QF-3' MODULES RELATIVE TO TORSION THEORIES AND OTHERS

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Let R be a ring with identity, and let Mod-R be the category of right R-modules. Let M be a right R-module. We denote by $E(M)$ the injective hull of M. M is called *QF-3' module*, if $E(M)$ is M-torsionless, that is, $E(M)$ is isomorphic to a submodule of a direct product ΠM of some copies of M.

A subfunctor of the identity functor of Mod-R is called a *preradical*. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R\ ;\ \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R\ ;\ \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. A right R-module M is called *σ-injective* (resp. *σ-projective*) if the functor $\text{Hom}_R(\ \ , M)$ (resp. $\text{Hom}_R(M,\ \ )$) preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$ (resp. $A \in \mathcal{F}_\sigma$). A right R-module M is called *σ-QF-3' module* if $E_\sigma(M)$ is M-torsionless, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$.

In this note, we characterize σ-QF-3' modules and give some related facts.

1. QF-3' modules relative to hereditary torsion theories

In [1], Y.Kurata and H.Katayama characterized QF-3' modules by using torsion theories. In this section we generarize QF-3' modules by using an idempotent radical. A preradical σ is *idempotent* (resp. *radical*) if $\sigma(\sigma(M)) = \sigma(M)$ (resp. $\sigma\,(M/\sigma(M)) = 0$) for any module M. For modules M and N, $k_N(M)$ denotes $\cap\{\ker f\ ;\ f \in \text{Hom}_R(M, N)\}$. It is well known that k_A is a radical for any module A and that $\mathcal{T}_{k_A} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(M, A) = 0\}$ and $\mathcal{F}_{k_A} = \{M \in \text{Mod-}R\ ;\ M \subseteq \Pi A\}$.

Theorem 1. *Let A be a module and σ a preradical. Then the following conditions* (1), (2) *and* (3) *are equivalent. If σ is an idempotent radical, then* (1), (2), (3) *and* (4) *are equivalent. Moreover if σ is a left exact radical and A is σ-torsion, then all conditions are equivalent.*

(1) *A is a σ-QF-3' module.*

(2) $k_A(E_\sigma(A)) = 0$

(3) $k_A(\_) = k_{E_\sigma(A)}(\_)$

(4) *$k_A(N) = N \cap k_A(M)$ holds for any module M and any submodule N such that M/N is σ-torsion.*

(5) *Let M be a module and N a submodule of M such that M/N is σ-torsion. Then for any nonzero $f \in \text{Hom}_R(N, A)$, there exists $p \in \text{Hom}_R(A, A)$ and $\bar{f} \in \text{Hom}_R(M, A)$ such that $p \cdot f = \bar{f} \cdot i \neq 0$.*

(6) *Let $0 \to N \xrightarrow{f} M \to L \to 0$ be an exact sequence such that L is σ-torsion. If $\text{Hom}_R(f, A) = 0$, then $\text{Hom}_R(N, A) = 0$.*

(7) *For any module M and a submodule N of M,*

The detailed version of this paper will be submitted for publication elsewhere.

(*i*) *If* $M \in \mathcal{T}_{k_A}$ *and* $M/N \in \mathcal{T}_\sigma$, *then* $N \in \mathcal{T}_{k_A}$.
(*ii*) *If* $N \in \mathcal{F}_{k_A}$ *and* $M/N \in \mathcal{F}_{k_A} \cap \mathcal{T}_\sigma$, *then* $M \in \mathcal{F}_{k_A}$.
(8) *If* $M \in \mathcal{F}_{k_A}$, *then* $E_\sigma(M) \in \mathcal{F}_{k_A}$.
(9) *If* N *is an essential submodule of a module* M *such that* $M/N \in \mathcal{T}_\sigma$ *and* $N \in \mathcal{F}_{k_A}$, *then* $M \in \mathcal{F}_{k_A}$.

As an application of Theorem 1, we give a characterization of the ring having the property that a right maximal quotient ring Q is torsionless.

Corollary 2. *Let* Q *be a maximal right quotient ring of* R. *Then the following conditions are equivalent.*
(1) Q *is torsionless (i.e.,* $Q \subset \Pi R$*).*
(2) $k_R(Q) = 0$
(3) $k_R(\_) = k_Q(\_)$
(4) $k_R(N) = N \cap k_R(M)$ *holds for a module* M *and any submodule* N *of* M *such that* $\mathrm{Hom}_R(M/N, E(R)) = 0$.

Proposition 3. *If* σ *is a left exact radical,* (7) *of (i) is equivalent to the condition* (10) $\mathcal{T}_{k_A} = \mathcal{T}_{k_{E_\sigma(A)}}$.

For a module M, $Z(M)$ denote the singular submodule of M, that is ,$Z(M) := \{m \in M\ ;\ (0 : m)$ is essential in $R\}$, where $(0 : m) = \{r \in R\ ;\ mr = 0\}$.

Proposition 4. *If* σ *is a left exact radical and* $A \in \mathcal{T}_\sigma \cap \mathcal{F}_Z$, *then* (7) *of (i) is equivalent to the condition* (1), *that is,* $E_\sigma(A) \subseteq \Pi A$ *is equivalent to the condition that* $\mathcal{T}_{k_A}$ *is closed under taking* σ*-dense submodules.*

A module N is called a *σ-essential extension* of M if N is an essential submodule of M such that M/N is σ-torsion.

Lemma 5. *Let* σ *be an idempotent radical and* M *a* σ*-essential extension of a module* N. *Then* $E_\sigma(M) = E_\sigma(N)$ *holds.*

Proposition 6. *Let* σ *be an idempotent radical. Then the class of* σ*-QF-3' modules is closed under taking* σ*-essential extensions.*

2. σ-LEFT EXACT PRERADICALS AND σ-HEREDITARY TORSION THEORIES

A preradical t is *left exact* if $t(N) = N \cap t(M)$ holds for any module M and any submodule N of M. In this section we generalize left exact preradicals by using torsion theories.

Let σ be a preradical. We call a preradical t *σ-left exact* if $t(N) = N \cap t(M)$ holds for any module M and any submodule N of M with $M/N \in \mathcal{T}_\sigma$. If a module A is σ-QF-3' and $t = k_A$, then t is a σ-left exact radical. Now we characterize σ-left exact preradicals.

Lemma 7. *For a preradical* t *and* σ, *let* $t_\sigma(M)$ *denote* $M \cap t(E_\sigma(M))$ *for any module* M. *Then* $t_\sigma(M)$ *is uniquely determined for any choice of* $E(M)$.

Lemma 8. *Let* t *be a preradical and* σ *an idempotent radical. Then* t_σ *is a* σ*-left exact preradical.*

Theorem 9. *Let σ be an idempotent radical. We consider the following conditions on a preradical t. Then the implications $(5) \leftarrow (1) \Leftrightarrow (2) \to (3) \Leftrightarrow (4)$ hold. If t is a radical, then $(4) \to (1)$ holds. If t is an idempotent preradical and σ is left exact, then $(5)(i) \to (1)$ holds. Thus if t is an idempotent radical and σ is a left exact radical, then all conditions are equivalent.*

(1) *t is a σ-left exact preradical.*

(2) *$t(M) = M \cap t(E_\sigma(M))$ holds for any module M.*

(3) *$\mathcal{F}_t$ is closed under taking σ-essential extension, that is, if M is an essential extension of a module $N \in \mathcal{F}_t$ with $M/N \in \mathcal{T}_\sigma$, then $M \in \mathcal{F}_t$.*

(4) *$\mathcal{F}_t$ is closed under taking σ-injective hulls, that is, if $M \in \mathcal{F}_t$, then $E_\sigma(M) \in \mathcal{F}_t$.*

(5) *For any module M and a submodule N of M,*

(*i*) *$\mathcal{T}_t$ is closed under taking σ-dense submodules, that is, if $M \in \mathcal{T}_t$ and $M/N \in \mathcal{T}_\sigma$, then $N \in \mathcal{T}_t$.*

(*ii*) *$\mathcal{F}_t$ is closed under taking σ-extensions, that is, if $N \in \mathcal{F}_t$ and $M/N \in \mathcal{F}_t \cap \mathcal{T}_\sigma$, then $M \in \mathcal{F}_t$.*

A *torsion theory* for Mod-R is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of Mod-R satisfying the following three conditions.

(*i*) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

(*ii*) If $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$.

(*iii*) If $\mathrm{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$.

We put $t(M) = \sum_{\mathcal{T} \ni N \subset M} N \ (= \bigcap_{M/N \in \mathcal{F}} N)$, then $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$ hold.

For a torsion theory $(\mathcal{T}, \mathcal{F})$, if $\mathcal{T}$ is closed under taking submodules, then $(\mathcal{T}, \mathcal{F})$ is called a *hereditary torsion theory.* $\mathcal{T}$ is closed under taking submodules if and only if $\mathcal{F}$ is closed under taking injective hulls.

Now we call $(\mathcal{T}, \mathcal{F})$ a *σ-hereditary torsion theory* if $\mathcal{T}$ is closed under taking σ-dense submodules. If σ is a left exact radical, $\mathcal{T}$ is closed under taking σ-dense submodules if and only if $\mathcal{F}$ is closed under taking σ-injective hulls by Theorem 9.

Proposition 10. *Let t be an idempotent preradical and σ a radical such that $\mathcal{F}_\sigma$ is included $\mathcal{F}_t$. If $\mathcal{F}_t$ is closed under taking σ-injective hulls, then $\mathcal{F}_t$ is closed under taking injective hulls.*

Thus if σ is a left exact radical, $\mathcal{T}_\sigma \supseteq \mathcal{T}_t$ and $(\mathcal{T}_t, \mathcal{F}_t)$ is a σ-hereditary torsion theory, then $(\mathcal{T}_t, \mathcal{F}_t)$ is a hereditary torsion theory.

Proposition 11. *If $\sigma(M)$ contains the singlar submodule $Z(M)$ for any module M, then a σ-left exact preradical is a left exact preradical.*

Theorem 12. *Let σ be a left exact radical. Then $(\mathcal{T}, \mathcal{F})$ is σ-hereditary if and only if there exists a σ-injective (σ-QF-3') module Q such that $\mathcal{T} = \{M \in \text{Mod-}R \ ; \ \mathrm{Hom}_R(M, Q) = 0\}$.*

Proposition 13. *Let σ be an idempotent radical and $(\mathcal{T}, \mathcal{F})$ a σ-hereditary torsion theory, where $\mathcal{T} = \{M \in \text{Mod-}R \ ; \ \mathrm{Hom}_R(M, Q) = 0\}$ for some σ QF-3' module Q in $\mathcal{F}$. Let M be a σ-torsion module. Then M is in $\mathcal{F}$ if and only if M is contained in a direct product of some copies of Q.*

3. CQF-3' MODULES RELATIVE TO TORSION THEORIES

A preradical t is called *epi-preserving* if $t(M/N) = (t(M) + N)/N$ for any module M and any submodule N of M. A short exact sequence $0 \to K(M) \to P(M) \to M \to 0$ is a *projective cover* of a module M if $P(M)$ is projective and $K(M)$ is small in $P(M)$.

In [2], F.F. Mbuntum and K. Varadarajan dualized QF-3' modules and characterized them. Let M be a module with a projective cover. M is called a *CQF-3' module* if $P(M)$ is M-generated, that is, $P(M)$ is isomorphic to a homomorphic image of a direct sum $\oplus M$ of some copies of M. In this section we generalize CQF-3' modules and characterize them.

A short exact sequence $0 \to K_\sigma(M) \to P_\sigma(M) \to M \to 0$ is called *σ-projective cover* of a module M if $P_\sigma(M)$ is σ-projective and $K_\sigma(M)$ is σ-torsion and small in $P_\sigma(M)$. If σ is an idempotent radical and a module M has a projective cover, then M has a σ-projective cover and it is given $K_\sigma(M) = k(M)/\sigma(K(M)), P_\sigma(M) = P(M)/\sigma(K(M))$. Now we call a module M with a projective cover a *σ-CQF-3'module* if $P_\sigma(M)$ is M-generated. Let $t_M(N)$ denote the sums of images of all homomorphisms from M to N for a module M and a module N.It is well known that t_A is an idempotent preradical for any module A and $\mathcal{T}_{t_A} = \{M \in \text{Mod-}R \; ; \; \oplus \text{A} \to \text{M} \to 0\}$ and $\mathcal{F}_{t_A} = \{M \in \textit{Mod-R} \; ; \; \text{Hom}(A, M) = 0\}$.

Theorem 14. *Let σ be a preradical, and suppose that a module A has a σ-projective cover $0 \to K_\sigma(A) \to P_\sigma(A) \to A \to 0$. Consider the following conditions.*

(1) *$P_\sigma(A)$ is a σ-CQF-3' module.*

(2) $t_A(P_\sigma(A)) = P_\sigma(A)$

(3) $t_A(\_) = t_{P_\sigma(A)}(\_)$

(4) *$t_A(\_)$ is a σ-epi-preserving preradical, that is, $t_A(M/N) = (t_A(M) + N)/N$ holds for any module M and any submodule $N \in \mathcal{F}_\sigma$.*

(5) (*i*) *$\mathcal{T}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-extensions, that is, $t_A(M) = M$ holds for any module M and any submodule N of M such that $M/N \in \mathcal{T}_{t_A}$ and $N \in \mathcal{F}_\sigma \cap \mathcal{T}_{t_A}$.*

(*ii*) *$\mathcal{F}_{t_A}$ is closed under taking $\mathcal{F}_\sigma$-factor modules, that is, $M/N \in \mathcal{F}_{t_A}$ holds for any module $M \in \mathcal{F}_{t_A}$ and any submodule $N \in \mathcal{F}_\sigma$ of M.*

(6) *$\mathcal{T}_{t_A}$ is closed under taking σ-projective covers, that is, $P_\sigma(M) \in \mathcal{T}_{t_A}$ holds for any module $M \in \mathcal{T}_{t_A}$.*

(7) *$\mathcal{T}_{t_A}$ is closed under taking σ-coessential extensions, that is, for any module M if there exists a small submodule N in $\mathcal{F}_\sigma$ such that $M/N \in \mathcal{T}_{t_A}$ then M is in $\mathcal{T}_{t_A}$.*

(8) *If $\text{Hom}_R(A, f) = 0$, then $\text{Hom}_R(A, M/N) = 0$ holds for any module M and any submodule $N \in \mathcal{F}_\sigma$.*

Then $(1) \to (2) \to (3) \to (1)$ *and* $(4) \to (1)$ *hold. If σ is idempotent, then* $(3) \to (4)$, $(1) \to (8)$ *and* $(6) \to (5), (7)$ *hold. If σ is a radical, then* $(7) \to (6)$, $(4) \to (2), (6)$ *hold. If σ is an epi-preserving radical and A is in $\mathcal{F}_\sigma$, then* $(8) \to (5)$ *holds, moreover if σ is idempotent then* $(5) \to (2)$ *holds.*

Thus if σ is an epi-preserving idempotent radical and A is in $\mathcal{F}_\sigma$, all conditions are equivalent.

Proposition 15. *Let σ be an epi-preserving idempotent radical. Then the following conditions on a module A are equivalent.*

(1) $\mathcal{F}_{t_A}$ *is closed under taking* $\mathcal{F}_\sigma$*-factor modules.*
(2) $\mathcal{F}_{t_A} = \mathcal{F}_{t_{P_\sigma(A)}}$

Lemma 16. *Let σ be an idempotent radical. If N is in $\mathcal{F}_\sigma$ and is a small submodule of M, then $P_\sigma(M/N) \cong P_\sigma(M)$ holds.*

Proposition 17. *Let σ be an idempotent radical. The class of σ CQF-3' modules is closed under taking σ-coessntial extensions, that is, if a module M has a small submodule $N \in \mathcal{F}_\sigma$ such that M/N is a σ-CQF-3' module, then M is also a σ-CQF-3' module.*

4. σ-EPI-PRESERVING PRERADICALS AND σ-COHEREDITARY TORSION THEORIES

In this section we characterize σ-epi-preserving preradicals when R is a right perfect ring.

Theorem 18. *Let R be a right perfect ring and σ an idempotent radical. Consider the following conditions on a preradical t.*

(1) *t is an σ-epi-preserving preradical, that is, $t(M/N) = (t(M) + N)/N$ holds for a module M and any submodule $N \in \mathcal{F}_\sigma$ of M.*

(2) *$\mathcal{T}_t$ is closed under taking σ-coessential extensions, that is, for any module M if there exists a small submodule N in $\mathcal{F}_\sigma$ such that $M/N \in \mathcal{T}_t$ then M is in $\mathcal{T}_t$.*

(3) *$\mathcal{T}_t$ is closed under taking σ-projective covers, that is, $P_\sigma(M) \in \mathcal{T}_t$ holds for any module $M \in \mathcal{T}_t$.*

(4) (*i*) *$\mathcal{F}_t$ is closed under taking $\mathcal{F}_\sigma$-factor modules, that is, $M/N \in \mathcal{F}_t$ holds for any module $M \in \mathcal{F}_t$ and any submodule $N \in \mathcal{F}_\sigma$ of M.*

(*ii*) *$\mathcal{T}_t$ is closed under taking $\mathcal{F}_\sigma$-extensions, that is, $t(M) = M$ holds for any module M and any submodule N of M such that $M/N \in \mathcal{T}_t$ and $N \in \mathcal{F}_\sigma \cap \mathcal{T}_t$.*

Then (4) $\leftarrow$ (1) $\rightarrow$ (2) $\Leftrightarrow$ (3) *hold. If t is an idempotent preradical,then* (3) $\rightarrow$ (1) *holds. If σ is an epi-preserving preradical and t is a radical, then* (4) $\rightarrow$ (1) *holds. Thus if σ is an epi-preserving idempotent radical and t is an idempotent radical, then all conditions are equivalent.*

We call a torsion theory $(\mathcal{T}, \mathcal{F})$ *σ-cohereditary torsion theory* if $\mathcal{F}$ is cosed under taking $\mathcal{F}_\sigma$-factor modules for an idempotent radical σ.

Theorem 19. *Let R be a right perfect ring and σ an epi-preserving idempotent radical. Then a torsion theory $(\mathcal{T}, \mathcal{F})$ is σ-cohereditary if and only if there exists an σ-projective (σ-CQF-3') module Q such that $\mathcal{F} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(Q, M) = 0\}$.*

Proposition 20. *Let R be a right perfect ring, σ be an idempotent radical and $(\mathcal{T}, \mathcal{F})$ be a σ-cohereditary torsion theory, where $\mathcal{F} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(Q, M) = 0\}$ for some σ-CQF-3' module $Q \in \mathcal{T}$. Let M be a σ-torsionfree module. Then $M \in \mathcal{T}$ if and only if M is generated by Q.*

5. σ-STABLE TORSION THEORY AND σ-COSTABLE TORSION THEORY

A torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ is called *stable* if $\mathcal{T}_t$ is closed under taking injective hulls. In this section we generalize stable torsion theory by using torsion theories.

Proposition 21. *Let σ be an idempotent radical and L a submodule of a module M. Then the implications* (1) $\to$ (2) $\to$ (3) *hold. Moreover, if σ is a left exact radical, then* (3) $\to$ (1) *holds.*

(1) *L is σ-complemented in M, that is, there exists a submodule K of M such that L is maximal in $\Gamma_K = \{L_i \ ; \ L_i \subseteq M, L_i \cap K = 0, M/(L_i + K) \in \mathcal{T}_\sigma\}$*

(2) $L = E_\sigma(L) \cap M$.

(3) *L is σ-essentially closed in M,that is, there is no σ-essential extension of L in M.*

We call a preradical t *σ-stable* if $\mathcal{T}_t$ is closed under taking σ-injective hulls. We put $\mathcal{X}_t(M) := \{X \ ; \ M/X \in \mathcal{T}_t\}$ and $N \cap \mathcal{X}_t(M) := \{N \cap X \ ; \ X \in \mathcal{X}_t(M)\}$.

Theorem 22. *Let t be an idempotent preradical and σ an idempotent radical. Then the following conditions* (1), (2) *and* (3) *are equivalent. Moreover, if σ is left exact and $\mathcal{T}_t$ is closed under taking σ-dense submodules, then all the following conditions are equivalent.*

(1) *t is σ-stable, that is, $\mathcal{T}_t$ is closed under taking σ-injective hulls.*

(2) *The class of σ-injective modules are closed under taking the unique maximal t-torsion submodules, that is, $t(M)$ is σ-injective for any σ-injective module M.*

(3) *$E_\sigma(t(M)) \subset t(E_\sigma(M))$ holds for any module M.*

(4) *$\mathcal{T}_t$ is closed under taking σ-essential extensions.*

(5) *If M/N is σ-torsion, then $N \cap \mathcal{X}_t(M) = \mathcal{X}_t(N)$ holds.*

(6) *For any module M, $t(M)$ is σ-complemented in M.*

(7) *For any module M, $t(M) = E_\sigma(t(M)) \cap M$ holds.*

(8) *For any module M, $t(M)$ is σ-essentially closed in M.*

(9) *For any σ-injective module E with $E/t(E) \in \mathcal{T}_\sigma$, $t(E)$ is a direct summand of E.*

(10) *$E_\sigma(t(M)) = t(E_\sigma(M))$ holds for any module M.*

If $\mathcal{T}_t$ is closed under taking σ-dense submodules, then (1) $\to$ (4) $\to$ (5) *hold. Moreover, if σ is left exact, then* (1) $\to$ (6) *and* (3) $\to$ (10) *hold.*

It is well known that if R is right noetherian, t is stable if and only if every indecomposable injective module is t-torsion or t-torsionfree. By using Theorem 1 in [3], we generalized this as follows.

Theorem 23. *Let t be an idempotent radical and σ a left exact radical. Then*

(1) *If t is σ-stable, then $(*)$ every indecomposable σ-injective module E with $E/T(E) \in \mathcal{T}_\sigma$ is either t-torsion or t-torsionfree.*

(2) *If the ring R satisfies the condition $(*)$ and the ascending chain conditions on σ-dense ideals of R, then $\mathcal{T}_t \cap \mathcal{T}_\sigma$ is closed under taking σ-injective hulls.*

We now dualize σ-stable torsion theory. Let R be a right perfect ring. We call a preradical t *σ-costable* if $\mathcal{F}_t$ is closed under taking σ-projective covers.

Theorem 24. *Let σ be an idempotent radical. Then a radical t is σ-costable if and only if the class of σ-projective modules is closed under taking the unique maximal t-torsionfree factor modules, that is, $P/t(P)$ is σ-projective for any σ-projective module P.*

6. σ-SINGULAR SUBMODULES

Let σ be a left exact radical. For a module M we put $Z_\sigma(M) := \{m \in M \ ; \ (0:m)$ is σ-essential in $R\}$ and call it *σ-singular submodule of M.* Since $R/(0:m) \in \mathcal{T}_Z \cap \mathcal{T}_\sigma$, then

$Z_\sigma(M) \subseteq Z(M) \cap \sigma(M) = Z(\sigma(M)) = \sigma(Z(M))$, and so $Z_\sigma(M) = \{m \in M\ ;\ mR \in \mathcal{T}_Z \cap \mathcal{T}_\sigma\}$. Since Z and σ is left exact, Z_σ is also left exact. We will call M *σ-singular* (resp. *σ-nonsingular*) if $Z_\sigma(M) = M$ (resp. $Z_\sigma(M) = 0$).

Proposition 25. *Let σ be an idempotent radical and E a σ-nonsingular module and $\mathcal{T} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(M, E) = 0\}$. Then $\mathcal{T}$ is closed under taking σ-essential extensions. Therefore a torsion theory $(\mathcal{T}, \mathcal{F})$ is σ-stable, where $\mathcal{F} = \{M \in \text{Mod-}R\ ;\ \text{Hom}_R(X, M) = 0$ for any $X \in \mathcal{T}\}$.*

Proposition 26. *Let σ a left exact radical. Then the following facts hold.*

(1) *If N is σ-essential in M, then $Z_\sigma(M/N) = M/N$.*

(2) *A right ideal of R is σ-essential in R if and only if $Z_\sigma(R/I) = R/I$.*

(3) *Let M be a σ-nonsingular module and N a submodule of M. Then N is σ-essential in M if and only if $Z_\sigma(M/N) = M/N$.*

(4) *For a module M, $Z_\sigma(M/Z_\sigma(M)) = M/Z_\sigma(M)$ holds if and only if $Z_\sigma(M)$ is σ-essential in M.*

(5) *For a simple right R-module S, S is σ-nonsingular if and only if S is σ-torsionfree or projective.*

(6) *If R is σ-nonsingular, then Z_σ is left exact radical.*

(7) *If M/N is σ-nonsingular for a module M and a submodule N of M, then N is σ-complemented in M. If M is σ-nonsingular, then the converse holds.*

7. σ-small and σ-radical

Let σ be a left exact radical. A submodule N of a module M is called *σ-dense* in M if M/N is σ-torsion. A module M is called *σ-cocritical* if M is σ-torsionfree and L is σ-dense in M for any nonzero submodule L of M. It is well known that nonzero submodule of σ-cocritical module M is essential in M. A module M is called *σ-noetherian* if for every ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq I_4 \subseteq \cdots\cdots \subseteq M$, (where $\cup I_j$ is σ-dense in M), there exists a positive integer k such that I_k is σ-dense in M. Let $J_\sigma(M)$ denote $\cap N_i(M/N_i$ is σ-cocritical).

Now we define σ-small submodule as follows. A submodule N of a module M is called *σ-small* in M if $M/(N + X) \in \mathcal{T}_\sigma$ implies $M/X \in \mathcal{T}_\sigma$ for any submodule X of M.

Theorem 27. *$J_\sigma(M)$ contains $\sum N(N$ is σ-small in $M)$. Conversely if M be a σ-noetherian module, then $J_\sigma(M)$ coincides with $\sum N(N$ is σ-small in $M)$.*

Remark 28. We can see in [4] that the definition of σ-small is different from ours.

(B.A.Benander's definition). N is σ-small in M if $M/(N' + X) \in \mathcal{T}_\sigma$ and $M/X \in \mathcal{F}_\sigma$, then $M = X$ for any X of M, where $\sigma(M/N) = N'/N$.

Benander's definition of σ-small is a stronger condition than ours.

In fact, if $M/(N + X) \in \mathcal{T}_\sigma$, then $M/(N' + X) \in \mathcal{T}_\sigma$. We put $X'/X := \sigma(M/X)$. Then $M/X' \in \mathcal{F}_\sigma$. Since $M/(N' + X) \in \mathcal{T}_\sigma$, $M/(N' + X') \in \mathcal{T}_\sigma$. Thus $M = X'$, and so $M/X \in \mathcal{T}_\sigma$, as desired.

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