PICARD GROUPS OF ADDITIVE FULL SUBCATEGORIES

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1. Introduction

Let k be a commutative ring and let A be a commutative k-algebra. We denote by A-Mod the category of all A-modules and all A-homomorphisms. Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. Since A is a k-algebra, every additive full subcategory $\mathfrak{C}$ is a k-category. A covariant functor $\mathfrak{C} \to \mathfrak{C}$ is called a k-linear automorphism of $\mathfrak{C}$ if it is a k-linear functor giving an auto-equivalence of the category $\mathfrak{C}$. We denote the set of all the isomorphism classes of k-linear automorphisms of $\mathfrak{C}$ by $\mathrm{Aut}_k(\mathfrak{C})$, which forms a group by defining the multiplication to be the composition of functors.

Our study was motivated by the following computational result. Recall that a local ring $(A, \mathfrak{m})$ is said to have only an isolated singularity if A_p is a regular local ring for all prime ideals $\mathfrak{p}$ except $\mathfrak{m}$.

Theorem 1. *Let A be a Cohen-Macaulay local k-algebra with dimension d. Suppose that A has only an isolated singularity. Then,*

$$\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \begin{cases} \mathrm{Aut}_{k\text{-}alg}(A) & (d \neq 2) \\ \mathrm{Aut}_{k\text{-}alg}(A) \ltimes C\ell(A) & (d = 2), \end{cases}$$

where $\mathrm{CM}(A)$ *is the additive full subcategory consisting of all maximal Cohen-Macaulay modules and $C\ell(A)$ denotes the divisor class group of A.*

In this note we generalize this computation to much wider classes of additive full subcategories $\mathfrak{C}$ of A-Mod, and we shall show a certain structure theorem for $\mathrm{Aut}_k(\mathfrak{C})$.

2. Automorphism groups

Throughout the paper, k is a commutative ring and A is a commutative k-algebra. When we say that $\mathfrak{C}$ is a full subcategory of A-Mod, we always assume that $\mathfrak{C}$ is closed under isomorphisms, and we simply write $X \in \mathfrak{C}$ to indicate that X is an object of $\mathfrak{C}$. Suppose that we are given an additive full subcategory $\mathfrak{C}$ of A-Mod and an additive covariant functor $F : \mathfrak{C} \to \mathfrak{C}$. Recall that F is a k-linear functor if it induces k-linear mappings $\mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_A(F(X), F(Y))$ for all $X, Y \in \mathfrak{C}$.

Definition 2. $\mathrm{Aut}_k(\mathfrak{C})$ is the group of all the isomorphism classes of k-linear automorphisms of $\mathfrak{C}$, i.e.

$$\mathrm{Aut}_k(\mathfrak{C}) = \left\{ F : \mathfrak{C} \to \mathfrak{C} \;\middle|\; \begin{array}{l} F \text{ is a } k\text{-linear covariant functor that} \\ \text{gives an equivalence of the category } \mathfrak{C} \end{array} \right\} / \cong .$$

The detailed version of this paper will be submitted for publication elsewhere.

Note that the multiplication in $\mathrm{Aut}_k(\mathfrak{C})$ is defined to be the composition of functors, hence the identity element of $\mathrm{Aut}_k(\mathfrak{C})$ is represented by the class of the identity functor on $\mathfrak{C}$.

We denote by $\mathrm{Aut}_{k\text{-alg}}(A)$ the group of all the k-algebra automorphisms of A. For $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$, we can define a covariant k-linear functor $\sigma_* : A\text{-Mod} \to A\text{-Mod}$ as in the following manner. For each A-module M, we define $\sigma_* M$ to be M as an abelian group on which the A-module structure is defined by $a \circ m = \sigma^{-1}(a)m$ for $a \in A$, $m \in M$. For an A-homomorphism $f : M \to N$, we define $\sigma_* f : \sigma_* M \to \sigma_* N$ to be the same mapping as f. Note that $\sigma_* f$ is an A-homomorphism, since $(\sigma_* f)(a \circ m) = f(\sigma^{-1}(a)m) = \sigma^{-1}(a)f(m) = a \circ (\sigma_*)f(m)$ for all $a \in A$ and $m \in M$. Notice that σ_* is a k-automorphism of the category A-Mod.

Definition 3. Let $\mathfrak{C}$ be an additive full subcategory of A-Mod. Then $\mathfrak{C}$ is said to be stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ if $\sigma_*(\mathfrak{C}) \subseteq \mathfrak{C}$ for all $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$.

Note that if $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ then $\sigma_*|_{\mathrm{C}}$ gives a k-automorphism of $\mathfrak{C}$ for all $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$. Therefore we have a natural group homomorphism $\Psi : \mathrm{Aut}_{k\text{-alg}}(A) \to \mathrm{Aut}_k(\mathfrak{C})$ which maps σ to the class of $\sigma_*|_{\mathrm{C}}$. It is easy to verify the following lemma.

Lemma 4. *Assume that $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ and that $A \in \mathfrak{C}$. Then the natural group homomorphism $\Psi : \mathrm{Aut}_{k\text{-alg}}(A) \to \mathrm{Aut}_k(\mathfrak{C})$ is an injection.*

By this lemma, we can regard $\mathrm{Aut}_{k\text{-alg}}(A)$ as a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$.

Definition 5. Let N be an A-module. Given a k-algebra homomorphism $\sigma : A \to A$, we define an $(A \otimes_k A)$-module N_σ by $N_\sigma = N$ as an abelian group on which the ring action is defined by $(a \otimes b) \cdot n = a\sigma(b)n$ for $a \otimes b \in A \otimes_k A$ and $n \in N$. In such a case, we can define a k-linear functor $\mathrm{Hom}_A(N_\sigma, -) : A\text{-Mod} \to A\text{-Mod}$, for which the A-module structure on $\mathrm{Hom}_A(N_\sigma, X)$ ($X \in A$-Mod) is defined by $(b \cdot f)(n) = f((1 \otimes b) \cdot n)$ for $f \in \mathrm{Hom}_A(N_\sigma, X)$, $b \in A$ and $n \in N$.

If σ is a k-algebra automorphism of A, then it is easy to see the following equality of functors holds:

$$(\sigma^{-1})_* \circ \mathrm{Hom}_A(N,\ ) = \mathrm{Hom}_A(N_\sigma,\ ).$$

The following theorem is one of the main results of this note.

Theorem 6 ([2, Theorem 2.5])**.** *Let A be a commutative k-algebra and let $\mathfrak{C}$ be an additive full subcategory of A-Mod such that $A \in \mathfrak{C}$. For a given k-linear automorphism $F \in \mathrm{Aut}_k(\mathfrak{C})$, there is a k-algebra automorphism $\sigma \in \mathrm{Aut}_{k\text{-alg}}(A)$ such that F is isomorphic to the composition of functors $\sigma_* \circ \mathrm{Hom}_A(N, -)|_{\mathrm{C}}$, where N is any object in $\mathfrak{C}$ satisfying $F(N) \cong A$ in $\mathfrak{C}$.*

Proof. We give below an outline of the proof. See [2, Theorem 2.5] for the detail.

Since A is commutative, the multiplication map $a_X : X \to X$ by an element $a \in A$ is an A-homomorphism for all objects $X \in \mathfrak{C}$. Thus we can define a natural transformation $\alpha(a) : F \to F$ by $\alpha(a)(X) = F(a_X) : F(X) \to F(X)$. Denote by $\mathrm{End}(F)$ the set of all the natural transformations $F \to F$, and this induces the mapping

$$\alpha : A \to \mathrm{End}(F)\ ; \quad a \mapsto F(a_{(\ )}).$$

Note that $\mathrm{End}(F)$ is a ring by defining the composition of natural transformations as the multiplication and it is also a k-algebra, since F is a k-linear functor. By using the fact that F is an auto-equivalence, it is straightforward to see that α is a k-algebra isomorphism.

Since F is a dense functor and $A \in \mathfrak{C}$, there is an object $N \in \mathfrak{C}$ such that $F(N) \cong A$. For such an object N, we can identify $\mathrm{End}_A(F(N))$ with A as k-algebra through the mapping $A \to \mathrm{End}_A(F(N))$ which sends $a \in A$ to the multiplication mapping $a_{F(N)}$ by a on $F(N)$. Thus we have a k-algebra homomorphism

$$\beta : \mathrm{End}(F) \to \mathrm{End}_A(F(N)) \cong A \ ; \quad \varphi \mapsto \varphi(N).$$

We easily see that β is a k-algebra isomorphism.

Now define a k-algebra automorphism $\sigma : A \to A$ as the composition of α and β;

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & \mathrm{End}(F) & \xrightarrow{\beta} & \mathrm{End}_A(F(N)) & \xrightarrow{\cong} & A \\ a & \longrightarrow & F(a_{(\ )}) & \longrightarrow & F(a_{(N)}) & \longrightarrow & \sigma(a). \end{array}$$

Then, for each object $X \in \mathfrak{C}$, we have isomorphisms of k-modules;

$$\begin{array}{ccccc} F(X) & \xrightarrow{\cong} & \mathrm{Hom}_A(F(N), F(X)) & \xrightarrow{\cong} & \mathrm{Hom}_A(N_{\sigma^{-1}}, X) \\ x & \longrightarrow & (x_{F(N)} : 1 \mapsto x) & \longrightarrow & F^{-1}(x_{F(N)}), \end{array}$$

whose composition we denote by φ_X. Since $F^{-1}(\sigma(a)_{F(N)}) = a_{(N)}$ holds for $a \in A$, we can show that φ_X is an A-module isomorphism for all $X \in \mathfrak{C}$. Since it is easily verified that φ_X is functorial in X, we have the isomorphism of functors $F \cong \mathrm{Hom}_A(N_{\sigma^{-1}}, \ \ )$, and the proof is completed. □

3. Picard groups

In this section, we study the group of all the A-linear automorphisms of an additive full subcategory of A-Mod. As in the previous section $\mathfrak{C}$ is an additive full subcategory of A-Mod. We always assume that $\mathfrak{C}$ contains A as an object.

By virtue of Theorem 6, we have the following corollary.

Corollary 7 ([2, Corollary 3.1]). *For any element* $[F] \in \mathrm{Aut}_A(\mathfrak{C})$*, there is an isomorphism of functors* $F \cong \mathrm{Hom}_A(N, -)|_{\mathrm{C}}$ *for some* $N \in \mathfrak{C}$.

Taking this corollary into consideration, we make the following definition.

Definition 8. We define $\mathrm{Pic}(\mathfrak{C})$ to be the set of all the isomorphism classes of A-modules $M \in \mathfrak{C}$ such that $\mathrm{Hom}_A(M, -)|_{\mathrm{C}}$ gives an auto-equivalence of the category $\mathfrak{C}$. That is,

$$\mathrm{Pic}(\mathfrak{C}) = \{M \in \mathfrak{C} \mid \mathrm{Hom}_A(M, -)|_{\mathrm{C}} \text{ gives an } (A\text{-linear}) \text{ equivalence } \mathfrak{C} \to \mathfrak{C}\}/\cong .$$

We define the group structure on $\mathrm{Pic}(\mathfrak{C})$ as follows: Let $[M]$ and $[N]$ be in $\mathrm{Pic}(\mathfrak{C})$. Since the composition $\mathrm{Hom}_A(M, -)|_{\mathrm{C}} \circ \mathrm{Hom}_A(N, -)|_{\mathrm{C}}$ is also an A-linear equivalence, it follows from Corollary 7 that there exists an $L \in \mathfrak{C}$ such that

$$\mathrm{Hom}_A(L, -)|_{\mathrm{C}} \cong \mathrm{Hom}_A(M, -)|_{\mathrm{C}} \circ \mathrm{Hom}_A(N, -)|_{\mathrm{C}}.$$

We define the multiplication in $\mathrm{Pic}(\mathfrak{C})$ by $[M]\cdot[N]=[L]$. Note that

$$\begin{aligned}\mathrm{Hom}_A(M,-)|_{\mathrm{C}}\circ\ \mathrm{Hom}_A(N,-)|_{\mathrm{C}} &\cong \mathrm{Hom}_A(M\otimes_A N,-)|_{\mathrm{C}}\\ &\cong \mathrm{Hom}_A(N,-)|_{\mathrm{C}}\circ\ \mathrm{Hom}_A(M,-)|_{\mathrm{C}},\end{aligned}$$

and hence $[M]\cdot[N]=[N]\cdot[M]$. In such a way $\mathrm{Pic}(\mathfrak{C})$ is an abelian group with the identity element $[A]$. We call $\mathrm{Pic}(\mathfrak{C})$ the Picard group of $\mathfrak{C}$.

Note from Yoneda's lemma that the multiplication in $\mathrm{Pic}(\mathfrak{C})$ is well-defined. Furthermore, the mapping $\mathrm{Pic}(\mathfrak{C})\to\mathrm{Aut}_A(\mathfrak{C})$ which sends $[M]$ to $\mathrm{Hom}_A(M,-)|_{\mathrm{C}}$ is an isomorphism of groups by Corollary 7. Since $\mathrm{Aut}_A(\mathfrak{C})$ is naturally a subgroup of $\mathrm{Aut}_k(\mathfrak{C})$, we can regard $\mathrm{Pic}(\mathfrak{C})$ as a subgroup $\mathrm{Aut}_k(\mathfrak{C})$ through the isomorphism $\mathrm{Pic}(\mathfrak{C})\cong\mathrm{Aut}_A(\mathfrak{C})$.

Assume furthermore that an additive full subcategory $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$. Then we have shown by the above argument together with Lemma 4 that $\mathrm{Aut}_k(\mathfrak{C})$ contains two subgroups, $\mathrm{Pic}(\mathfrak{C})$ and $\mathrm{Aut}_{k\text{-alg}}(A)$. Moreover, Theorem 6 implies that these two subgroups generate the group $\mathrm{Aut}_k(\mathfrak{C})$. Thus it is straightforward to see that the following theorem holds.

Theorem 9 ([2, Theorem 4.9])**.** *Assume that an additive full subcategory $\mathfrak{C}$ is stable under $\mathrm{Aut}_{k\text{-alg}}(A)$ and assume that $A\in\mathfrak{C}$. Then there is an isomorphism of groups*

$$\mathrm{Aut}_k(\mathfrak{C})\cong\mathrm{Aut}_{k\text{-alg}}(A)\ltimes\mathrm{Pic}(\mathfrak{C}).$$

Now we give several examples for $\mathrm{Pic}(\mathfrak{C})$.

Example 10 ([2, Example 3.8, 3.11])**.** We denote by A-mod the full subcategory consisting of all finitely generated A-modules. We also denote by $\mathrm{Proj}(A)$ (resp. $\mathrm{proj}(A)$) the full subcategory consisting of all projective A-modules (resp. all finitely generated projective A-modules). If A is an integral domain, we denote by $\mathrm{Tf}(A)$ (resp. $\mathrm{tf}(A)$) the full subcategory consisting of all torsion free A-modules (resp. all finitely generated torsion free A-modules). Let $\mathfrak{C}$ be one of the full subcategories A-Mod, A-mod, $\mathrm{Proj}(A)$, $\mathrm{proj}(A)$, $\mathrm{Tf}(A)$ and $\mathrm{tf}(A)$. Then we have an isomorphism $\mathrm{Pic}(\mathfrak{C})\cong\mathrm{Pic}\ A$, where $\mathrm{Pic}\ A$ denotes the (classical) Picard group of the ring A, i.e. $\mathrm{Pic}\ A=\{\text{invertible } A\text{-modules}\}/\cong$. See also [2, Proposition 3.7].

Example 11 ([2, Example 3.9, 3.10])**.** Let A be a Krull domain and let $\mathrm{Ref}(A)$ be the full subcategory consisting of all reflexive A-lattices. (Respectively, let A be a Noetherian normal domain and let $\mathrm{ref}(A)$ be the full subcategory consisting of all finitely generated reflexive A-modules.) Then there is an isomorphism $\mathrm{Pic}(\mathrm{Ref}(A))\cong C\ell(A)$ (resp. $\mathrm{Pic}(\mathrm{ref}(A))\cong C\ell(A)$), where $C\ell(A)$ denotes the divisor class group of A.

Example 12 ([2, Example 3.12])**.** Let $(A,\mathfrak{m})$ be a Noetherian local ring. We consider the full subcategory $d^{\geq 1}(A)$ of A-Mod which consists of all the finitely generated A-modules M satisfying depth $M\geq 1$. If depth $A\geq 1$, then $\mathrm{Pic}(d^{\geq 1}(A))$ is a trivial group.

4. Picard group of CM(A)

In this section, let $(A,\mathfrak{m})$ be a Cohen-Macaulay local k-algebra, i.e. A is a Noetherian local k-algebra with maximal ideal $\mathfrak{m}$ and satisfies the equality depth $A=\dim A$. We focus on the additive full subcategory $\mathrm{CM}(A)$ consisting of all the maximal Cohen-Macaulay modules over A and we give the reason why Theorem 1 holds. See [3] for the details of $\mathrm{CM}(A)$.

For the Picard group of $\mathrm{CM}(A)$, we have the following result.

Theorem 13 ([2, Theorem 5.2]). *Let A be a Cohen-Macaulay local k-algebra of any dimension. Suppose that A is regular in codimension two, i.e. A_{p} is a regular local ring for any prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 2$. Then $\mathrm{Pic}(\mathrm{CM}(A))$ is a trivial group.*

Proof. If $\dim A = 0$, then $\mathrm{CM}(A) = A$-mod and hence $\mathrm{Pic}(\mathrm{CM}(A)) = \mathrm{Pic}\ A$ is a trivial group by Example 10. If $\dim A = 1$, then $\mathrm{CM}(A) = d^{\geq 1}(A)$ and we have shown in Example 12 that $\mathrm{Pic}(\mathrm{CM}(A))$ is again a trivial group. If $\dim A = 2$, then our assumption means that A is a regular local ring hence a UFD. Note that $\mathrm{CM}(A) = \mathrm{ref}(A)$ in this case. Therefore $\mathrm{Pic}(\mathrm{CM}(A)) \cong C\ell(A)$ is a trivial group.

In the rest we assume $d = \dim A \geq 3$. Let $[M] \in \mathrm{Pic}(\mathrm{CM}(A))$. Assuming that M is not free, we shall show a contradiction. Take a free cover F of M and we obtain an exact sequence $0 \longrightarrow \Omega(M) \longrightarrow F \longrightarrow M \longrightarrow 0$. Note that the first syzygy module $\Omega(M)$ belongs to $\mathrm{CM}(A)$. Apply $\mathrm{Hom}_A(M, -)$ to the sequence, and we get an exact sequence

$$0 \to \mathrm{Hom}_A(M, \Omega(M)) \to \mathrm{Hom}_A(M, F) \to \mathrm{Hom}_A(M, M) \xrightarrow{f} \mathrm{Ext}^1_A(M, \Omega(M)) \ .$$

Notice that $f \neq 0$, since we have assumed that M is not free. Because of the assumption, we see that $\mathrm{Ext}^1_A(M, \Omega(M))_{\mathrm{p}} = 0$ for all prime ideals $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = 2$. This implies that $\dim \mathrm{Ext}^1_A(M, \Omega(M)) \leq d - 3$, hence the image $\mathrm{Im}(f)$ is a nontrivial A-module of dimension at most $d - 3$. In particular, we have depth $\mathrm{Im}(f) \leq d - 3$.

On the other hand, since $\mathrm{Hom}_A(M, -)|_{\mathrm{CM}(A)}$ is a functor from $\mathrm{CM}(A)$ to itself, the modules $\mathrm{Hom}_A(M, \Omega(M))$, $\mathrm{Hom}_A(M, F)$ and $\mathrm{Hom}_A(M, M)$ have depth d. Hence we conclude from the depth argument [1, Proposition 1.2.9] that depth $\mathrm{Im}(f) \geq d - 2$. This is a contradiction, and the proof is completed. □

As in Theorem 1, let A be a Cohen-Macaulay local k-algebra of dimension d that has only an isolated singularity. We give a proof for the equalities in Theorem 1. If $d \neq 2$, then we see from Theorem 13 that $\mathrm{Pic}(\mathrm{CM}(A))$ is a trivial group, hence $\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \mathrm{Aut}_{k\text{-alg}}(A)$ by Theorem 9. On the other hand, if $d = 2$ then A is a normal domain and we have $\mathrm{CM}(A) = \mathrm{ref}(A)$, hence $\mathrm{Pic}(\mathrm{CM}(A)) \cong C\ell(A)$ by Example 11. Therefore $\mathrm{Aut}_k(\mathrm{CM}(A)) \cong \mathrm{Aut}_{k\text{-alg}}(A) \ltimes C\ell(A)$ by Theorem 9.

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