ABOUT DECOMPOSITION NUMBERS OF J_4

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Abstract. The decomposition numbers with some unknown parameters of non-principal blocks of the largest Janko group J_4 [5] for characteristic 3 are determined. We also concerned with the decomposition numbers of maximal 2-local subgroups [6] of J_4 in odd characteristics. We used the character table library in GAP[4]

Key Words: Sporadic group J_4, Green correspondence, Modular representation.

1. Notation

Let G be a finite group. Let p be an odd prime such that p devides the order of G. We denote by $Bl_p^+(G)$ a set of p-block of G with positive defect. For $A \in Bl_p^+(G)$, we denote by $\mathrm{Irr}(A)$ a set of irreducible ordinary characters in A and by $\mathrm{IBr}(A)$ a set of irreducible Brauer characters in A. Let $k(A)$ and $l(A)$ be numbers of irrecucible characters in $\mathrm{Irr}(A)$ and $\mathrm{IBr}(A)$, respectively. Let I_G be the trivial character of G. Let $b_0(G)$ be the principal block of G i.e. $b_0(G) \in Bl_p^+(G)$ and $I_G \in \mathrm{Irr}(b_0(G))$. We denote by $D(A)$ the decomposition matrix of A with respect to $\mathrm{Irr}(A) = \{\chi_1, \cdots, \chi_{k(A)}\}$ and $\mathrm{IBr}(A) = \{\varphi_1, \cdots, \varphi_{l(A)}\}$. So $D(A)$ is the $k(A) \times l(A)$-matrix $\{d_{ij}\}$ such that $\chi_i = \sum_{j=1}^{l(A)} d_{ij}\varphi_j$ for $i = 1, \ldots, k(A)$ on p'-elements in G.

Let k be an algebraically closed field. Let H be a subgroup of G. We called a kG–module M is a trivial source module if M is a direct summand of the induced module of the trivial kH–module. Since trivial source modules have some good property, it is important to find may trivial source modules. In particular, simple trivial source modules are very important.

2. Fong's theorem

Let X be a normal p'-subgroup of G. Let b be a p–block of X. Since X is p'-group, $\mathrm{Irr}(b)$ has only one irreducible character ξ. Let $T = T(b)$ be an inertial group of b in G. If a p–block B of T is a direct summand of $e_b kT$ as a k–algebra, we call that B covers b. We denote by $Bl(T\,|\,b)$ the set of all p–blocks of T which cover b. In [3], Fong showed the following two theorems.

Theorem 1. *(2B in* [3]*) Let A be a p–block in $Bl(G\,|\,b)$. Then there is a p–block B in $Bl(T\,|\,b)$, such that the following are true:*

(i) *A and B have a defect group in common.*

The detailed version of this paper will be submitted for publication elsewhere.

(ii) *There is a 1-1 height-preserving correspondence between the irreducible ordinary characters of A and B.*
(iii) *There is a 1-1 correspondence between the irreducible modular characters of A and B.*
(iv) *With respect to these correspondences of characters, the matrices of decomposition numbers and Cartan invariants of A and B are same.*

Let s be the Schur multiplier of T/X.

Theorem 2. *(2D in [3]) Let B be a p–block in $Bl(T\,|\,b)$. Then there is a group $\widehat{T}$ with a cyclic normal p'-subgroup Z and p–block $\widehat{B}$ in $Bl(\widehat{T}\,|\,\widehat{b})$ where $\widehat{b}$ is a p–block of Z such that the following are true:*
(i) *B and $\widehat{B}$ have isomorphic defect groups.*
(ii) *There is a 1-1 height-preserving correspondence between the irreducible ordinary characters of B and $\widehat{B}$.*
(iii) *There is a 1-1 correspondence between the irreducible modular characters of B and $\widehat{B}$.*
(iv) *With respect to these correspondences of characters, the matrices of decomposition numbers and Cartan invariants of B and $\widehat{B}$ are same.*

The group $\widehat{T}$ has the following structure:
(a) *Z is the center of $\widehat{T}$.*
(b) *$\widehat{T}/Z \cong T/X$.*
(c) *The order of Z is s.*

In case that the irreducible character ξ is linear and T is a semidirect product of X with T/X. It is easy to see that the p–block $\widehat{b}$ is the principal block. Thus we can identify p–blocks in $Bl(\widehat{T}\,|\,\widehat{b})$ with p–blocks in $Bl(\widehat{T}/Z) = Bl(T/X)$. So the next corollary follows.

Corollary 3. *If ξ is a linear character and T is a semidirect product of X with T/X, then there is a bijection between p–blocks in $Bl(T\,|\,b)$ and $Bl(T/X)$ such that the same statements in Theorem 2 hold.*

3. Decomposition Matrix of J_4

Let G be the largest Janko group J_4. The order of G is $2^{31}\cdot 3^3\cdot 5\cdot 7\cdot 11^3\cdot 23\cdot 29\cdot 31\cdot 37\cdot 43$. There are non-conjugate involutions s, t in G such that $K := C_G(s) \cong 2_+^{1+12}.3M_{22}{:}2$, $C_G(t) \cong 2^{11}{:}M_{22}{:}2$. The centralizer $C_G(t)$ is contained in a subgroup $H \cong 2^{11}{:}M_{24}$. The subgroups H and K are the maximal subgroups of G.

In [6], B. Kleidman and R. A. Wilson investigate these groups in detail. The character tables of these groups are found by GAP[4]. We apply Fong's theorem for getting the decomposition numbers of these maximal 2-local subgroups H and K.

Proposition 4. *Let p be an odd prime. Then all $D(B)$ and $D(C)$ where $B \in Bl_p^+(H)$ and $C \in Bl_p^+(K)$ are determined.*

In case that $p = 3$, let
$Bl_3^+(H) = \{B_{3a},\ B_{3b},\ B_2,\ B_{1a}, \dots, B_{1p}\}$

$Bl_3^+(K) = \{C_{3a},\ C_{3b},\ C_{2a},\ C_{2b}\ C_{1a}, \dots, C_{1h}\}$
$Bl_3^+(6.M_{22}{:}2) = \{X_{3a},\ X_{3b},\ X_{2a},\ X_{2b}\ X_{1a}, \dots, X_{1d}\}$
where indeces of each p-blocks indicate its defect.

(1) $D(B_{1*})$ and $D(C_{1*})$ are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(2) $D(B_{3a}) = D(b_0(M_{24}))$, $D(B_{3b}) = D(b_0(M_{12}.2))$, $D(B_2) = D(b_0(2^4{:}A_8))$

(3) $D(C_{3a}) = D(X_{3a})$, $D(C_{3b}) = D(X_{3b})$, $D(C_{2a}) = D(X_{2a})$, $D(C_{2b}) = D(X_{2b})$

From above decomposition matrices, I can calculate projective indecomposable characters of H and K. By inducing these characters to G, we can get projective characters of G. Using tensor of characters and Green correspondence between G and H or G and K, we can prove the following proposition.

Proposition 5. *Let $p = 3$ then $Bl_p^+(G) = \{A_{3a}, A_{3b}, A_2, A_{1a}, A_{1b}, A_{1c}, A_{1d}\}$. The almost all decomposition numbers (=entries of decomposition matrix) of non-principal blocks are determined the followsing. The indeces which we put on the irreducible ordinary characters are same one in* [2].

(1) $D(A_{1*})$ *are* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ *or* $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(2) $$D(A_2) = \begin{pmatrix} 1 &&&& \\ 0 & 1 &&& \\ 0 & 0 & 1 && \\ 0 & 0 & 0 & 1 & \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

where $\mathrm{Irr}(A_2) = \{\chi_{14}, \chi_{21}, \chi_{25}, \chi_{27}, \chi_{28}, \chi_{30}, \chi_{31}, \chi_{35}, \chi_{41}\}$.

(3) $$D(A_{3b}) = \left(\begin{array}{ccccc|cccc} 1 &&&&&&&& \\ 0 & 1 &&&&&&& \\ 1 & 0 & 1 &&&&&& \\ 0 & 1 & 0 & 1 &&&&& \\ 0 & 0 & 1 & 0 & 1 &&&& \\ 0 & 0 & 0 & 1 & 1 &&&& \\ \hline 0 & 0 & 0 & 0 & \alpha & 1 &&& \\ 0 & 0 & 0 & 0 & \alpha+1 & 1 &&& \\ 0 & 0 & 0 & 0 & \alpha+1 & 1 &&& \\ 0 & 0 & 1 & 1 & \beta & 0 & 1 && \\ 0 & 0 & 0 & 0 & \alpha+\beta+\gamma+1 & 1 & 1 & 1 & \\ 0 & 0 & 0 & 0 & \alpha+\beta+\gamma+1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & \beta+2\gamma+1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2\beta+2\gamma+2 & 0 & 2 & 1 & 1 \end{array}\right)$$

where $0 \leqq \alpha \leqq 3$, $0 \leqq \beta \leqq 7$, $0 \leqq \alpha + \beta + \gamma \leqq 15$, $0 \leqq \beta + \gamma \leqq 12$ *and* $\mathrm{Irr}(A_{3b}) = \{\chi_2, \chi_3, \chi_{12}, \chi_{13}, \chi_{17}, \chi_{18}, \chi_{22}, \chi_{23}, \chi_{24}, \chi_{26}, \chi_{38}, \chi_{39}, \chi_{44}, \chi_{50}\}$.

4. Green correspondence and Trivial Source Module

In G, there are 2 simple modules M_a and M_b with dimension 1,333. In this section, we see that these modules are trivial source modules.

Let χ_2 and χ_3 in $\mathrm{Irr}(A_{3b})$ of degree 1,333. Then these two characters are corresponding to M_a and M_b.

Let P be a Sylow 3-subgroup which is isomorphic to the extraspecial group of the order 27. The center of P denote by $Z := Z(P)$. We can get the following inclusion.

$$Z \subset P \subset N_G(P) \cong (2 \times P : 8) : 2 \subset N_G(Z) \cong 6.M_{22}\colon 2 \subset K \subset G$$

Let F and f be the Green correspondence with respect to $(G, P, N_G(Z))$ and $(G, P, N_G(P))$, respectively.

Proposition 6. *The simple modules M_a and M_b are trivial source.*

Proof:

Since the restriction of χ_2 to K is a direct sum of $640a \in \mathrm{Irr}(C_{3b})$ and $693c \in \mathrm{Irr}(C_{1f})$ and the restriction of $640a$ to $N_G(Z)$ is a direct sum of $10a \in \mathrm{Irr}(X_{3b})$ and $210c + 420a \in \mathrm{Irr}(X_{2b})$ by GAP, $F(M_a) = 10a$. Moreover we can check that the restriction of $10a$ to $N_G(P)$ are a direct sum of $1a$ and $9a$ by MAGMA[1]. Thus $f(M_a) = 1a$. So M_a is the direct summand of the induced module $1a^G$ and M_a is the trivial source module. For M_b, we can prove by the same way. □

There is an irreducible character θ in B_{3a} with degree 45. This character is corresponding to simple trivial source module. Since θ^G has a direct summand $\chi_2 + \chi_{44}$, there are a trivial source module M which is corresponding to $\chi_2 + \chi_{44}$. So we can prove that the top and the Socle of M are isomorphic to M_a. I hope that I can determine the unknown number β and γ by the investigation of the Loewy structure of M.

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