Let $R$ be a ring with identity, and let Mod-$R$ be the category of right $R$-modules. Let $M$ be a right $R$-module. We denote by $E(M)$ the injective hull of $M$. $M$ is called QF-3' module, if $E(M)$ is $M$-torsionless, that is, $E(M)$ is isomorphic to a submodule of a direct product $IM$ of some copies of $M$.

A subfunctor of the identity functor of Mod-$R$ is called a preradical. For a preradical $\sigma$, $T_\sigma := \{M \in \text{Mod-}R : \sigma(M) = M\}$ is the class of $\sigma$-torsion right $R$-modules, and $F_\sigma := \{M \in \text{Mod-}R : \sigma(M) = 0\}$ is the class of $\sigma$-torsionfree right $R$-modules. A right $R$-module $M$ is called $\sigma$-injective (resp. $\sigma$-projective) if the functor $\text{Hom}_R(\ , M)$ (resp. $\text{Hom}_R(M, \ )$) preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in T_\sigma$ (resp. $A \in F_\sigma$). A right $R$-module $M$ is called $\sigma$-QF-3’ module if $E_\sigma(M)$ is $M$-torsionless, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$.

In this note, we characterize $\sigma$-QF-3’ modules and give some related facts.

1. QF-3’ Modules Relative to Hereditary Torsion Theories

In [1], Y. Kurata and H. Katayama characterized QF-3’ modules by using torsion theories. In this section we generalize QF-3’ modules by using an idempotent radical. A preradical $\sigma$ is idempotent (resp. radical) if $\sigma(\sigma(M)) = \sigma(M)$ (resp. $\sigma(M/\sigma(M)) = 0$) for any module $M$. For modules $M$ and $N$, $k_N(M)$ denotes $\cap\{\ker f ; f \in \text{Hom}_R(M,N)\}$. It is well known that $k_A$ is a radical for any module $A$ and that $T_{k_A} = \{M \in \text{Mod-}R ; \text{Hom}_R(M, A) = 0\}$ and $F_{k_A} = \{M \in \text{Mod-}R ; M \subseteq \Pi A\}$.

**Theorem 1.** Let $A$ be a module and $\sigma$ a preradical. Then the following conditions (1), (2) and (3) are equivalent. If $\sigma$ is an idempotent radical, then (1), (2), (3) and (4) are equivalent. Moreover if $\sigma$ is a left exact radical and $A$ is $\sigma$-torsion, then all conditions are equivalent.

1. $A$ is a $\sigma$-QF-3’ module.
2. $k_A(E_\sigma(A)) = 0$
3. $k_A(\omega) = k_{E_\sigma(A)}(\omega)$
4. $k_A(N) = N \cap k_A(M)$ holds for any module $M$ and any submodule $N$ such that $M/N$ is $\sigma$-torsion.
5. Let $M$ be a module and $N$ a submodule of $M$ such that $M/N$ is $\sigma$-torsion. Then for any nonzero $f \in \text{Hom}_R(N,A)$, there exists $p \in \text{Hom}_R(A,A)$ and $\bar{f} \in \text{Hom}_R(M,A)$ such that $p \cdot f = \bar{f} \cdot i \neq 0$.
6. Let $0 \to N \overset{f}{\to} M \to L \to 0$ be an exact sequence such that $L$ is $\sigma$-torsion. If $\text{Hom}_R(f, A) = 0$, then $\text{Hom}_R(N, A) = 0$.
7. For any module $M$ and a submodule $N$ of $M$.

The detailed version of this paper will be submitted for publication elsewhere.
(i) If $M \in \mathcal{T}_{k_A}$ and $M/N \in \mathcal{T}_{\sigma}$, then $N \in \mathcal{T}_{k_A}$.
(ii) If $N \in \mathcal{F}_{k_A}$ and $M/N \in \mathcal{F}_{k_A} \cap \mathcal{T}_{\sigma}$, then $M \in \mathcal{F}_{k_A}$.
(8) If $M \in \mathcal{F}_{k_A}$, then $E_\sigma(M) \in \mathcal{F}_{k_A}$.
(9) If $N$ is an essential submodule of a module $M$ such that $M/N \in \mathcal{T}_{\sigma}$ and $N \in \mathcal{F}_{k_A}$, then $M \in \mathcal{F}_{k_A}$.

As an application of Theorem 1, we give a characterization of the ring having the property that a right maximal quotient ring $Q$ is torsionless.

**Corollary 2.** Let $Q$ be a maximal right quotient ring of $R$. Then the following conditions are equivalent.

1. $Q$ is torsionless (i.e., $Q \subseteq \Pi R$).
2. $k_R(Q) = 0$
3. $k_R(\cdot) = k_Q(\cdot)$
4. $k_R(N) = N \cap k_R(M)$ holds for a module $M$ and any submodule $N$ of $M$ such that $\text{Hom}_R(M/N, E(R)) = 0$.

**Proposition 3.** If $\sigma$ is a left exact radical, (7) of (i) is equivalent to the condition (10) $\mathcal{T}_{k_A} = \mathcal{T}_{k_{E_\sigma}(A)}$.

For a module $M$, $Z(M)$ denote the singular submodule of $M$, that is, $Z(M) := \{m \in M ; (0 : m) \text{ is essential in } R\}$, where $(0 : m) = \{r \in R ; \, mr = 0\}$.

**Proposition 4.** If $\sigma$ is a left exact radical and $A \in \mathcal{T}_{\sigma} \cap \mathcal{F}_{Z}$, then (7) of (i) is equivalent to the condition (1), that is, $E_\sigma(A) \subseteq \Pi A$ is equivalent to the condition that $\mathcal{T}_{k_A}$ is closed under taking $\sigma$-dense submodules.

A module $N$ is called a $\sigma$-essential extension of $M$ if $N$ is an essential submodule of $M$ such that $M/N$ is $\sigma$-torsion.

**Lemma 5.** Let $\sigma$ be an idempotent radical and $M$ a $\sigma$-essential extension of a module $N$. Then $E_\sigma(M) = E_\sigma(N)$ holds.

**Proposition 6.** Let $\sigma$ be an idempotent radical. Then the class of $\sigma$-QF-3’ modules is closed under taking $\sigma$-essential extensions.

2. $\sigma$-LEFT EXACT PRERADICALS AND $\sigma$-HEREDITARY TORSION THEORIES

A preradical $t$ is left exact if $t(N) = N \cap t(M)$ holds for any module $M$ and any submodule $N$ of $M$. In this section we generalize left exact preradicals by using torsion theories.

Let $\sigma$ be a preradical. We call a preradical $t$ $\sigma$-left exact if $t(N) = N \cap t(M)$ holds for any module $M$ and any submodule $N$ of $M$ with $M/N \in \mathcal{T}_{\sigma}$. If a module $A$ is $\sigma$-QF-3’ and $t = k_{k_A}$, then $t$ is a $\sigma$-left exact radical. Now we characterize $\sigma$-left exact preradicals.

**Lemma 7.** For a preradical $t$ and $\sigma$, let $t_\sigma(M)$ denote $M \cap t(E_\sigma(M))$ for any module $M$. Then $t_\sigma(M)$ is uniquely determined for any choice of $E(M)$.

**Lemma 8.** Let $t$ be a preradical and $\sigma$ an idempotent radical. Then $t_\sigma$ is a $\sigma$-left exact preradical.
Theorem 9. Let $\sigma$ be an idempotent radical. We consider the following conditions on a preradical $t$. Then the implications $(5) \iff (1) \iff (2) \iff (3) \iff (4)$ hold. If $t$ is a radical, then $(4) \implies (1)$ holds. Thus if $t$ is an idempotent preradical and $\sigma$ is left exact, then $(5)(i) \implies (1)$ holds. Thus if $t$ is an idempotent radical and $\sigma$ is a left exact radical, then all conditions are equivalent.

1. $t$ is a $\sigma$-left exact preradical.
2. $t(M) = M \cap t(E_\sigma(M))$ holds for any module $M$.
3. $F_t$ is closed under taking $\sigma$-essential extensions, that is, if $M$ is an essential extension of a module $N \in F_t$ with $M/N \in T_\sigma$, then $M \in F_t$.
4. $F_t$ is closed under taking $\sigma$-injective hulls, that is, if $M \in F_t$, then $E_\sigma(M) \in F_t$.
5. For any module $M$ and a submodule $N$ of $M$,
   (i) $T_\sigma$ is closed under taking $\sigma$-dense submodules, that is, if $M \in T_\sigma$ and $M/N \in T_\sigma$, then $N \in T_\sigma$.
   (ii) $F_t$ is closed under taking $\sigma$-extensions, that is, if $N \in F_t$ and $M/N \in F_t \cap T_\sigma$, then $M \in F_t$.

A torsion theory for $\text{Mod-}R$ is a pair $(T,F)$ of classes of objects of $\text{Mod-}R$ satisfying the following three conditions.
(i) $\text{Hom}_R(T,F) = 0$ for all $T \in T$ and $F \in F$.
(ii) If $\text{Hom}_R(M,F) = 0$ for all $F \in F$, then $M \in T$.
(iii) If $\text{Hom}_R(T,N) = 0$ for all $T \in T$, then $N \in F$.

We put $t(M) = \sum_{T \in N < M} N_\cap N$, then $T = T_\sigma$ and $F = F_t$ hold.

For a torsion theory $(T,F)$, if $T$ is closed under taking submodules, then $(T,F)$ is called a hereditary torsion theory. $T$ is closed under taking submodules if and only if $F$ is closed under taking injective hulls.

Now we call $(T,F)$ a $\sigma$-hereditary torsion theory if $T$ is closed under taking $\sigma$-dense submodules. If $\sigma$ is a left exact radical, $T$ is closed under taking $\sigma$-dense submodules if and only if $F$ is closed under taking $\sigma$-injective hulls by Theorem 9.

Proposition 10. Let $t$ be an idempotent preradical and $\sigma$ a radical such that $F_\sigma$ is included $F_t$. If $F_t$ is closed under taking $\sigma$-injective hulls, then $F_t$ is closed under taking injective hulls.

Thus if $\sigma$ is a left exact radical, $T_\sigma \supseteq T_t$ and $(T_t,F_t)$ is a $\sigma$-hereditary torsion theory, then $(T_t,F_t)$ is a hereditary torsion theory.

Proposition 11. If $\sigma(M)$ contains the singular submodule $Z(M)$ for any module $M$, then a $\sigma$-left exact preradical is a left exact preradical.

Theorem 12. Let $\sigma$ be a left exact radical. Then $(T,F)$ is $\sigma$-hereditary if and only if there exists a $\sigma$-injective ($\sigma$-QF-$3'$) module $Q$ such that $T = \{M \in \text{Mod-}R : \text{Hom}_R(M,Q) = 0\}$.

Proposition 13. Let $\sigma$ be an idempotent radical and $(T,F)$ a $\sigma$-hereditary torsion theory, where $T = \{M \in \text{Mod-}R : \text{Hom}_R(M,Q) = 0\}$ for some $\sigma$ QF-$3'$ module $Q$ in $F$. Let $M$ be a $\sigma$-torsion module. Then $M$ is in $F$ if and only if $M$ is contained in a direct product of some copies of $Q$. 
3. CQF-3’ modules relative to torsion theories

A preradical \( t \) is called \textit{epi-preserving} if \( t(M/N) = (t(M) + N)/N \) for any module \( M \) and any submodule \( N \) of \( M \). A short exact sequence \( 0 \to K(M) \to P(M) \to M \to 0 \) is a \textit{projective cover} of a module \( M \) if \( P(M) \) is projective and \( K(M) \) is small in \( P(M) \).

In [2], F.F. Mbuntum and K. Varadarajan dualized QF-3’ modules and characterized them. Let \( M \) be a module with a projective cover. \( M \) is called a \textit{CQF-3’ module} if \( P(M) \) is \( M \)-generated, that is, \( P(M) \) is isomorphic to a homomorphic image of a direct sum \( \oplus M \) of some copies of \( M \). In this section we generalize CQF-3’ modules and characterize them.

A short exact sequence \( 0 \to K_\sigma(M) \to P_\sigma(M) \to M \to 0 \) is called \( \sigma \)-\textit{projective cover} of a module \( M \) if \( P_\sigma(M) \) is \( \sigma \)-projective and \( K_\sigma(M) \) is \( \sigma \)-torsion and small in \( P_\sigma(M) \). If \( \sigma \) is an idempotent radical and a module \( M \) has a projective cover, then \( M \) has a \( \sigma \)-projective cover and it is given \( K_\sigma(M) = k(M)/\sigma(K(M)), P_\sigma(M) = P(M)/\sigma(K(M)) \). Now we call a module \( M \) with a projective cover a \( \sigma \)-\textit{CQF-3’ module} if \( P_\sigma(M) \) is \( M \)-generated. Let \( t_M(N) \) denote the sums of images of all homomorphisms from \( M \) to \( N \) for any module \( M \). If \( t_A \) is an idempotent preradical for any module \( A \) and \( T_{tA} = \{ M \in \text{Mod}-R \mid \oplus A \to M \to 0 \} \) and \( F_{tA} = \{ M \in \text{Mod}-R \mid \text{Hom}(A, M) = 0 \} \).

**Theorem 14.** Let \( \sigma \) be a preradical, and suppose that a module \( A \) has a \( \sigma \)-projective cover \( 0 \to K_\sigma(A) \to P_\sigma(A) \to A \to 0 \). Consider the following conditions.

1. \( P_\sigma(A) \) is a \( \sigma \)-CQF-3’ module.
2. \( t_A(P_\sigma(A)) = P_\sigma(A) \)
3. \( t_A(\lambda) = t_{P_\sigma(A)}(\lambda) \)
4. \( t_A(\lambda) \) is a \( \sigma \)-epi-preserving preradical, that is, \( t_A(M/N) = (t_A(M) + N)/N \) holds for any module \( M \) and any submodule \( N \in F_\sigma \).
5. \( (i) T_{tA} \) is closed under taking \( F_\sigma \)-extensions, that is, \( t_A(M) = M \) holds for any module \( M \) and any submodule \( N \) of \( M \) such that \( M/N \in T_{tA} \) and \( N \in F_\sigma \cap T_{tA} \).
6. \( (ii) F_{tA} \) is closed under taking \( F_\sigma \)-factor modules, that is, \( M/N \in F_{tA} \) holds for any module \( M \in F_{tA} \) and any submodule \( N \in F_\sigma \) of \( M \).
7. \( T_{tA} \) is closed under taking \( \sigma \)-projective covers, that is, \( P_\sigma(M) \in T_{tA} \) holds for any module \( M \in T_{tA} \).
8. \( (i) R_{tA} \) is closed under taking \( \sigma \)-coessential extensions, that is, for any module \( M \) if there exists a small submodule \( N \) in \( F_\sigma \) such that \( M/N \in R_{tA} \) then \( M \) is in \( T_{tA} \).

Then (1) \( (2) \rightarrow (3) \rightarrow (1) \) and (4) \( (4) \rightarrow (1) \) hold. If \( \sigma \) is idempotent, then (3) \( (4) \rightarrow (1) \), (1) \( (8) \rightarrow (6) \rightarrow (5) \), (7) hold. If \( \sigma \) is a radical, then (7) \( (6) \rightarrow (6) \rightarrow (2) \), (6) hold. If \( \sigma \) is an epi-preserving idempotent radical and \( A \) is in \( F_\sigma \), then (8) \( (5) \rightarrow (5) \) holds, moreover if \( \sigma \) is an epi-preserving idempotent radical then (5) \( (2) \rightarrow (2) \) holds.

Thus if \( \sigma \) is an epi-preserving idempotent radical and \( A \) is in \( F_\sigma \), all conditions are equivalent.

**Proposition 15.** Let \( \sigma \) be an epi-preserving idempotent radical. Then the following conditions on a module \( A \) are equivalent.
Lemma 16. Let \( \sigma \) be an idempotent radical. If \( N \) is in \( \mathcal{F}_\sigma \) and is a small submodule of \( M \), then \( P_\sigma(M/N) \cong P_\sigma(M) \) holds.

Proposition 17. Let \( \sigma \) be an idempotent radical. The class of \( \sigma \) CQF-3’ modules is closed under taking \( \sigma \)-coessential extensions, that is, if a module \( M \) has a small submodule \( N \in \mathcal{F}_\sigma \) such that \( M/N \) is a \( \sigma \)-CQF-3’ module, then \( M \) is also a \( \sigma \)-CQF-3’ module.

4. \( \sigma \)-EPI-PRESERVING PRERADICALS AND \( \sigma \)-COHEREDITARY TORSION THEORIES

In this section we characterize \( \sigma \)-epi-preserving preradicals when \( R \) is a right perfect ring.

Theorem 18. Let \( R \) be a right perfect ring and \( \sigma \) an idempotent radical. Consider the following conditions on a preradical \( t \).

1. \( t \) is an \( \sigma \)-epi-preserving preradical, that is, \( t(M/N) = (t(M) + N)/N \) holds for a module \( M \) and any submodule \( N \in \mathcal{F}_\sigma \) of \( M \).

2. \( \mathcal{T}_t \) is closed under taking \( \sigma \)-coessential extensions, that is, for any module \( M \) if there exists a small submodule \( N \in \mathcal{F}_\sigma \) such that \( M/N \in \mathcal{T}_t \) then \( M \) is in \( \mathcal{T}_t \).

3. \( \mathcal{T}_t \) is closed under taking \( \sigma \)-projective covers, that is, \( P_\sigma(M) \in \mathcal{T}_t \) holds for any module \( M \in \mathcal{T}_t \).

4. \( \mathcal{F}_t \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules, that is, \( M/N \in \mathcal{F}_t \) holds for any module \( M \in \mathcal{F}_t \) and any submodule \( N \in \mathcal{F}_\sigma \) of \( M \).

\[ (i) \mathcal{F}_t \text{ is closed under taking } \mathcal{F}_\sigma \text{-factor modules, that is, } M/N \in \mathcal{F}_t \text{ holds for any module } M \in \mathcal{F}_t \text{ and any submodule } N \in \mathcal{F}_\sigma \text{ of } M. \]

\[ (ii) \mathcal{T}_t \text{ is closed under taking } \mathcal{F}_\sigma \text{-extensions, that is, } t(M) = M \text{ holds for any module } M \text{ and any submodule } N \text{ of } M \text{ such that } M/N \in \mathcal{T}_t \text{ and } N \in \mathcal{F}_\sigma \cap \mathcal{T}_t. \]

Then (4) \( \iff \) (1) \( \iff \) (3) hold. If \( t \) is an idempotent preradical, then (3) \( \rightarrow \) (1) holds. If \( \sigma \) is an epi-preserving idempotent radical and \( t \) is a radical, then (4) \( \rightarrow \) (1) holds. Thus if \( \sigma \) is an epi-preserving idempotent radical and \( t \) is an idempotent radical, then all conditions are equivalent.

We call a torsion theory \((\mathcal{T}, \mathcal{F})\) \( \sigma \)-cohereditary torsion theory if \( \mathcal{F} \) is closed under taking \( \mathcal{F}_\sigma \)-factor modules for an idempotent radical \( \sigma \).

Theorem 19. Let \( R \) be a right perfect ring and \( \sigma \) an epi-preserving idempotent radical. Then a torsion theory \((\mathcal{T}, \mathcal{F})\) is \( \sigma \)-cohereditary if and only if there exists an \( \sigma \)-projective \( (\sigma\text{-CQF-3'}) \) module \( Q \) such that \( \mathcal{F} = \{ M \in \text{Mod-}R ; \text{Hom}_R(Q, M) = 0 \} \).

Proposition 20. Let \( R \) be a right perfect ring, \( \sigma \) be an idempotent radical and \((\mathcal{T}, \mathcal{F})\) be a \( \sigma \)-cohereditary torsion theory, where \( \mathcal{F} = \{ M \in \text{Mod-}R ; \text{Hom}_R(Q, M) = 0 \} \) for some \( \sigma \)-CQF-3’ module \( Q \in \mathcal{T} \). Let \( M \) be a \( \sigma \)-torsionfree module. Then \( M \in \mathcal{T} \) if and only if \( M \) is generated by \( Q \).

5. \( \sigma \)-STABLE TORSION THEORY AND \( \sigma \)-COSTABLE TORSION THEORY

A torsion theory \((\mathcal{T}_t, \mathcal{F}_t)\) is called stable if \( \mathcal{T}_t \) is closed under taking injective hulls. In this section we generalize stable torsion theory by using torsion theories.
Proposition 21. Let $\sigma$ be an idempotent radical and $L$ a submodule of a module $M$. Then the implications $(1) \rightarrow (2) \rightarrow (3)$ hold. Moreover, if $\sigma$ is a left exact radical, then $(3) \rightarrow (1)$ holds.

1. $L$ is $\sigma$-complemented in $M$, that is, there exists a submodule $K$ of $M$ such that $L$ is maximal in $\Gamma_K = \{L_i ; L_i \subseteq M, L_i \cap K = 0, M/(L_i + K) \in T_\sigma\}

2. $L = E_\sigma(L) \cap M$.

3. $L$ is $\sigma$-essentially closed in $M$, that is, there is no $\sigma$-essential extension of $L$ in $M$.

We call a preradical $t$ $\sigma$-stable if $T_t$ is closed under taking $\sigma$-injective hulls. We put $\mathcal{X}_t(M) := \{X ; M/X \in T_t\}$ and $N \cap \mathcal{X}_t(M) := \{N \cap X ; X \in \mathcal{X}_t(M)\}$.

Theorem 22. Let $t$ be an idempotent preradical and $\sigma$ an idempotent radical. Then the following conditions $(1)$, $(2)$ and $(3)$ are equivalent. Moreover, if $\sigma$ is left exact and $T_t$ is closed under taking $\sigma$-dense submodules, then all the following conditions are equivalent.

1. $t$ is $\sigma$-stable, that is, $T_t$ is closed under taking $\sigma$-injective hulls.
2. The class of $\sigma$-injective modules is closed under taking the unique maximal $t$-torsion submodules, that is, $t(M)$ is $\sigma$-injective for any $\sigma$-injective module $M$.
3. $E_\sigma(t(M)) \subset t(E_\sigma(M))$ holds for any module $M$.
4. $T_t$ is closed under taking $\sigma$-essential extensions.
5. If $M/N$ is $\sigma$-torsion, then $N \cap \mathcal{X}_t(M) = \mathcal{X}_t(N)$ holds.
6. For any module $M$, $t(M)$ is $\sigma$-complemented in $M$.
7. For any module $M$, $t(M) = E_\sigma(t(M)) \cap M$ holds.
8. For any module $M$, $t(M)$ is $\sigma$-essentially closed in $M$.
9. For any $\sigma$-injective module $E$ with $E/t(E) \in T_\sigma$, $t(E)$ is a direct summand of $E$.
10. $E_\sigma(t(M)) = t(E_\sigma(M))$ holds for any module $M$.

If $T_t$ is closed under taking $\sigma$-dense submodules, then $(1) \rightarrow (4) \rightarrow (5)$ hold. Moreover, if $\sigma$ is left exact, then $(1) \rightarrow (6)$ and $(3) \rightarrow (10)$ hold.

It is well known that if $R$ is right noetherian, $t$ is stable if and only if every indecomposable injective module is $t$-torsion or $t$-torsionfree. By using Theorem 1 in [3], we generalized this as follows.

Theorem 23. Let $t$ be an idempotent radical and $\sigma$ a left exact radical. Then

1. If $t$ is $\sigma$-stable, then $(*)$ every indecomposable $\sigma$-injective module $E$ with $E/T(E) \in T_\sigma$ is either $t$-torsion or $t$-torsionfree.
2. If the ring $R$ satisfies the condition $(*)$ and the ascending chain conditions on $\sigma$-dense ideals of $R$, then $T_t \cap T_\sigma$ is closed under taking $\sigma$-injective hulls.

We now dualize $\sigma$-stable torsion theory. Let $R$ be a right perfect ring. We call a preradical $t$ $\sigma$-costable if $F_t$ is closed under taking $\sigma$-projective covers.

Theorem 24. Let $\sigma$ be an idempotent radical. Then a radical $t$ is $\sigma$-costable if and only if the class of $\sigma$-projective modules is closed under taking the unique maximal $t$-torsionfree factor modules, that is, $P/t(P)$ is $\sigma$-projective for any $\sigma$-projective module $P$.

6. $\sigma$-singular submodules

Let $\sigma$ be a left exact radical. For a module $M$ we put $Z_\sigma(M) := \{m \in M ; (0 : m) is \sigma\text{-essential in } R\}$ and call it $\sigma$-singular submodule of $M$. Since $R/(0 : m) \in T_\sigma \cap T_\sigma$, then
\[Z_\sigma(M) \subseteq Z(M) \cap \sigma(M) = Z(\sigma(M)) = \sigma(Z(M)),\] and so \(Z_\sigma(M) = \{m \in M : mR \in T_Z \cap T_\sigma\}.\) Since \(Z\) and \(\sigma\) is left exact, \(Z_\sigma\) is also left exact. We will call \(M\) \(\sigma\)-singular (resp. \(\sigma\)-nonsingular) if \(Z_\sigma(M) = M\) (resp. \(Z_\sigma(M) = 0\)).

**Proposition 25.** Let \(\sigma\) be an idempotent radical and \(E\) a \(\sigma\)-nonsingular module and \(T = \{M \in \text{Mod-}R : \text{Hom}_R(M, E) = 0\}\). Then \(T\) is closed under taking \(\sigma\)-essential extensions. Therefore a torsion theory \((T, F)\) is \(\sigma\)-stable, where \(F = \{M \in \text{Mod-}R : \text{Hom}_R(X, M) = 0\}\) for any \(X \in T\).

**Proposition 26.** Let \(\sigma\) a left exact radical. Then the following facts hold.

1. If \(N\) is \(\sigma\)-essential in \(M\), then \(Z_\sigma(M/N) = M/N\).
2. A right ideal of \(R\) is \(\sigma\)-essential in \(R\) if and only if \(Z_\sigma(R/I) = R/I\).
3. Let \(M\) be a \(\sigma\)-nonsingular module and \(N\) a submodule of \(M\). Then \(N\) is \(\sigma\)-essential in \(M\) if and only if \(Z_\sigma(M/N) = M/N\).
4. For a module \(M\), \(Z_\sigma(M/Z_\sigma(M)) = M/Z_\sigma(M)\) holds if and only if \(Z_\sigma(M)\) is \(\sigma\)-essential in \(M\).
5. For a simple right \(R\)-module \(S\), \(S\) is \(\sigma\)-nonsingular if and only if \(S\) is \(\sigma\)-torsionfree or projective.
6. If \(R\) is \(\sigma\)-nonsingular, then \(Z_\sigma\) is left exact radical.
7. If \(M/N\) is \(\sigma\)-nonsingular for a module \(M\) and a submodule \(N\) of \(M\), then \(N\) is \(\sigma\)-complemented in \(M\). If \(M\) is \(\sigma\)-nonsingular, then the converse holds.

7. \(\sigma\)-small and \(\sigma\)-radical

Let \(\sigma\) be a left exact radical. A submodule \(N\) of a module \(M\) is called \(\sigma\)-dense in \(M\) if \(M/N\) is \(\sigma\)-torsion. A module \(M\) is called \(\sigma\)-cocritical if \(M\) is \(\sigma\)-torsionfree and \(L\) is \(\sigma\)-dense in \(M\) for any nonzero submodule \(L\) of \(M\). It is well known that nonzero submodule of \(\sigma\)-cocritical module \(M\) is essential in \(M\). A module \(M\) is called \(\sigma\)-noetherian if for every ascending chain \(I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \cdots\) \(\subseteq M\), \((\text{where } \cup I_j \text{ is } \sigma\)-dense in \(M\)) there exists a positive integer \(k\) such that \(I_k\) is \(\sigma\)-dense in \(M\). Let \(J_\sigma(M)\) denote \(\cap \text{Nil}_i(M/N_i)\) is \(\sigma\)-cocritical).

Now we define \(\sigma\)-small submodule as follows. A submodule \(N\) of a module \(M\) is called \(\sigma\)-small in \(M\) if \(M/(N + X) \in T_\sigma\) implies \(M/X \in T_\sigma\) for any submodule \(X\) of \(M\).

**Theorem 27.** \(J_\sigma(M)\) contains \(\sum N(N\) is \(\sigma\)-small in \(M\)). Conversely if \(M\) be a \(\sigma\)-noetherian module, then \(J_\sigma(M)\) coincides with \(\sum N(N\) is \(\sigma\)-small in \(M\)).

**Remark 28.** We can see in [4] that the definition of \(\sigma\)-small is different from ours.

(B.A.Benander’s definition). \(N\) is \(\sigma\)-small in \(M\) if \(M/(N' + X) \in T_\sigma\) and \(M/X \in F_\sigma\), then \(M = X\) for any \(X\) of \(M\), where \(\sigma(M/N) = N'/N\).

Benander’s definition of \(\sigma\)-small is a stronger condition than ours.

In fact, if \(M/(N + X) \in T_\sigma\), then \(M/(N' + X) \in T_\sigma\). We put \(X'/X := \sigma(M/X)\). Then \(M/X' \in F_\sigma\). Since \(M/(N' + X) \in T_\sigma\), \(M/(N' + X') \in T_\sigma\). Thus \(M = X'\), and so \(M/X \in T_\sigma\), as desired.

**References**


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