ON FILTERED SEMI-DUALIZING BIMODULES

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Abstract. In this paper, we study the homological property of Rees modules of finitely generated filtered modules. In particular we state on Gorenstein dimension (more generally G_C-dimension in the sense of T. Araya, R. Takahashi, and Y. Yoshino [1]) of Rees modules.

1. Introduction

Semi-dualizing bimodule was introduced by T. Araya, R. Takahashi and Y. Yoshino in [1], which is a generalization of semi-dualizing module in commutative ring theory. For a semi-dualizing bimodule C and a finitely generated module M, they also introduced G_C-dim M, which is a generalization of Gorenstein dimension of M, and extended the notion of Cohen-Macaulay dimension for modules over commutative Noetherian local rings to that for bounded complexes over non-commutative Noetherian rings. On the other hand, in [3] with K. Nishida, we showed the following:

Theorem A. *Let Λ be a filtered ring, and M a finitely generated filtered Λ-module with good filtration. Then the Gorenstein dimension of M is less than or equal to the Gorenstein dimension of associated graded module of M.*

In Section 2, we study the filtered semi-dualizing bimodules and give a generalization of Theorem A without proof.

In Section 3, we state on Gorenstein-dimension of Rees modules. For a filtered (Λ, Λ')-bimodule C, we show that if the associated graded bimodule $\mathrm{gr}C$ of C is semi-dualizing, then Rees bimodule $\widetilde{C}$ of C is semi-dualzing (proposition 23), and we compare $G_{\mathrm{gr}C}$-$\dim \mathrm{gr}M$ with $G_{\widetilde{C}}$-$\dim\widetilde{M}$ for a finitely generated filtered Λ-module M.

In the rest of this section, we shall recall some definitions and properties on filtered ring theory.

Definition 1. Let Λ be a ring. A family $\mathcal{F}\Lambda = \{ \mathcal{F}_p\Lambda \mid p \in \mathbb{Z} \}$ of additive subgroups of Λ is called a *(positive) filtration* of Λ, if

(1) $\mathcal{F}_p\Lambda \subset \mathcal{F}_{p+1}\Lambda$ for all $p \in \mathbb{Z}$,
(2) $\mathcal{F}_p\Lambda = 0$, if $p < 0$
(3) $1 \in \mathcal{F}_0\Lambda$,
(4) $(\mathcal{F}_p\Lambda)(\mathcal{F}_q\Lambda) \subset \mathcal{F}_{p+q}\Lambda$ for all $p, q \in \mathbb{Z}$, and
(5) $\Lambda = \bigcup_{p\in\mathbb{Z}} \mathcal{F}_p\Lambda$.

The detailed version of this paper will be submitted for publication elsewhere.

A ring Λ is called a *(positive) filtered ring*, if it has a filtration. If a ring Λ has a filtration $\mathcal{F}\Lambda$, then $\bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a *graded ring* with multiplication $\sigma_p(a)\sigma_q(b) = \sigma_{p+q}(ab)$ where $\sigma_p : \mathcal{F}_p\Lambda \longrightarrow \mathcal{F}_p\Lambda/\mathcal{F}_{p-1}\Lambda$ is a canonical map, and $a \in \mathcal{F}_p\Lambda,\ b \in \mathcal{F}_q\Lambda$. We denote by $\mathrm{gr}\Lambda$ the above associated graded ring of Λ.

Definition 2. Let Λ be a filtered ring with a filtration $\mathcal{F}\Lambda$, and M a Λ-module. A family $\mathcal{F}M = \{\, \mathcal{F}_pM \mid p \in \mathbb{Z}\,\}$ of additive subgroups of M is called a *filtration* of M, if

(1) $\mathcal{F}_pM \subset \mathcal{F}_{p+1}M$ for all $p \in \mathbb{Z}$,
(2) $\mathcal{F}_pM = 0$ for $p \ll 0$,
(3) $(\mathcal{F}_p\Lambda)(\mathcal{F}_qM) \subset \mathcal{F}_{p+q}M$ for all $p, q \in \mathbb{Z}$, and
(4) $M = \bigcup_{p\in\mathbb{Z}} \mathcal{F}_pM$.

A Λ-module M is called a *filtered* Λ*-module* if M has a filtration. If a left Λ-module M has a filtration $\mathcal{F}M$, then $\bigoplus_{p\in\mathbb{Z}} \mathcal{F}_pM/\mathcal{F}_{p-1}M$ is a graded left $\mathrm{gr}\Lambda$-module with action $\sigma_p(a)\tau_q(x) = \tau_{p+q}(ax)$ where $a \in \mathcal{F}_p\Lambda,\ x \in \mathcal{F}_qM$, and $\tau_q : \mathcal{F}_qM \longrightarrow \mathcal{F}_qM/\mathcal{F}_{q-1}M$ is a canonical map. We denote by $\mathrm{gr}M$ the above associated graded $\mathrm{gr}\Lambda$-module of M.

Let Λ, Λ' be filtered rings. A (Λ, Λ')-bimodule M is called a *filtered bimodule* if there exists a family $\mathcal{F}M$ of subgroups of M such that $({}_\Lambda M, \mathcal{F}M)$ and $(M_{\Lambda'}, \mathcal{F}M)$ are filtered modules.

Definition 3. Let Λ be a filtered ring with a filtration $\mathcal{F}\Lambda$. Then the graded ring $\oplus_{p\in\mathbb{Z}}\mathcal{F}_p\Lambda$ is called the *Rees ring* of $(\Lambda, \mathcal{F})$, and denoted by $\widetilde{\Lambda}$. Similarly, for a filtered left module $(M, \mathcal{F}M)$ over a filtered ring $(\Lambda, \mathcal{F}\Lambda)$, the graded left $\widetilde{\Lambda}$-module $\oplus_{p\in\mathbb{Z}}\mathcal{F}_pM$ is called the *Rees module* of M, and denoted by $\widetilde{M}$.

Let Λ be a filtered ring. Then the Rees ring $\widetilde{\Lambda}$ has the canonical central regular element $X = (\delta_{1p})_{p\in\mathbb{Z}} \in \widetilde{\Lambda}$ where δ_{ij} is the Kronecker's delta. Suppose that $(M, \mathcal{F}M)$ is a filtered Λ-module. Then,

(1) $\widetilde{\Lambda}/X\widetilde{\Lambda} \cong \mathrm{gr}\Lambda$ (as graded ring) and $\widetilde{M}/X\widetilde{M} \cong \mathrm{gr}_{\mathcal{F}}M$ (as graded module).
(2) $\widetilde{M}/(1-X)\widetilde{M} \cong M$

Definition 4. Let $(\Lambda, \mathcal{F}\Lambda)$ be a filtered ring. A filtration $\mathcal{F}M$ of a Λ-module M is called *good*, if there exist $p_1, \cdots, p_r \in \mathbb{Z}$ and $m_1, \cdots, m_r \in M$ such that for all $p \in \mathbb{Z}$

$$\mathcal{F}_pM = \sum_{i=1}^{r}(\mathcal{F}_{p-p_i}\Lambda)m_i.$$

From the above definition, we can easily check the following:

(1) For a filtered Λ-module $(M, \mathcal{F}M)$, $\mathcal{F}M$ is good if and only if $\mathrm{gr}_{\mathcal{F}}M$ is a finitely generated $\mathrm{gr}\Lambda$-module if and only if $\widetilde{M}$ is a finitely generated $\widetilde{\Lambda}$-module.
(2) Suppose that $(\Lambda, \mathcal{F}\Lambda)$ is a filtered ring. If M is a finitely generated Λ-module, then M has a good filtration.

Definition 5. Let $(M, \mathcal{F}M)$, $(N, \mathcal{F}N)$ be filtered Λ-modules. A Λ-homomorphism $f : M \longrightarrow N$ is called a *filtered homomorphism*, if $f(\mathcal{F}_pM) \subset \mathcal{F}_pN$ for all $p \in \mathbb{Z}$. Further, f is called *strict*, if $f(\mathcal{F}_pM) = \mathrm{Im}f \cap \mathcal{F}_pN$ for all $p \in \mathbb{Z}$.

Remark 6. (1) The composition of two filtered homomorphisms is also a filtered homomorphism, but it need not be strict even if both of them are strict.

(2) Let $f : M \longrightarrow N$ be a filtered homomorphism, then f indeuces canonical additive maps $f_p : \mathcal{F}_pM/\mathcal{F}_{p-1}M \longrightarrow \mathcal{F}_pN/\mathcal{F}_{p-1}N$ given by $x + \mathcal{F}_{p-1}M \longmapsto f(x) + \mathcal{F}_{p-1}N$. It is clear that $\mathrm{gr}f = \oplus_{p\in\mathbb{Z}} f_p$ defines a graded homomorphism from $\mathrm{gr}M$ to $\mathrm{gr}N$. Note that $(\mathrm{gr}g)(\mathrm{gr}f) = \mathrm{gr}(gf)$ for any filtered homomorphisms $f : M \longrightarrow N$, $g : N \longrightarrow L$. Similarly, $\tilde{f} = \oplus_{p\in\mathbb{Z}} f|_{\mathcal{F}_pM}$ defines a graded homomorphism from $\widetilde{M}$ to $\widetilde{N}$, and $\tilde{g}\tilde{f} = \widetilde{gf}$ holds.

Lemma 7. *Let* $(*)\ :\ L \xrightarrow{f} M \xrightarrow{g} N$ *be a sequence of filtered modules and filtered homomorphisms such that* $gf = 0$. *Then*

(1) *The sequence*

$$\mathrm{gr}(*)\ :\ \mathrm{gr}L \xrightarrow{\mathrm{gr}f} \mathrm{gr}M \xrightarrow{\mathrm{gr}g} \mathrm{gr}N$$

is exact if and only if $(*)$ *is exact and* f, g *are strict.*

(2) *The sequence*

$$\widetilde{(*)}\ :\ \widetilde{L} \xrightarrow{\tilde{f}} \widetilde{M} \xrightarrow{\tilde{g}} \widetilde{N}$$

is exact if and only if $(*)$ *is exact and* f *is strict.*

Lemma 8. *(*[2] *Chapter III Proposition 2.2.4) Let* M *and* N *be filtered* Λ*-modules. Then* $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(M, N)$ *is a subfactor of* $\mathrm{Ext}^i_\Lambda(\mathrm{gr}\,M, \mathrm{gr}\,N)$ *for each* $i \geq 0$.

2. Semi-dualizing Filtered Modules

First, we recall the definition of semi-dualizing bimodules.

Definition 9. ([1] Definition 2.1) Let R, R' be Noetherian rings. An (R, R')-bimodule C is called a *semi-dualizing bimodule* if the following conditions hold:

(1) The right homothety R'-bimodule morphism $R' \longrightarrow \mathrm{Hom}_R(C, C)$ is a bijection,
(2) The left homothety R-bimodule morphism $R \longrightarrow \mathrm{Hom}_{R'}(C, C)$ is a bijection,
(3) $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$, and
(4) $\mathrm{Ext}^i_{R'}(C, C) = 0$ for all $i > 0$.

Definition 10. ([1] Definition 2.2) Let R, R' be Noetherian rings and C a semi-dualizing (R, R')-bimodule. An R-module M is called *C-reflexive* if the following conditions hold:

(1) $\mathrm{Ext}^i_R(M, C) = 0$ for all $i > 0$,
(2) $\mathrm{Ext}^i_{R'}(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$, and
(3) The natural morphism

$$M \longrightarrow \mathrm{Hom}_{R'}(\mathrm{Hom}_R(M, C), C)$$

is a bijection.

Definition 11. ([1] Definition 2.3) Let C be a semi-dualizing (R, R')-bimodule and M an R-module. If there exists an exact sequence

$$0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

where each X_i is a C-reflexive R-module, M is called that G_C-dimension is less than or equal to n (denoted by G_C-dim$M \leq n$). If G_C-dim$M \leq n$ and G_C-dim$M \not\leq n-1$, then we say G_C-dimension of M is equal to n (denoted by G_C-dim $M = n$).

Remark 12. (1) In [1], a semi-dualizing bimodule was defined over a left Noetherian ring R and a right Noetherian ring R'. In this paper we assume that both R and R' are (left and right) Noetherian rings.

(2) The ring R itself is a semidualizing (R, R)-bimodule and the R-reflexive modules coincide with the modules whose Gorenstein dimension are equal to 0. Moreover, in the case of $C = {}_R R_R$, we have G_C-dim $M = G$-dim M.

The following lemma is indispensable for the study of filtered semi-dualizing bimodules.

Lemma 13. *Let* $(C, \mathcal{F})$ *be a filtered* (Λ, Λ')*-bimodule such that* ${}_{\mathrm{gr}\Lambda}\mathrm{gr}_{\mathcal{F}}C$ *and* $\mathrm{gr}_{\mathcal{F}}C_{\mathrm{gr}\Lambda}$ *are finitely generated. If* $\mathrm{gr}_{\mathcal{F}}C$ *is a semi-dualizing* $(\mathrm{gr}\,\Lambda, \mathrm{gr}\,\Lambda')$*-bimodule, then* C *is a semi-dualizing bimodule.*

Proof. Assume that $f : \Lambda' \longrightarrow \mathrm{Hom}_\Lambda(C, C)$ is the right homothety Λ'-bimodule morphism, and $\varphi : \mathrm{gr}\,\Lambda' \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C, \mathrm{gr}\,C)$ is the right homothety $\mathrm{gr}\,\Lambda'$-bimodule morphism. Since there is a natural graded monomorphism

$$\psi : \mathrm{gr}\,\mathrm{Hom}_\Lambda(C, C) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C, \mathrm{gr}\,C),$$

we get the following commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}\,\Lambda' & \xrightarrow[\varphi]{} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C, \mathrm{gr}\,C) \\ {\scriptstyle \mathrm{gr}f}\big\downarrow & & \big\| \\ \mathrm{gr}\,\mathrm{Hom}_\Lambda(C, C) & \xrightarrow[\psi]{} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C, \mathrm{gr}\,C) \end{array}$$

Since $\varphi = \psi \circ \mathrm{gr}f$ is an isomorphism from the assumption, ψ is an epimorphism. Thus f is a Λ'-isomorphism. It follows from the lemma 8 that $\mathrm{gr}\,\mathrm{Ext}^i_\Lambda(C, C)$ is a subfactor of $\mathrm{Ext}^i_\Lambda(\mathrm{gr}\,C, \mathrm{gr}\,C)$ for each $i \geq 0$. Therefore $\mathrm{Ext}^i_\Lambda(C, C) = 0$ for all $i > 0$. Similarly, we can prove that the left homothety morphhism $g : \Lambda \longrightarrow \mathrm{Hom}_{\Lambda'}(C, C)$ is a bijection and $\mathrm{Ext}^i_{\Lambda'}(C, C) = 0$ for all $i > 0$. Therefore C is a semi-dualizing bimodule. □

Definition 14. We say that a filtered (Λ, Λ')-bimodule C is a *semi-dualizing filtered bimodule* if $\mathrm{gr}\,C$ is a semi-dualizing $(\mathrm{gr}\,\Lambda, \mathrm{gr}\,\Lambda')$-bimodule.

All semi-dualizing filtered bimodules are semi-dualizing bimodules by lemma 13. In the rest of this section, C is a semi-dualizing filtered (Λ, Λ')-bimodule.

In [1], it is proved that G_C-dim$M \leq k$ if and only if G_C-dim$\Omega^k M = 0$, where $\Omega^k M$ is the k-th syzygy of M (Lemma 2.7). Applying this lemma, we can prove the following result in a completely similar way to the proof of Theorem A. So we give only the result without proof.

Proposition 15. *Let M be a filtered Λ-module. Then the following inequality holds:*

$$G_C\text{-dim}M \leq G_{\mathrm{gr}C}\text{-dim}\,\mathrm{gr}M$$

3. G_C-dimension for Rees Modules

Throughout this section, we denote by X (resp. X') the canonical central regular element $(\delta_{1p})_{p\in\mathbb{Z}} \in \widetilde{\Lambda}$ (resp. $(\delta_{1p})_{p\in\mathbb{Z}} \in \widetilde{\Lambda'}$) where δ_{ij} is the Kronecker's delta. First of all, we shall recall the Rees theorem, that is

Theorem 16 (Rees theorem)**.** *([4] Theorem 9.37) Let R be a ring, $T \in R$ a central regular element, and M a T-torsionfree left R-module (T-torsionfree means that left multiplication by T is injectoin). Then*

$$\mathrm{Ext}^n_{R/TR}(A, M/TM) \cong \mathrm{Ext}^{n+1}_R(A, M)$$

for any left R/TR-module A and $n \geq 0$.

Since $\widetilde{M}/X\widetilde{M} \cong \mathrm{gr}M$, $\widetilde{M}$ is X-torsionfree for any $M \in \mathrm{filt}\Lambda$, and $\widetilde{\Lambda}/X\widetilde{\Lambda} \cong \mathrm{gr}_{\mathcal{F}}\Lambda$, we can get the following:

$$\mathrm{Ext}^n_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C) \cong \mathrm{Ext}^{n+1}_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C})$$

In order to prove our main theorem, we give some easy lemmata without proofs.

Lemma 17. *Assume that M, N are filtered Λ-modules. Then there exists a natural isomorphism $\widetilde{\mathrm{Hom}_{\Lambda}(M, N)} \cong \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{N})$*

Lemma 18. *Assume that $M \in \mathrm{filt}\Lambda$ and*

$$\mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C) = \mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C) = 0.$$

Then, the natural map $\varphi : M \longrightarrow \mathrm{Hom}_{\Lambda'}(\mathrm{Hom}_{\Lambda}(M, C), C)$ is bijective if and only if the natural map $\Phi : \mathrm{gr}M \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda'}(\mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}M, \mathrm{gr}C), \mathrm{gr}C)$ is bijective.

Lemma 19. *Let $M \in \mathrm{filt}\Lambda$. Then, the natural map*

$$\varphi : M \longrightarrow \mathrm{Hom}_{\Lambda'}(\mathrm{Hom}_{\Lambda}(M, C), C)$$

is strict isomorphism if and only if the natural map

$$\Phi : \widetilde{M} \longrightarrow \mathrm{Hom}_{\widetilde{\Lambda'}}(\mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}), \widetilde{C})$$

is isomorphism.

Remark 20. Since X is an $\widetilde{M}$-regular element for all $M \in \mathrm{filt}\Lambda$, there exists an exact sequence:

$$0 \longrightarrow \widetilde{M} \xrightarrow{\mu_X} \widetilde{M} \longrightarrow \widetilde{M}/X\widetilde{M} \longrightarrow 0$$

where μ_X is the left multiplication by X. Applying $\mathrm{Hom}_{\widetilde{\Lambda}}(-,\widetilde{C})$, we get a long exact sequence:

$$(\dagger)\quad \cdots \longrightarrow \mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M},\widetilde{C}) \longrightarrow \mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \xrightarrow{\overline{\mu_X}} \mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \longrightarrow \mathrm{Ext}^{i+1}_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M},\widetilde{C}) \longrightarrow \cdots$$

Since $\overline{\mu_X}$ is a right multiplication by X and $\mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C})$ is graded $\widetilde{\Lambda}'$-module such that $(\mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}))_{(p)} = 0$ for $p \ll 0$, we get the following:

Lemma 21. *With the same notations as above, if* $\mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \neq 0$*, then*

$$\mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) \xrightarrow{\overline{\mu_X}} \mathrm{Ext}^i_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C})$$

is not surjective.

Form the lemma 21, we can immediately get the following:

Lemma 22. *With the same notations as avobe,* $\mathrm{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{M},\widetilde{C}) = 0$ *if and only if* $\mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda}(\mathrm{gr}M,\mathrm{gr}C) = 0$.

Proposition 23. *Let M be a filnitely generated filtered (Λ,Λ')-bimodule. If M is a semi-dualizing filtered bimodule then $\widetilde{M}$ is a semi-dualizing $(\widetilde{\Lambda},\widetilde{\Lambda}')$-bimodule.*

Proof. By the lemma 22, we have

$$\mathrm{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C}) = 0 \text{ if and only if } \mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda}(\mathrm{gr}C,\mathrm{gr}C) = 0, \text{ and}$$

$$\mathrm{Ext}^{i>0}_{\widetilde{\Lambda}'}(\widetilde{C},\widetilde{C}) = 0 \text{ if and only if } \mathrm{Ext}^{i>0}_{\mathrm{gr}\Lambda'}(\mathrm{gr}C,\mathrm{gr}C) = 0.$$

Assume that $f : \Lambda' \longrightarrow \mathrm{Hom}_\Lambda(C,C)$ is the right homothety Λ'-bimodule morphism, $g : \mathrm{gr}\,\Lambda' \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C,\mathrm{gr}\,C)$ is the right homothety $\mathrm{gr}\,\Lambda'$-bimodule morphism, and $h : \widetilde{\Lambda}' \longrightarrow \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C})$ is the right homothety $\widetilde{\Lambda}'$-bimodule morphism. Since there is a natural graded monomorphisms

$$\varphi : \mathrm{gr}\,\mathrm{Hom}_\Lambda(C,C) \longrightarrow \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C,\mathrm{gr}\,C),$$

we get the following two commutative diagrams:

$$\begin{array}{ccc} \mathrm{gr}\,\Lambda' & \xrightarrow[g]{} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C,\mathrm{gr}\,C) \\ {\scriptstyle \mathrm{gr}f}\downarrow & & \Vert \\ \mathrm{gr}\,\mathrm{Hom}_\Lambda(C,C) & \xrightarrow[\varphi]{} & \mathrm{Hom}_{\mathrm{gr}\Lambda}(\mathrm{gr}\,C,\mathrm{gr}\,C) \end{array}$$

$$\begin{array}{ccc} \widetilde{\Lambda}' & \xrightarrow[h]{} & \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C}) \\ {\scriptstyle \widetilde{f}}\downarrow & & \Vert \\ \widetilde{\mathrm{Hom}_\Lambda(C,C)} & \xrightarrow[\psi]{} & \mathrm{Hom}_{\widetilde{\Lambda}}(\widetilde{C},\widetilde{C}) \end{array}$$

Note that f is a strict Λ'-isomorphism if g is a bijection as in the proof of the lemma 13. Hence $\tilde{f}$ is an isomorphism. Since ψ is also an isomorphism from the remark 17, h is an isomorphism. Similarly, we can show that the left homothety $\widetilde{\Lambda}$-bimodule morphism is bijective. Therefore $\widetilde{C}$ is semi-dualizing.

□

Now we can show the main theorem of this paper.

Theorem 24. *Let M be a filtered Λ-module. Then $G_{\widetilde{C}}\text{-dim}\widetilde{M} = 0$ if and only if $G_{\text{gr}C}\text{-dimgr}M = 0$.*

Proof. Assume that $G_{\widetilde{C}}\text{-dim}\widetilde{M} = 0$. Since $\text{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}) = 0$, we have $\text{Ext}^{i>0}_{\text{gr}\Lambda}(\text{gr}M, \text{gr}C) = 0$ from the lemma 22. Moreover we get the following short exact seqence from the (†) in the remark 20:

$$0 \longrightarrow (\widetilde{M})^* \longrightarrow (\widetilde{M})^* \longrightarrow \text{Ext}^1_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C}) \longrightarrow 0$$

By the remark 17, we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\widetilde{M})^* & \longrightarrow & (\widetilde{M})^* & \longrightarrow & \text{Ext}^1_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{M^*} & \longrightarrow & \widetilde{M^*} & \longrightarrow & \widetilde{M^*}/X\widetilde{M^*} & \longrightarrow & 0 \quad (\dagger)^* \end{array}$$

Thus, $\text{gr}M^* \cong \widetilde{M^*}/X\widetilde{M^*} \cong \text{Ext}^1_{\widetilde{\Lambda}}(\widetilde{M}/X\widetilde{M}, \widetilde{C}) \cong (\text{gr}M)^*$ by Rees theorem. By taking the long exact sequence of $(\dagger)^*$, we get $\text{Ext}^{i>0}_{\text{gr}\Lambda'}((\text{gr}M)^*, \text{gr}\mathit{\Lambda}) = 0$. Hence it follows from the lemma 18 and 19 that the natural map

$$\Phi : \text{gr}M \longrightarrow \text{Hom}_{\text{gr}\Lambda'}(\text{Hom}_{\text{gr}\Lambda}(\text{gr}M, \text{gr}C), \text{gr}C)$$

is an isomorphism. Therefore $G_{\text{gr}C}\text{-dim}\,\text{gr}M = 0$.

Conversely, assume that $G_{\text{gr}C}\text{-dim}\,\text{gr}M = 0$. By the remark 20, the lemma 22 and the Rees theorem, we can show

$$\text{Ext}^{i>0}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}) = \text{Ext}^{i>0}_{\widetilde{\Lambda'}}((\widetilde{M})^*, \widetilde{C}) = 0.$$

Therefore $G_{\widetilde{C}}\text{-dim}\,\tilde{M} = 0$. Since $G_{\text{gr}C}\text{-dim}\,\text{gr}M = 0$,

$$\text{Ext}^{i>0}_{\text{gr}\Lambda}(\text{gr}M, \text{gr}C) = \text{Ext}^{i>0}_{\text{gr}\Lambda'}(\text{Hom}_{\text{gr}\Lambda}(\text{gr}M, \text{gr}C), \text{gr}C) = 0$$

and the natural map $\Phi : \text{gr}M \longrightarrow \text{Hom}_{\text{gr}\Lambda'}(\text{Hom}_{\text{gr}\Lambda}(\text{gr}M, \text{gr}C), \text{gr}C)$ is an isomorphism. Therefore the natural map

$$\Psi : \widetilde{M} \longrightarrow \text{Hom}_{\widetilde{\Lambda'}}(\text{Hom}_{\widetilde{\Lambda}}(\widetilde{M}, \widetilde{C}), \widetilde{C})$$

is an isomorphism from the lemma 18 and 19. Hence, $G_{\widetilde{C}}\text{-dim}\widetilde{M} = 0$.

□

We can show the following by induction on $G_{\text{gr}C}\text{-dim}\,\text{gr}M$.

Corollary 25. *For any finitely generated filtered Λ-module M with good filtration,*

$$G_{\mathrm{gr}C}\text{-}\dim \mathrm{gr}M = G_{\widetilde{C}}\text{-}\dim \widetilde{M}$$

holds.

In particular, in the case of $C = {}_{\Lambda}\Lambda_{\Lambda}$, we can get the following.

Corollary 26. $G\text{-}\dim \mathrm{gr}M = G\text{-}\dim \widetilde{M}$ *for all finitely generated filtered Λ-module M with good filtration.*

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