THE NUMBER OF COMPLETE EXCEPTIONAL SEQUENCES

TOKUJI ARAYA

Abstract. A complete exceptional sequence is very useful to investigate the category of finitely generated modules over a finite dimensional algebra. The aim of this note is to show how to find the all complete exceptional sequences over the path algebra of Dynkin quiver of type $(\mathbf{A}_n)$.

1. Introduction

Let Λ be the path algebra of Dynkin quiver of type $(\mathbf{A}_n)$ over a field k. We denote by $\operatorname{mod}\Lambda$ the category of finitely generated left Λ-modules. The concept of exceptional sequences was introduced by Gorodentsev and Rudakov [1]. It is very useful to investigate $\operatorname{mod}\Lambda$. A finitely generated left Λ-module E is called *exceptional* if $\operatorname{Hom}_\Lambda(E,E) \cong k$ and $\operatorname{Ext}^1_\Lambda(E,E) = 0$. We remark that E is exceptinal if and only if it is indecomposable. Indeed Λ is the path algebra of $(\mathbf{A}_n)$. A pair (E,F) of exceptional modules is called an *exceptional pair* if $\operatorname{Hom}_\Lambda(F,E) = \operatorname{Ext}^1_\Lambda(F,E) = 0$. A sequence $\epsilon = (E_1, E_2, \cdots, E_r)$ of exceptional modules is called an *exceptional sequence* of length r if (E_i, E_j) is an exceptional pair for each $i < j$. An exceptinal sequence ϵ is called *complete* if the length of ϵ is equal to n. (Here, n is the number of simple modules in $\operatorname{mod}\Lambda$). We put $\mathfrak{E}$ the set of complete exceptional sequences. Siedel [2, Proposition 1.1] proved that the cardinarity of $\mathfrak{E}$ is equal to $(n+1)^{n-1}$. There are a number of complete exceptional sequences. But it is not easy to find all complete exceptional sequence. The main purpose is to get how to find the complete exceptinal sequences completely by using the conbinatorics.

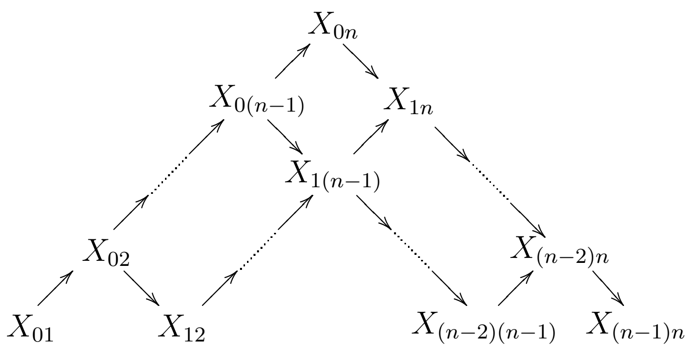
2. Main result

First of all, we give a remark that $\mathfrak{E}$ is independent of the orientation of $(\mathbf{A}_n)$. Indeed, let Λ' be a path algebra of Dynkin quiver of type $(\mathbf{A}_n)$ whose orientation is not equal to Λ, and let $\mathfrak{E}'$ be the set of complete exceptional sequences in $\operatorname{mod}\Lambda'$. In this case, Λ and Λ' are derived equivalent and there exists a equivalence $\varphi : \mathcal{D}^b(\operatorname{mod}\Lambda) \to \mathcal{D}^b(\operatorname{mod}\Lambda')$. Therefore we can get the one to one correspondence $\psi : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda'$ by φ and the suspention functor in $\mathcal{D}^b(\operatorname{mod}\Lambda')$. One can easily check that ψ gives the one to one correspondence between $\mathfrak{E}$ and $\mathfrak{E}'$. Thus we may assume the orientation of $(\mathbf{A}_n)$ as follows;

$$\overset{e_1}{\bullet} \longrightarrow \overset{e_2}{\bullet} \longrightarrow \cdots \longrightarrow \overset{e_n}{\bullet}$$

Let Γ be the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$. We identify the set Γ_0 of vertices in Γ with the class $\{X_{ij} \mid 0 \le i < j \le n\}$ of indecomposable Λ-modules. Then Γ is as follows;

The detailed version of this paper will be submitted for publication elsewhere.



We consider a circle with $n+1$ points labelled $0, 1, 2, \cdots, n$ counter clockwise on it. We put $c(i,j)$ the chord between the points i and j. We denote by C_{n+1} the set of chords in the circle. Since $C_{n+1} = \{c(i,j) |\ 0 \leq i < j \leq n\}$, there exists a one to one correspondence $\Phi : \Gamma_0 \to C_{n+1}$ defined by $\Phi(X_{ij}) = c(i,j)$.

For $\epsilon = (E_1, E_2, \cdots, E_n), \epsilon' = (E'_1, E'_2, \cdots, E'_n) \in \mathfrak{E}$, we define $\epsilon \sim \epsilon'$ by $\bigoplus_{i=0}^n E_i \cong \bigoplus_{i=0}^n E'_i$. Then $\sim$ is an equivalent relation on $\mathfrak{E}$. We shall prove the following theorem.

Theorem 1. Φ *gives a one to one correspondence between* $\mathfrak{E}/\sim$ *and the set of non crossing spanning trees by* $\Phi(\epsilon) := \{\Phi(E_1), \Phi(E_2), \cdots, \Phi(E_n)\}$ *for each* $\epsilon = (E_1, E_2, \cdots, E_n)$.

Here, we call a graph T a *non crossing spanning tree* if the following conditions are satisfied;

(i) the chords in T form a tree,
(ii) the chords in T meet only at endpoints.

It is known the number of noncrossing spanning trees. We get the following corollary.

Corollary 2. *The cardinarity of* $\mathfrak{E}/\sim$ *is equal to* $\dfrac{1}{2n+1}\dbinom{3n}{n}$.

Proof of Theorem 1. For $X \in \Gamma_0$, we consider the following four classes.

$$\begin{aligned}
\mathcal{H}_+(X) &= \{Y \in \Gamma_0 |\ \mathrm{Hom}_\Lambda(X,Y) \neq 0\},\\
\mathcal{H}_-(X) &= \{Y \in \Gamma_0 |\ \mathrm{Hom}_\Lambda(Y,X) \neq 0\},\\
\mathcal{E}_+(X) &= \{Y \in \Gamma_0 |\ \mathrm{Ext}^1_\Lambda(X,Y) \neq 0\},\\
\mathcal{E}_-(X) &= \{Y \in \Gamma_0 |\ \mathrm{Ext}^1_\Lambda(Y,X) \neq 0\}.
\end{aligned}$$

Then one can check the followings by using Auslander-Reiten sequence;

$$\begin{aligned}
\mathcal{H}_+(X_{i,j}) &= \{X_{s,t} |\ i \leq s \leq j-1, j \leq t \leq n\},\\
\mathcal{H}_-(X_{i,j}) &= \{X_{s,t} |\ 0 \leq s \leq i, i+1 \leq t \leq j\},\\
\mathcal{E}_+(X_{i,j}) &= \{X_{s,t} |\ 0 \leq s \leq i-1, i \leq t \leq j-1\},\\
\mathcal{E}_-(X_{i,j}) &= \{X_{s,t} |\ i+1 \leq s \leq j, j+1 \leq t \leq n\}.
\end{aligned}$$

Furthermore, we consider the following four classes for each $X \in \Gamma_0$;

$$\begin{aligned}
\mathfrak{P}(X) &= \{Y|\ \text{Both } (X,Y) \text{ and } (Y,X) \text{ are exceptional pair.}\}, \\
\mathfrak{P}_+(X) &= \left\{ \text{Y} \middle| \begin{array}{l} (X,Y) \text{ is an exceptinal pair,} \\ (Y,X) \text{ is not an exceptinal pair.} \end{array} \right\}, \\
\mathfrak{P}_-(X) &= \left\{ \text{Y} \middle| \begin{array}{l} (Y,X) \text{ is an exceptinal pair,} \\ (X,Y) \text{ is not an exceptinal pair.} \end{array} \right\}, \\
\overline{\mathfrak{P}(X)} &= \{Y|\ \text{Both } (X,Y) \text{ and } (Y,X) \text{ are not exceptional pair.}\}.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathfrak{P}(X) &= \Gamma_0 \setminus (\mathcal{H}_+(X) \cup \mathcal{E}_+(X) \cup \mathcal{H}_-(X) \cup \mathcal{E}_-(X)), \\
\mathfrak{P}_+(X) &= (\mathcal{H}_+(X) \cup \mathcal{E}_+(X)) \setminus (\mathcal{H}_-(X) \cup \mathcal{E}_-(X)), \\
\mathfrak{P}_-(X) &= (\mathcal{H}_-(X) \cup \mathcal{E}_-(X)) \setminus (\mathcal{H}_+(X) \cup \mathcal{E}_+(X)), \\
\overline{\mathfrak{P}(X)} &= (\mathcal{H}_+(X) \cup \mathcal{E}_+(X)) \cap (\mathcal{H}_-(X) \cup \mathcal{E}_-(X)),
\end{aligned}$$

we get the followings for each $X_{i,j} \in \Gamma_0$;

$$\begin{aligned}
\mathfrak{P}(X_{i,j}) &= \{X_{s,t}|\ 0 \le s < t \le i\} \cup \{X_{s,t}|\ i+1 \le s < t \le j-1\} \\
&\quad \cup \{X_{s,t}|\ j \le s < t \le n\} \cup \{X_{s,t}|\ 0 \le s \le i-1, j+1 \le t \le n\}, \\
\mathfrak{P}_+(X_{i,j}) &= \{X_{s,i}|\ 0 \le s \le i-1\} \cup \{X_{i,t}|\ j+1 \le t \le n\} \cup \{X_{s,j}|\ i+1 \le s \le j-1\}, \\
\mathfrak{P}_-(X_{i,j}) &= \{X_{i,t}|\ i+1 \le s \le j-1\} \cup \{X_{s,j}|\ 0 \le s \le j-1\} \cup \{X_{j,t}|\ j+1 \le s \le n\}, \\
\overline{\mathfrak{P}(X_{i,j})} &= \{X_{s,t}|\ 0 \le s \le i-1, i+1 \le t \le j-1\} \\
&\quad \cup \{X_{s,t}|\ i+1 \le s \le j-1, j+1 \le t \le n\}.
\end{aligned}$$

We apply Φ for each above classes, we get followings;

$$\begin{aligned}
\Phi(\mathfrak{P}(X_{i,j})) &= \{c(s,t)|\ 0 \le s < t \le i\} \cup \{c(s,t)|\ i+1 \le s < t \le j-1\} \\
&\quad \cup \{c(s,t)|\ j \le s < t \le n\} \cup \{c(s,t)|\ 0 \le s \le i-1, j+1 \le t \le n\}, \\
\Phi(\mathfrak{P}_+(X_{i,j})) &= \{c(s,i)|\ 0 \le s \le i-1\} \cup \{c(i,t)|\ j+1 \le t \le n\} \\
&\quad \cup \{c(s,j)|\ i+1 \le s \le j-1\}, \\
\Phi(\mathfrak{P}_-(X_{i,j})) &= \{c(i,t)|\ i+1 \le s \le j-1\} \cup \{c(s,j)|\ 0 \le s \le j-1\} \\
&\quad \cup \{c(j,t)|\ j+1 \le s \le n\}, \\
\Phi(\overline{\mathfrak{P}(X_{i,j})}) &= \{c(s,t)|\ 0 \le s \le i-1, i+1 \le t \le j-1\} \\
&\quad \cup \{c(s,t)|\ i+1 \le s \le j-1, j+1 \le t \le n\}.
\end{aligned}$$

Thus we have followings;

- $Y \in \mathfrak{P}(X) \Leftrightarrow$ $\Phi(Y)$ does not meet to $\Phi(X)$.
- $Y \in \mathfrak{P}_+(X) \Leftrightarrow$ $\Phi(Y)$ meets $\Phi(X)$ for some vertex i and $\Phi(Y)$ is the chord moved $\Phi(X)$ around a vertex i counterclockwise across the interior of the circle.
- $Y \in \mathfrak{P}_-(X) \Leftrightarrow$ $\Phi(Y)$ meets $\Phi(X)$ for some vertex i and $\Phi(Y)$ is the chord moved $\Phi(X)$ around a vertex i clockwise across the interior of the circle.

- $Y \in \overline{\mathfrak{P}(X)} \Leftrightarrow \Phi(Y)$ meets to $\Phi(X)$ at interior of the circle.

Therefore for any $\epsilon \in \mathfrak{E}$, each chords in $\Phi(\epsilon)$ do not meet each other at interior of the circle.

For $X_1, X_2, \cdots, X_r \in \Gamma_0$, suppose $\{\Phi(X_1), \Phi(X_2), \cdots, \phi(X_r)\}$ makes a cycle. We may assume $\Phi(X_\ell)$ meets $\Phi(X_{\ell+1})$ at a vertex i_ℓ for each $\ell = 1, 2, \cdots, r$ (where $X_{r+1} = X_1$) and $i_1 > i_2 > \cdots > i_r$. Then, $(X_1, X_2), (X_2, X_3), \cdots, (X_{r-1}, X_r)$ and (X_r, X_1) are exceptional pairs but $(X_2, X_1), (X_3, X_2), \cdots, (X_r, X_{r-1})$ and (X_1, X_r) are not exceptional pairs. Therefore any permutatin of $(X_1, X_2, \cdots, X_r)$ is not an exceptional sequence.

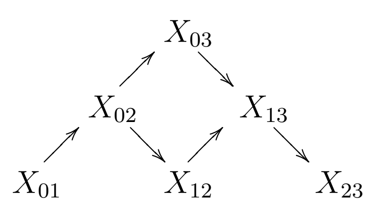
Thus we get $\Phi(\epsilon)$ is a non crossing spanning tree for any $\epsilon \in \mathfrak{E}$.

For $\epsilon = (E_1, E_2, \cdots, E_n), \epsilon' = (E'_1, E'_2, \cdots, E'_n) \in \mathfrak{E}$ suppose $\Phi(\epsilon) = \Phi(\epsilon')$. Then $\{\Phi(E_1), \Phi(E_2), \cdots, \phi(E_r)\} = \{\Phi(E_1), \Phi(E_2), \cdots, \phi(E_r)\}$. Since $\Phi : \Gamma_0 \to C_{n+1}$ is one to one, we get $\epsilon \sim \epsilon'$.

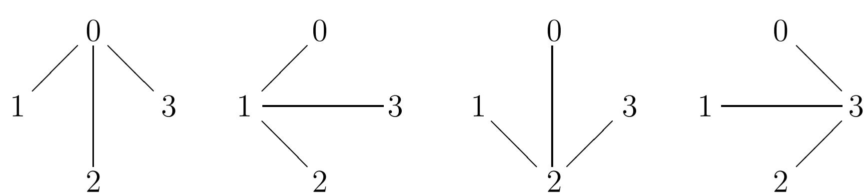
Conversely, suppose $T = \{c_1, c_2, \cdots, c_n\} \subset C_{n+1}$ is a non crossing spanning tree. We put $X_i := \Phi^{-1}(c_i)$ for each i. If there exists a pair (X_i, X_j) $(i \neq j)$ such that both (X_i, X_j) and (X_j, X_i) are not exceptional pair, then c_i crosses c_j at interior. Thus, there does not exist a such pair.

If there exists a subsequence $\{X_{a_1}, X_{a_2}, \cdots X_{a_r}\}$ such that (X_{a_1}, X_{a_2}), (X_{a_2}, X_{a_3}), $\cdots$, $(X_{a_{r-1}}, X_{a_r})$, and (X_{a_r}, X_{a_1}) are exceptional pairs but (X_{a_2}, X_{a_1}), (X_{a_3}, X_{a_2}), $\cdots$, $(X_{a_r}, X_{a_{r-1}})$, and (X_{a_1}, X_{a_r}) are not exceptional pairs, then $\{c_{a_1}, c_{a_2}, \cdots, c_{a_r}\}$ makes a cycle. Therefore there exists a parmutation σ such that $(X_{\sigma(1)}, X_{\sigma(2)}, \cdots X_{\sigma(n)})$ is a complete exceptional sequence. □

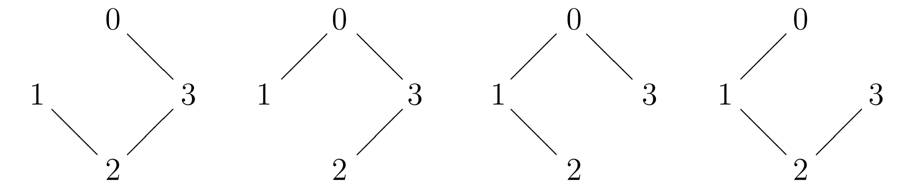
Example 3. If $n = 3$, the following quiver is the Auslander-Reiten quiver of $\operatorname{mod}\Lambda$.



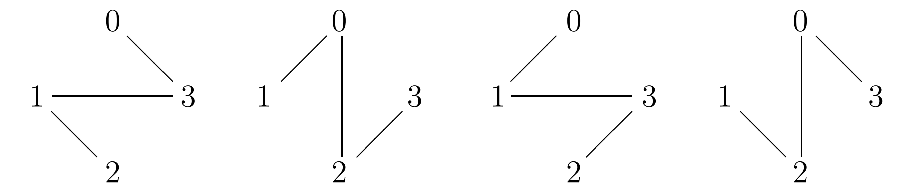
In this case, there are 16 complete exceptional sequences and 12 non crossing spanning trees. The followings are the complete exceptional sequences and corresponding non crossing spanning trees.



(X_{01}, X_{02}, X_{03}) (X_{12}, X_{13}, X_{01}) (X_{23}, X_{02}, X_{12}) (X_{03}, X_{13}, X_{23})



(X_{03}, X_{23}, X_{12}) (X_{01}, X_{03}, X_{23}) (X_{12}, X_{01}, X_{03}) (X_{23}, X_{12}, X_{01})



(X_{03}, X_{12}, X_{13}) (X_{23}, X_{01}, X_{02}) (X_{13}, X_{23}, X_{01}) (X_{02}, X_{03}, X_{12})

(X_{12}, X_{03}, X_{13}) (X_{01}, X_{23}, X_{02}) (X_{13}, X_{01}, X_{23}) (X_{02}, X_{12}, X_{03})

References

[1] A. L. Gorodentsev and A. N. Rudakov, *Exceptional vector bundles on projective spaces*, Duke. math. J. **54** (1987), 115–130.

[2] U. Siedel, *Exceptional sequences for quivers of Dynkin type*, Comm. Algebra **29** (3) (2001), 1373–1386.

Nara University of Education,
Takabatake-cho, Nara-city 630-8528, Japan
E-mail address: araya@math.okayama-u.ac.jp