SOME REGULAR DIRECT-SUM DECOMPOSITIONS IN MODULE THEORY

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Abstract. We review recent results about a weak form of the Krull-Schmidt Theorem that holds in some classes of modules.

1. Introduction

This is a survey about some direct-sum decompositions of modules with regular and interesting behaviors presented in two talks given in Shizuoka at the "Fortyfirst Symposium on Ring Theory and Representation Theory" (September 5-7, 2008). In particular, the first half of the paper will be devoted to describing some notions that have proved to be useful in the study of direct-sum decompositions. The symbol R will always denote an arbitrary associative ring with identity $1_R \neq 0_R$, and modules will be unital right R-modules unless otherwise stated explicitly.

Our aim is to describe the direct-sum decompositions $M_R = M_1 \oplus \cdots \oplus M_n$ of a fixed module M_R into a direct sum of finitely many direct summands $M_1, \ldots, M_n$. Several behaviors can take place. The best case we can have is when we have uniqueness up to isomorphism, as in the case of the celebrated *Krull-Schmidt Theorem*, which we all know:

Theorem 1. [Krull-Schmidt Theorem] *Every module M of finite composition length is a direct sum of indecomposable modules. If*

$$M = M_1 \oplus \cdots \oplus M_t = N_1 \oplus \cdots \oplus N_s$$

are two decompositions of M into direct sums of indecomposables, then $t = s$ and there is a permutation σ of $\{1, 2, \ldots, t\}$ such that $M_i \cong N_{\sigma(i)}$ for every $i = 1, 2, \ldots, t$.

A theorem of this kind appeared for the first time in a paper of Frobenius and Stickelberger [19], who proved the structure theorem of finite abelian groups (finite abelian groups are direct sums of cyclic subgroups whose orders are powers of primes, and these powers of primes are uniquely determined by the group). The Krull-Schmidt Theorem was later generalized by Azumaya in 1950 to infinite direct sums of modules with local endomorphism ring [3]. Important work on the Krull-Schmidt-Azumaya Theorem can be found in Harada [20], who introduced the use of factor categories in this setting. For an interesting survey on these results and their relation with the exchange property and extending modules, see [24].

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Uniqueness of direct-sum decomposition is an exception in Module Theory, and we will give in §4.1 an easy example of failure of the Krull-Schmidt Theorem for finitely generated modules over a noetherian commutative integral domain. A different possibility we can have decomposing a module M_R is that the module M_R possesses only finitely many direct-sum decompositions up to isomorphism. This is the case of finite-rank torsion-free abelian groups [23].

Another possible case we can meet is that of the modules M_R that do not decompose in a unique way up to isomorphism, but their direct sums enjoy some kind of regularity. We see in §4.2 that this happens for modules with a semilocal endomorphism ring, for instance for artinian modules. Several other possibilities can occur: a module can be a direct sum of indecomposables or not, can be decomposable but with no indecomposable direct summands, and so on.

2. Commutative monoids, order-units, and the Bergman-Dicks Theorem

Fix a class $\mathcal{C}$ of right R-modules. We want to study the direct-sum decompositions of the modules belonging to $\mathcal{C}$. We will assume that $\mathcal{C}$ is closed under isomorphism, direct summands and finite direct sums. For every module A_R, let $\langle A_R\rangle := \{\, B_R \mid B_R \cong A_R \,\}$ denote the *isomorphism class* of the module A_R. Set $V(\mathcal{C}) := \{\, \langle A_R\rangle \mid A_R \in \mathcal{C}\,\}$. Assume that $V(\mathcal{C})$ is a set[1]. Define $\langle A_R\rangle + \langle B_R\rangle := \langle A_R \oplus B_R\rangle$ for every $A_R, B_R \in \mathcal{C}$. Then $V(\mathcal{C})$ becomes an additive commutative monoid, which is clearly the algebraic structure that describes the direct-sum decompositions in $\mathcal{C}$.

In our first example (the Krull-Schmidt Theorem, that is, Theorem 1), $\mathcal{C}$ is the class of all right R-modules M_R of finite composition length, and $V(\mathcal{C})$ turns out to be a *free* commutative monoid, that is, a monoid isomorphic to $\mathbb{N}_0^{(X)}$ for some set X. In this example, X can be any set of representatives, up to isomorphism, of the indecomposable R-modules of finite composition length.

All the monoids we will consider in this paper are commutative, and the operation will be denoted as addition. Thus our monoids will be commutative additive semigroups with a zero element 0. For such a monoid M, $U(M)$ will denote the set of all invertible elements with respect to the addition, that is, all elements $a \in M$ with an opposite $-a$ in M. A commutative monoid M is said to be *reduced* if $U(M) = \{0\}$. For every monoid M, the quotient monoid $M/U(M) = \{\, x + U(M) \mid x \in M\,\}$ is a reduced monoid. For any class $\mathcal{C}$ of modules, the commutative monoid $V(\mathcal{C})$ is reduced. The converse appears in the following wonderful theorem, due to Bergman [4, Theorems 6.2 and 6.4] and Bergman-Dicks [5, p. 315]. See [12, Corollary 5].

[1]This is an odd assumption, because in Axiomatic Set Theory, where elements of sets are sets, $V(\mathcal{C})$ can never be a set. More precisely, $V(\mathcal{C})$ cannot be a set by Zermelo's Sum Axiom (Union Axiom) of General Set Theory, which is the axiom that guarantees that the union of a set of sets is still a set ("for any set S there exists the set whose elements are the elements of the elements of S"). This set theoretical difficulty can be avoided fixing once for all a set of representatives of $V(\mathcal{C})$ up to isomorphism. Hence, when we say "assume that $V(\mathcal{C})$ is a set" we mean "assume that $V(\mathcal{C})$ can be put in one-to-one correspondence with a set", that is, a class whose cardinality can be measured with a cardinal number.

Theorem 2. *Let k be a field and M a reduced commutative monoid. Then there exist a right and left hereditary k-algebra R and a class $\mathcal{C}$ of finitely generated projective right R-modules with $\mathcal{C}$ closed under isomorphism, direct summands and finite direct sums, and $V(\mathcal{C}) \cong M$.*

This theorem gives, in a sense, a complete answer to what can be done with our description of direct-sum decompositions in a class $\mathcal{C}$ of modules making use of the monoid $V(\mathcal{C})$.

If, instead of the direct-sum decompositions of the modules in a class $\mathcal{C}$, we want to study the direct-sum decompositions of one fixed module A_R, t he following refinement of the construction of $V(\mathcal{C})$ is sufficient. Given a fixed module A_R, we can construct the class $\mathrm{add}(A_R)$ whose elements are all modules B_R isomorphic to a direct summand of A_R^n for some integer $n \geq 0$. This is the smallest class of right R-modules containing A_R and closed under isomorphism, direct summands and finite direct sums. For instance, if A_R is the right module R_R, then $\mathrm{add}(R_R)$ is the class proj-R of all finitely generated projective right R-modules. For any ring R and module A_R, we will denote with $V(R)$ and $V(A_R)$ the monoids $V(\text{proj-}R)$ and $V(\mathrm{add}(A_R))$, respectively. Clearly, for every module A_R, the element $\langle A_R\rangle$ of the monoid $V(A_R)$ is a special element: it is an *order-unit* in the commutative monoid. Let us briefly present order-units, monoids with order-unit, and the category of commutative monoids with order-unit.

An element u of a commutative additive monoid M is an *order-unit* if, for every $x \in M$, there exist $y \in M$ and an integer $n \geq 0$ with $x + y = nu$. For instance, the element $\langle R_R\rangle$ of the commutative additive reduced monoid $V(R)$ is an order-unit. More generally, as we have said above, $\langle A_R\rangle$ is an order-unit in the monoid $V(A_R)$. The category of commutative monoids with order-unit has as its objects the pairs (M, u), where M is a commutative monoid and $u \in M$ is an order-unit, and as morphisms $f : (M, u) \to (M', u')$ the monoid homomorphisms $f : M \to M'$ that preserve the order-units, that is, such that $f(u) = u'$. Notice that V is a functor of the category of associative rings with identity into the category of commutative monoids with order-unit.

Clearly, as the commutative monoid $V(\mathcal{C})$ describes the direct-sum decompositions of the modules in a fixed class $\mathcal{C}$, so the commutative monoid with order-unit $(V(\mathrm{add}A_R), \langle A_R\rangle)$ describes the direct-sum decompositions of a fixed module A_R.

For any given module A_R, we can consider the endomorphism ring $E := \mathrm{End}(A_R)$ and the covariant functor

$$\mathrm{Hom}_R(A_R, -) : \text{Mod-}R \to \text{Mod-}E.$$

By restriction, the functor $\mathrm{Hom}_R(A_R, -)$ induces a categorical equivalence between the full subcategory of Mod-R whose class of objects is $\mathrm{add}(A_R)$ and the full subcategory of Mod-E whose class of objects is proj-E [10, Theorem 4.7]. This equivalence induces an isomorphism $(V(\mathrm{add}(A_R)), \langle A_R\rangle) \cong (V(E), \langle E_E\rangle)$ of monoids with order-unit. Therefore, in the study of "pathologies" of direct-sums, we can suppose $A_R = R_R$, that is, it suffices to study direct-sum decompositions of finitely generated projective modules.

Similarly, notice that the contravariant functor

$$\mathrm{Hom}_R(-, R) : \text{Mod-}R \to R\text{-Mod}$$

induces by restriction a duality between the full subcategory of Mod-R whose class of objects is proj-R and the full subcategory of R-Mod whose class of objects R-proj consists of all finitely generated projective *left* R-modules. This duality induces an isomorphism of monoids with order-unit $(V(\text{proj-}R), \langle R_R\rangle) \cong (V(R\text{-proj}), \langle {}_RR\rangle)$. In other words, in the definition of the monoid $V(R)$ there is no difference considering right or left finitely generated projective modules. The monoid $V(R)$ is the object of study of Non-Stable Algebraic K-Theory, as the Grothendieck group $K_0(R)$ is the object of study of (classical) Algebraic K-Theory. Here the Grothendieck group $K_0(R)$ is the enveloping group of $V(R)$, and its elements are the *stable isomorphism classes* $[P_R]$ of the finitely generated projective R-modules P_R. There is a pre-order (= reflexive, transitive and translation-invariant relation) on $K_0(R)$, for which the positive cone (= set of non-negative elements of $K_0(R)$) is the image of the universal mapping $\psi_R\colon V(R) \to K_0(R)$. If $J(R)$ denotes the Jacobson radical of R, the canonical projection $p\colon R \to R/J(R)$ induces a pullback diagram

$$\begin{array}{ccc} V(R) & \overset{V(p)}{\longrightarrow} & V(R/J(R)) \\ \psi_R \downarrow & & \downarrow \psi_{R/J(R)} \\ K_0(R) & \overset{K_0(p)}{\longrightarrow} & K_0(R/J(R)) \end{array}$$

in the category of commutative monoids [2].

We can adapt the Bergman-Dicks Theorem (Theorem 2) to monoids with order-units as follows.

Theorem 3. *Let k be a field and let M be a commutative reduced monoid with order-unit u. Then there exists a right and left hereditary k-algebra R such that (M, u) and $(V(R), \langle R_R\rangle)$ are isomorphic as monoids with order-unit.*

3. Local morphisms and semilocal rings

3.1. **Local morphisms.** In Algebraic Geometry and Commutative Algebra, local morphisms are defined as the ring morphisms $\varphi\colon R \to S$, between local commutative rings $(R, \mathcal{M})$ and $(S, \mathcal{N})$, for which $\varphi(\mathcal{M}) \subseteq \mathcal{N}$. Here $\mathcal{M}$ and $\mathcal{N}$ denote the maximal ideals of R and S respectively. More generally, let R and S be arbitrary associative rings with identity (not necessarily commutative and not necessarily local). We will say that a ring morphism $\varphi\colon R \to S$ is *local* if, for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R. These two definitions coincide in the case of R and S local commutative rings. The notion of local morphism for non-commutative rings was introduced, in the case in which S was a division ring, by Cohn [8].

Here is a list of trivial properties of local morphisms. Their proofs follow immediately from the definition. Let $\varphi\colon R \to S$, $\psi\colon S \to T$ be ring morphisms.

(1) If φ is a local morphsm, then $\ker(\varphi) \subseteq J(R)$.

(2) If φ is onto and is a local morphism, then $\varphi(J(R)) = J(S)$, and the induced morphism $\mathbf{M}_n(\varphi)\colon \mathbf{M}_n(R) \to \mathbf{M}_n(S)$ between the $n \times n$ matrix rings is local for every $n > 1$.

(3) If φ and ψ are local morphisms, then so is $\psi \circ \varphi$.

(4) If the composite morpisms $\psi \circ \varphi$ is local, then φ local.

(5) If I is any two-sided ideal of R contained in the Jacobson radical $J(R)$, the canonical projection $R \to R/I$ is a local morphism.

3.2. **Semilocal rings, dual Goldie dimension.** A ring R is a *semilocal* ring if $R/J(R)$ is a semisimple artinian ring. (In Commutative Algebra a commutative ring is semilocal if it has only finitely many maximal ideals. The two definitions coincide in the case of commutative rings, but notice that a semilocal non-commutative ring can have infinitely many maximal right ideals, as the example of the ring $\mathbf{M}_n(k)$ of $n \times n$ matrices over an infinite field k shows.)

The relation between the notions of semilocal ring and local morphism is given by the following theorem, due to Camps and Dicks [6].

Theorem 4. *A ring R is semilocal if and only if there exists a local morphism of R into a semilocal ring, if and only if there exists a local morphism of R into a semisimple artinian ring.*

The notion of semilocal ring is also related to the notion of dual Goldie dimension. Goldie dimension can be defined not only for modules M_R, but more generally for any modular lattice L with a greatest element 1 and a least element 0 [10, § 2.6]. If $\mathcal{L}(M_R)$ denotes the lattice of all submodules of a module M_R, the Goldie dimension $\dim(M_R)$ of the module M_R coincides with the Goldie dimension $\dim(\mathcal{L}(M_R))$ of the lattice $\mathcal{L}(M_R)$. The *dual Goldie dimension* $\operatorname{codim}(M_R)$ of a module M_R is by definition the Goldie dimension of the dual (=opposite) lattice of the lattice $\mathcal{L}(M_R)$. The next result describes the relation between the notions of semilocal ring and dual Goldie dimension of a ring.

Proposition 5. *A ring R is semilocal if and only if the dual Goldie dimension of the right R-module R_R is finite, if and only if the dual Goldie dimension of the left R-module ${}_RR$ is finite. Moreover, if these equivalent conditions hold, then*

$$\operatorname{codim}(R_R) = \operatorname{codim}({}_RR) = \dim(R/J(R)).$$

In this proposition, note that R is semilocal exactly when $R/J(R)$ is semisimple artinian, that is, when $R/J(R)$ is a direct sum of simple modules, and in this case the Goldie dimension $\dim(R/J(R))$ of $R/J(R)$ is simply the number of direct summands in a direct-sum decomposition of $R/J(R)$ into simple submodules, that is, into simple right ideals of $R/J(R)$. The next theorem is related to Theorem 4 and Proposition 5.

Theorem 6. [6] *If $R \to S$ is a local morphism between two rings R and S, then* $\operatorname{codim}(R) \leq \operatorname{codim}(S)$.

3.3. **Modules with semilocal endomorphism rings.** The reason why we are interested in semilocal rings is that we want to study modules whose endomorphism ring is semilocal. Having a semilocal endomorphism ring is a finiteness condition on modules. For instance, a module with semilocal endomorphism ring is always a direct sum of finitely many indecomposable modules, it is not a direct sum of infinitely many non-zero modules, and it is directly finite. The class of the modules with semilocal endomorphism rings is closed under direct summands and finite direct sums. We will see in §4.2 that direct-sum decompositions of modules with semilocal endomorphism rings are described by reduced Krull monoids, and this implies a regularity in the behavior of direct-sum decompositions.

We begin with a proposition that shows how the property of having a semilocal endomorphism ring is related to restriction of scalars.

Proposition 7. [15] *Let $R \to S$ be a ring morphism, and let M_S be an S-module with* $\mathrm{End}(M_R)$ *semilocal. Then* $\mathrm{End}(M_S)$ *is semilocal.*

The proof is incredibly easy. The embedding

$$\mathrm{End}(M_S) \to \mathrm{End}(M_R)$$

is a local morphism, because an S-endomorphism is an S-automorphism if and only if it is an R-automorphism. Hence Theorem 4 applies.

4. Examples. Krull monoids.

This Section 4 is devoted to analyzing some examples of modules with semilocal endomorphism rings.

4.1. **Noetherian modules, artinian modules.** Our first example of class of modules with semilocal endomorphism rings is the class of all artinian right modules over a fixed ring R. Recall that the Krull-Schmidt Theorem (Theorem 1) holds for modules of finite composition lenght. Now a module has finite composition length if and only if it is both noetherian and artinian. A very natural question is therefore whether "noetherian" or "artinian" are sufficient conditions for the Krull-Schmidt Theorem to hold.

It is very easy to construct examples of noetherian modules for which the Krull-Schmidt Theorem does not hold. For instance, take a non-local noetherian commutative integral domain of Krull dimension ≥ 2, for example $R = k[x, y]$ (the ring of polynomials in two indeterminates x and y with coefficients in a field k). Then R has two distinct maximal ideals M_1, M_2, necessarily non-principal. Thus $R_R = M_1 + M_2$. The exact sequence $0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to R_R \to 0$ splits, so that

$$M_1 \oplus M_2 \cong R_R \oplus (M_1 \cap M_2). \tag{4.1}$$

But M_1 and M_2 are non-cyclic modules, and R_R is a cyclic module, so that the two direct-sum decompositions (4.1) are not isomorphic.

It was Krull who first asked in 1932 whether "the Krull-Schmidt Theorem holds for artinian modules" [22]. That is, any artinian module is a direct sum of indecomposables, but is such a direct-sum decomposition unique up to isomorphism? The first examples showing that there exist artinian modules with non-isomorphic direct-sum decompositions were given by Facchini, Herbera, Levy and Vámos in [16]. Nevertheless, direct-sum decompositions of artinian modules, and more generally of any class of modules with semilocal endomorphism rings are regular, because their behavior is described by a Krull monoids.

4.2. **Krull monoids and regular decompositions.** Krull monoids are the analogue for commutative monoids of what Krull domains are in Commutative Algebra. They were introduced by Chouinard in [7]. In Commutative Algebra we can fix a field F, take a family of valuations on F, consider the corresponding valutation subrings, and their intersection, when it is of finite character, is called a *Krull domain*. Then we can consider the fractional ideals, construct the divisor class semigroup, and so on. We have a perfectly similar case when we deal with commutative monoids instead of commutative integral domains. We can fix an abelian group G, take a family of valuations on G, consider the corresponding valutation submonoids, and their intersection, when it is of finite character, is called a *Krull monoid*. Then we can consider the fractional ideals, construct the divisor class semigroup of the Krull monoid, and so on. For the details, see [7]. For us, now, it is sufficient to know that the finitely generated reduced Krull monoids are the monoids isomorphic to monoids of the form $G \cap \mathbb{N}_0^t$, where $t \geq 0$ is an integer and G is a *subgroup* of the free abelian group $\mathbb{Z}^t$.

Theorem 8. [14, 26] *For every artinian module A_R, the monoid $V(A_R)$ is a finitely generated reduced Krull monoid with order-unit $\langle A_R \rangle$. Conversely, for every finitely generated reduced Krull monoid V with an order-unit u there exists an artinian module A_R with*

$$(V(A_R), \langle A_R \rangle) \cong (V, u).$$

More generally, for any class $\mathcal{C}$ of modules with semilocal endomorphism rings with $\mathcal{C}$ closed under isomorphism, direct summands and finite direct sums, the monoid $V(\mathcal{C})$ turns out to be a reduced Krull monoid [11]. Notice the geometric regularity implied by Krull monoids. In the language of Minkowski's Geometry of Numbers, a subgroup G of $\mathbb{Z}^t$ is represented by a "lattice", that is, a structure with a very regular geometric pattern (Here we are using the word *lattice* with a meaning completely different from the meaning employed until now in this paper.) If V is a reduced Krull monoid, then $V \cong \mathbb{N}_0^t \cap G$ is the intersection of the lattice $G \subseteq \mathbb{Z}^t$ with the positive cone $\mathbb{N}_0^t$. The failure of the Krull-Schmidt Theorem is minimal in this case, due only to the presence of the border of $\mathbb{N}_0^t \cap G$. Hence, when $V(A_R)$ is a Krull monoid that is not free, Krull-Schmidt uniqueness fails, but direct-sum decompositions still have a very regular geometric pattern.

4.3. **Further examples.** Let us pass to present other examples of modules with semilocal endomorphism rings. The following result is well known. For a proof, see [15, Proposition 3.1].

Proposition 9. *Every finitely generated module over a commutative semilocal ring has a semilocal endomorphism ring.*

Here is an extension of the previous proposition.

Proposition 10. [15, Theorem 3.3] *Every finitely presented module over a semilocal ring has a semilocal endomorphism ring.*

Notice that we have extended the class of rings (from commutative semilocal rings to arbitrary semilocal rings), but we have to restrict the class of modules (from finitely generated modules to finitely presented modules). Proposition 10 cannot be extended to finitely generated modules over non-commutative rings: there exist finitely generated modules over non-commutative semilocal rings whose endomorphism rings are not semilocal [15, Example 3.5].

Here are further examples of modules with semilocal endomorphism rings. We say that a module M is *quotient finite dimensional* if every homomorphic image of M has finite Goldie dimension.

Corollary 11. [15, Corollary 5.8] *Every submodule of a quotient finite dimensional injective module has a semilocal endomorphism ring.*

Recall that a module M is *uniserial* if, for any submodules A and B of M, either $A \subseteq B$ or $B \subseteq A$. Thus a module M is uniserial if and only if the lattice $\mathcal{L}(M)$ of its submodules is linearly ordered under set inclusion. Clearly, uniserial modules are quotient finite dimensional. A module is *serial* if it is a direct sum of uniserial submodules. Hence a module is serial and has finite Goldie dimension if and only if it is a direct sum of finitely many uniserial submodules.

Corollary 12. [15, Corollary 5.10] *Let E be an injective serial right module of finite Goldie dimension. Then the endomorphism ring of every submodule of E is semilocal.*

For further examples of modules with semilocal endomorphism rings, see [15] and [21].

5. Monogeny class, epigeny class

5.1. **Biuniform modules.** We say that two right R-modules A_R and B_R *belong to the same monogeny class*, and write $[A_R]_m = [B_R]_m$, if there exist a monomorphism $A_R \to B_R$ and a monomorphism $B_R \to A_R$. Similarly, we say that A_R and B_R *belong to the same epigeny class*, and write $[A_R]_e = [B_R]_e$, if there exist an epimorphism $A_R \to B_R$ and an epimorphism $B_R \to A_R$.

Recall that a module A_R is said to be: *uniform* if it has Goldie dimension 1, that is, it is non-zero and the intersection of any two non-zero submodules is a non-zero submodule; *couniform* if it has dual Goldie dimension 1, that is, it is non-zero and the sum of any two proper submodules is a proper submodule; *biuniform* if it uniform and couniform. For instance, uniserial non-zero modules are biuniform modules.

Theorem 13. [10, Theorem 9.1] *Let A_R be a biuniform module over an arbitrary ring R and let $E = \mathrm{End}(A_R)$ be its endomorphism ring. Let $I = \{ f \in E \mid f$ is not injective$\}$ and $K = \{ f \in E \mid f$ is not surjective$\}$. Then I and K are two-sided completely prime ideals of E, and every proper right ideal of E and every proper left ideal of E is contained either in I or in K. Moreover, exactly one of the following two conditions hold:*
(a) *Either E is a local ring, or*
(b) *$E/J(E) \cong E/I \times E/K$, where E/I and E/K are division rings.*

From Theorem 13 we get the following weak form of the Krull-Schmidt Theorem, proved by the author in [9, Theorem 1.9].

Theorem 14. *Let $U_1, \dots, U_n, V_1, \dots, V_t$ be biuniform right modules over an arbitrary ring R. Then the direct sums $U_1 \oplus \cdots \oplus U_n$ and $V_1 \oplus \cdots \oplus V_t$ are isomorphic if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[U_i]_m = [V_{\sigma(i)}]_m$ and $[U_i]_e = [V_{\tau(i)}]_e$ for every $i = 1, 2, \dots, n$.*

This theorem allowed us to solve a problem posed by Warfield in [25].

5.2. **Cyclically presented modules over local rings.** We will now present some results proved in [1]. Recall that a right module over a ring R is said to be *cyclically presented* if it is isomorphic to R/aR for some $a \in R$. For any ring R with identity, $U(R)$ will denote the group of all invertible elements of R.

If R/aR and R/bR are cyclically presented modules over a local ring R, we say that R/aR and R/bR *have the same lower part*, and write $[R/aR]_l = [R/bR]_l$, if there exist $u, v \in U(R)$ and $r, s \in R$ with $au = rb$ and $bv = sa$. (The reason why we give this definition is that in this way two cyclically presented modules over a local ring turn out to have the same lower part exacly when their Auslander-Bridger transposes have the same epigeny class; cf. [1].)

We will now describe the endomorphism ring of a cyclically presented module. Clearly, the endomorphism ring $\mathrm{End}_R(R/aR)$ of a non-zero cyclically presented module R/aR is isomorphic to E/aR, where $E := \{ r \in R \mid ra \in aR \}$ is the *idealizer* of aR.

Theorem 15. *Let a be a non-zero non-invertible element of a local ring R, let E be the idealizer of aR, and let E/aR be the endomorphism ring of the cyclically presented right R-module R/aR. Set $I := \{ r \in R \mid ra \in aJ(R) \}$ and $K := J(R) \cap E$. Then I and K are completely prime two-sided ideals of E containing aR, the union $(I/aR) \cup (K/aR)$ is the set of all non-invertible elements of E/aR, and every proper right ideal of E/aR and every proper left ideal of E/aR is contained either in I/aR or in K/aR. Moreover, exactly one of the following two conditions hold:*

(a) *Either E/aR is a local ring, or*

(b) *I and K are not comparable, $J(E/aR) = (I \cap K)/aR$, and $(E/aR)/J(E/aR)$ is canonically isomorphic to the direct product of the two division rings E/I and E/K.*

Theorem 16. (Weak Krull-Schmidt Theorem) *Let $a_1, \dots, a_n, b_1, \dots, b_t$ be non-invertible elements of a local ring R. Then*

$$R/a_1R \oplus \cdots \oplus R/a_nR \quad \textit{and} \quad R/b_1R \oplus \cdots \oplus R/b_tR$$

are isomorphic right R-modules if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[R/a_iR]_l = [R/b_{\sigma(i)}R]_l$ and $[R/a_iR]_e = [R/b_{\tau(i)}R]_e$ for every $i = 1, 2, \dots, n$.

This has an immediate consequence as far as equivalence of matrices is concerned. Recall that two $m \times n$ matrices A, B with entries in a ring R are *equivalent*, denoted $A \sim B$, if there exist an $m \times m$ invertible matrix P and an $n \times n$ invertible matrix Q with $B = PAQ$. We denote by $\mathrm{diag}(a_1, \dots, a_n)$ the $n \times n$ diagonal matrix whose (i, i) entry is a_i and whose other entries are zero.

Corollary 17. *Let $a_1, \dots, a_n, b_1, \dots, b_n$ be elements of a local ring R. Then* $\mathrm{diag}(a_1, \dots, a_n) \sim \mathrm{diag}(b_1, \dots, b_n)$ *if and only if there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ with*

$$[R/a_iR]_l = [R/b_{\sigma(i)}R]_l \quad \textit{and} \quad [R/a_iR]_e = [R/b_{\tau(i)}R]_e$$

for every $i = 1, 2, \dots, n$.

6. Kernels of morphisms, couniformly presented modules

6.1. **Kernels of morphisms between indecomposable injective modules.** The next results are taken from [17]. We say that two modules A_R and B_R *have the same upper part*, and write $[A_R]_u = [B_R]_u$, if there exist a homomorphism $\varphi \colon E(A_R) \to E(B_R)$ and a homomorphism $\psi \colon E(B_R) \to E(A_R)$ such that $\varphi^{-1}(B_R) = A_R$ and $\psi^{-1}(A_R) = B_R$. Here $E(-)$ denotes the injective envelope.

We need some further notation for the statement of the next theorem. Let E_1, E_2, E_1', E_2' be indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_1 \to E_2, \varphi'\colon E_1' \to E_2'$ be two non-injective morphisms. Any morphism $f\colon \ker\varphi \to \ker\varphi'$ extends to a morphism $f_1\colon E_1 \to E_1'$. Hence f_1 induces a morphism $\widetilde{f}_1\colon E_1/\ker\varphi \to E_1'/\ker\varphi'$, which extends to a morphism $f_2\colon E_2 \to E_2'$. Thus we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \to & \ker\varphi & \to & E_1 & \xrightarrow{\varphi} & E_2 \\ & & \downarrow f & & \downarrow f_1 & & \downarrow f_2 \\ 0 & \to & \ker\varphi' & \to & E_1' & \xrightarrow{\varphi'} & E_2'. \end{array}$$

Notice that f_1 and f_2 are not uniquely determined by f.

Theorem 18. *Let E_1 and E_2 be two indecomposable injective right modules over an arbitrary ring R, and let $\varphi\colon E_1 \to E_2$ be a non-zero non-injective morphism. Set $S := \mathrm{End}_R(\ker\varphi)$, $I\colon = \{\, f \in S \mid f \text{ is not injective}\,\} = \{\, f \in S \mid f_1 \text{ is not injective}\,\}$ and $K\colon = \{\, f \in S \mid f_2 \text{ is not injective}\,\} = \{\, f \in S \mid f_1^{-1}(\ker\varphi) \text{ properly contains } \ker\varphi\,\}$. Then I and K are two completely prime two-sided ideals of S, and one of the following two conditions hold:*
(a) *Either S is a local ring, or*
(b) *$S/J(S)) \cong S/I \times S/K$, where S/I and S/K are division rings.*

Theorem 19. (Weak Krull-Schmidt Theorem) *Let $\varphi_i\colon E_{i,1} \to E_{i,2}$ $(i = 1, 2, \dots, n)$ and $\varphi_j'\colon E_{j,1}' \to E_{j,2}'$ $(i = 1, 2, \dots, t)$ be $n+t$ non-injective morphisms between indecomposable injective modules $E_{i,1}, E_{i,2}, E_{j,1}', E_{j,2}'$ over an arbitrary ring R. Then $\oplus_{i=1}^n \ker\varphi_i \cong \oplus_{j=1}^t \ker\varphi_j'$ if and only if $n = t$ and there exist two permutations σ, τ of $\{1, 2, \dots, n\}$ such that $[\ker\varphi_i]_m = [\ker\varphi_{\sigma(i)}']_m$ and $[\ker\varphi_i]_u = [\ker\varphi_{\tau(i)}']_u$ for every $i = 1, 2, \dots, n$.*

Hence, also in this case we find the same behavior: at most two maximal ideals and the same weak form of the Krull-Schmidt Theorem. Now we will present a further class of modules over arbitrary rings with exactly the same behavior. It extends the class of cyclically presented modules over local rings we have met with in §5.2.

6.2. **Couniformly presented modules.** These modules have been introduced and studied in [13].

It is easily seen that a projective right module P_R is couniform, that is, has dual Goldie dimension one (cf. §5.1) if and only if P_R is the projective cover of a simple module, if and only if $\mathrm{End}(P_R)$ is a local ring, if and only if there exists an idempotent $e \in R$ with $P_R \cong eR$ and eRe a local ring, if and only if P_R is a finitely generated module with a unique maximal submodule [1, Lemma 8.7].

We say that a module M_R is *couniformly presented* if it is non-zero and there exists an exact sequence

$$0 \to C_R \xrightarrow{\iota} P_R \to M_R \to 0 \qquad (6.1)$$

with P_R projective and both C_R and P_R couniform modules. Under these hypotheses, (6.1) will be called a *couniform presentation* of the couniformly presented module M_R.

For such a module M_R, every endomorphism f of M_R lifts to an endomorphism f_0 of the projective cover P_R of M_R, and we will denote by f_1 the restriction of f_0 to C_R. Hence we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0 \\ & & f_1 \downarrow & & \downarrow f_0 & & \downarrow f & & \\ 0 & \to & C_R & \xrightarrow{\iota} & P_R & \to & M_R & \to & 0. \end{array}$$

Theorem 20. *Let $0 \to C_R \to P_R \to M_R \to 0$ be a couniform presentation of a couniformly presented module M_R. Let $K := \{ f \in \mathrm{End}(M_R) \mid f$ is not surjective$\}$ and $I := \{ f \in \mathrm{End}(M_R) \mid f_1 \colon C_R \to C_R$ is not surjective$\}$. Then K and I are completely prime two-sided ideals of $\mathrm{End}(M_R)$, and the union $K \cup I$ is the set of all non-invertible elements of $\mathrm{End}(M_R)$. Moreover, exactly one of the following two conditions hold:*

(a) *Either $\mathrm{End}(M_R)$ is a local ring, or*

(b) *$J(\mathrm{End}(M_R)) = K \cap I$, and $\mathrm{End}(M_R)/J(\mathrm{End}(M_R))$ is canonically isomorphic to the direct product of the two division rings $\mathrm{End}(M_R)/K$ and $\mathrm{End}(M_R)/I$.*

If M_R and M'_R are two couniformly presented modules with couniform presentations $0 \to C_R \to P_R \to M_R \to 0$ and $0 \to C'_R \to P'_R \to M'_R \to 0$ respectively, we say that M_R and M'_R *have the same lower part,* and write $[M_R]_\ell = [M'_R]_\ell$, if there are two homomorphisms $f_0 \colon P_R \to P'_R$ and $f'_0 \colon P'_R \to P_R$ such that $f_0(C_R) = C'_R$ and $f'_0(C'_R) = C_R$. (The definition of "having the same lower part" had been given in §5.2 only for cyclically presented modules over local rings. Here we are giving it for arbitrary couniformly presented modules over arbitrary rings.)

Theorem 21. (Weak Krull-Schmidt Theorem for couniformly presented modules) *Let $M_1, \dots, M_n, N_1, \dots, N_t$ be couniformly presented right R-modules. Then the modules $M_1 \oplus \cdots \oplus M_n$ and $N_1 \oplus \cdots \oplus N_t$ are isomorphic if and only if $n = t$ and there are two permutations σ, τ of $\{1, 2, \dots, n\}$ with $[M_i]_\ell = [N_{\sigma(i)}]_\ell$ and $[M_i]_e = [N_{\tau(i)}]_e$ for every $i = 1, \dots, n$.*

6.3. **Relation between upper part and lower part.** We have seen that kernels of morphisms between indecomposable injective modules are described by their monogeny class and their upper part. Couniformly presented modules are described by their epigeny class and their lower part. Let us explain the reason of this symmetry.

Let R be a fixed ring. Let $\{ E_\lambda \mid \lambda \in \Lambda \}$ be a set of representatives up to isomorphism of all indecomposable injective right R-modules. Set $E_R := E(\oplus_{\lambda \in \Lambda} E_\lambda)$ and $S := \mathrm{End}(E_R)$, so that ${}_SE_R$ turns out to be an S-R-bimodule and $H := \mathrm{Hom}(-, {}_SE_R) \colon \text{Mod-}R \to S\text{-Mod}$ is an additive contravariant exact functor.

If $\mathcal{K}$ is the full subcategory of Mod-R whose objects are finite direct sums of kernels of morphisms between uniform (equivalently, indecomposable) injective right R-modules, and $\mathcal{C}$ is the full subcategory of S-Mod whose objects are finite direct sums of cokernels of morphisms between couniform projective left S-modules, then the restriction $H = \mathrm{Hom}(-, {}_SE_R) \colon \mathcal{K} \to \mathcal{S}$ is a duality. It exchanges monogeny and epigeny (and upper part and lower part) as stated in the next proposition.

Proposition 22. *Let K_R and K'_R be the kernels of two non-zero non-injective morphisms between uniform injective right R-modules. Then:*

(a) $[K_R]_m = [K'_R]_m$ *if and only if* $[H(K_R)]_e = [H(K'_R)]_e$.
(b) $[K_R]_u = [K'_R]_u$ *if and only if* $[H(K_R)]_\ell = [H(K'_R)]_\ell$.

7. Seeking a general theory

We have seen three pair-wise incomparable classes of modules with the same behavior: (1) The class of biuniform modules. It contains the class of uniserial modules. These modules are described by their monogeny classes and their epigeny classes. (2) The class of all couniformly presented modules. It contains the class of all cokernels of morphisms between projective couniform modules, which in turn contains the class of all cyclically presented modules when the base ring R is local. These modules are described by their lower parts and their epigeny classes. (3) The class of all kernels of morphisms between uniform injective right R-modules. They are described by the monogeny classes and the upper parts, and there is a duality between this class and the class of all cokernels of morphisms between projective couniform modules. It would be easy to construct further examples of classes of modules with exactly the same behavior. For instance, fix two simple non-isomorphic right R-modules S_1 and S_2. Then the class of all artinian right R-modules with socle isomorphic to $S_1 \oplus S_2$ has this kind of behavior.

P. Příhoda and the author have found a general theory, a general setting able to describe all these particular classes [18]. We say that a ring S *has type* n if the factor ring $S/J(S)$ is a direct product of n division rings, and we say that a right module M_R over a ring R *has type* n if its endomorphism ring $\mathrm{End}(M_R)$ is a ring of type n. A ring R has type 1 if and only if it is a local ring, if and only if there is a local morphism of R into a division ring.

Lemma 23. *The following conditions are equivalent for a ring S with Jacobson radical $J(S)$ and a positive integer n.*

(i) *n is the smallest of the integers m such that there exists a local morphism of the ring S into a direct product of m division rings.*

(ii) *S has exactly n distinct maximal right ideals, and they are all two-sided ideals in S.*

(iii) *The ring S has type n.*

The natural question is: if $\mathcal{T}$ is the full subcategory of Mod-R whose class of objects consists of all indecomposable right R-modules of type 2, does a weak Krull-Schmidt Theorem hold for $\mathcal{T}$?

Let $\mathcal{C}$ be a full subcategory of Mod-R whose objects are indecomposable modules. A *completely prime ideal* $\mathcal{P}$ of $\mathcal{C}$ consists of a subgroup $\mathcal{P}(A,B)$ of $\mathrm{Hom}_R(A,B)$ for every pair of objects $A, B \in \mathrm{Ob}\,\mathcal{C}$ such that for every $A, B, C \in \mathrm{Ob}\,\mathcal{C}$, every $f\colon A \to B$ and every $g\colon B \to C$ one has that $gf \in \mathcal{P}(A,C)$ if and only if either $f \in \mathcal{P}(A,B)$ or $g \in \mathcal{P}(B,C)$. In all the previous situations, we have a pair of completely prime ideals $\mathcal{P}, \mathcal{Q}$ of $\mathcal{C}$ with the property that, for every object $A \in \mathrm{Ob}\,\mathcal{C}$, and endomorphism $f \in \mathrm{End}(A)$ of A is an automorphism of A if and only if $f \notin \mathcal{P}(A,A) \cup \mathcal{Q}(A,A)$.

If $\mathcal{C}$ is a full subcategory of $\mathcal{T}$, M is an object of $\mathcal{I}$, and I is a fixed ideal of $\mathrm{End}_R(M)$, let $\mathcal{I}$ be the ideal of the category $\mathcal{C}$ defined as follows: a morphism $f\colon X \to Y$ is in $\mathcal{I}(X,Y)$ if and only if $\beta f \alpha \in I$ for every $\alpha\colon M \to X$ and every $\beta\colon Y \to M$. We call $\mathcal{I}$ the *ideal of* $\mathcal{C}$ *associated to* I. It is the greatest among the ideals $\mathcal{I}'$ of $\mathcal{C}$ with $\mathcal{I}'(M,M) \subseteq I$, and in this case, as it is easily seen, $\mathcal{I}(M,M) = I$.

We can associate to the category $\mathcal{C}$ a graph $G(\mathcal{C})$. The edges of $G(\mathcal{C})$ are the isomorphisms classes $\langle M \rangle := \{ Y \in \mathrm{Ob}(\mathcal{C}) \mid Y \cong M \text{ in Mod-}R \}$, where M ranges in $\mathrm{Ob}(\mathcal{C})$; the vertices of $G(\mathcal{C})$ are the ideals $\mathcal{I}$ in the category $\mathcal{C}$ associated to a maximal ideal I of $\mathrm{End}(M_R)$ for some $M \in \mathrm{Ob}(\mathcal{C})$; for every $M \in \mathrm{Ob}(\mathcal{C})$, the endomorphism ring $\mathrm{End}(M_R)$ has exactly two maximal ideals I_1, I_2, and the edge $\langle M \rangle$ connects the vertices $\mathcal{I}_1$ and $\mathcal{I}_2$.

Theorem 24. *Let* $\mathcal{C}$ *be a full subcategory of* $\mathcal{T}$. *A weak Krull-Schmidt Theorem holds for* $\mathcal{C}$ *if and only if the graph* $G(\mathcal{C})$ *does not contain a subgraph isomorphic to the complete graph* K_4.

For suitable rings R, the graph $G(\mathcal{T})$ contains a copy of the complete graph K_4, so that a weak Krull-Schmidt Theorem does not hold for $\mathcal{T}$.

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