_{r-1} \mathfrak{t}(g_r^n)) & \text{for } 1 \leq r \leq n-1; \\ (-1)^n \mathfrak{o}(g_n^n) \otimes_{n-1} b + b \otimes_{n-1} \mathfrak{t}(g_n^n) & \text{for } r = n; \\ \mathfrak{o}(g_{n+1}^n) \otimes_0 c + (-1)^n a \otimes_n \mathfrak{t}(g_{n+1}^n) & \text{for } r = n+1. \end{cases}$$

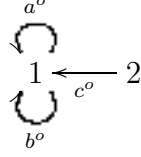
**Proposition 4.4.** *The algebra  $\mathcal{A}$  of Example 4.1 is a Koszul algebra.*

*Proof.* We apply the functor  $\mathcal{A}/\mathfrak{t} \otimes_{\mathcal{A}} -$  to the resolution  $(P^*, d^*)$  of Theorem 4.3 to give a (minimal) projective resolution of  $\mathcal{A}/\mathfrak{t}$  as a right  $\mathcal{A}$ -module. Thus  $\mathcal{A}/\mathfrak{t}$  has a linear projective resolution, and so  $\mathcal{A}$  is a Koszul algebra.  $\square$

Since,  $\mathcal{A}$  is a Koszul algebra, it now follows from [9] that the image of the ring homomorphism  $\phi_{\mathcal{A}/\mathfrak{t}} = \mathcal{A}/\mathfrak{t} \otimes_{\mathcal{A}} - : \text{HH}^*(\mathcal{A}) \rightarrow E(\mathcal{A})$  is the graded centre  $Z_{\text{gr}}(E(\mathcal{A}))$  of  $E(\mathcal{A})$ , where  $Z_{\text{gr}}(E(\mathcal{A}))$  is the subalgebra generated by all homogeneous elements  $z$  such that  $zg = (-1)^{|g||z|}gz$  for all  $g \in E(\mathcal{A})$ . Thus  $\phi_{\mathcal{A}/\mathfrak{t}}$  induces an isomorphism

$\mathrm{HH}^*(\mathcal{A})/\mathcal{N} \cong Z_{\mathrm{gr}}(E(\mathcal{A}))/\mathcal{N}_Z$ , where  $\mathcal{N}_Z$  denotes the ideal of  $Z_{\mathrm{gr}}(E(\mathcal{A}))$  generated by all homogeneous nilpotent elements.

From [30, Theorem 2.2],  $E(\mathcal{A})$  is the Koszul dual of  $\mathcal{A}$  and is given explicitly by quiver and relations as  $E(\mathcal{A}) \cong K\mathcal{Q}^{\mathrm{op}}/I^\perp$ , where  $\mathcal{Q}$  is the quiver of  $\mathcal{A}$  and  $I^\perp$  is the ideal generated by the orthogonal relations to those of  $I$ . Specifically, for this example,  $E(\mathcal{A})$  has quiver



and  $I^\perp = \langle a^o b^o + b^o a^o, b^o c^o \rangle$ , where, for an arrow  $\alpha \in \mathcal{Q}$ , we denote by  $\alpha^o$  the corresponding arrow in  $\mathcal{Q}^{\mathrm{op}}$ . Moreover, the left modules over  $E(\mathcal{A})$  are the right modules over  $K\mathcal{Q}/\langle ab + ba, bc \rangle$ .

It is now easy to calculate  $Z_{\mathrm{gr}}(E(\mathcal{A}))$  to give the following theorem. The structure of  $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$  for  $\mathrm{char} K = 2$  was given by Xu in [57].

**Theorem 4.5.** *Let  $\mathcal{A}$  be the algebra of Example 4.1.*

1.

$$Z_{\mathrm{gr}}(E(\mathcal{A})) \cong \begin{cases} K \oplus K[a, b]b & \text{if } \mathrm{char} K = 2 \\ K \oplus K[a^2, b^2]b^2 & \text{if } \mathrm{char} K \neq 2, \end{cases}$$

where  $b$  is in degree 1 and  $ab$  is in degree 2.

2.

$$\mathrm{HH}^*(\mathcal{A})/\mathcal{N} \cong \begin{cases} K \oplus K[a, b]b & \text{if } \mathrm{char} K = 2 \\ K \oplus K[a^2, b^2]b^2 & \text{if } \mathrm{char} K \neq 2, \end{cases}$$

where  $b$  is in degree 1 and  $ab$  is in degree 2.

3.  $\mathrm{HH}^*(\mathcal{A})/\mathcal{N}$  is not finitely generated as an algebra.

This example now raises the new question as to whether we can give necessary and sufficient conditions on a finite-dimensional algebra for its Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra.

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