DESCENT OF DIVISOR CLASS GROUPS OF KRULL DOMAINS

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ABSTRACT. For a pair of Krull domains (A, B) such that $\mathcal{Q}(B)/\mathcal{Q}(A)$ is a field extension of their quotient fields and $\mathcal{Q}(A) \cap B = A$, we study on the relation between the divisor class groups $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ by A. Magid's diagram showing finite generation of class groups of rings of invariants (cf. [4]). We define the descent properties with the existence of the canonical morphism of class groups in the sense of Magid and obtain a ladder property of these descents. This can be applied to regular actions of algebraic tori on affine normal varieties and characterizes freeness of monomials of prime relative invariants on these varieties. Furthermore we define certain subgroups of class groups of normal domains and their invariant subrings which determine a class of modules of relative invariants to be free.

Key Words: Krull Domain, Class Group, Relative Invariant, Algebraic Torus.

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1. Introduction

In this paper, we denote by (A, B) a pair of Krull domains such that the quotient field $\mathcal{Q}(B)$ of B is an extension of $\mathcal{Q}(A)$ satisfying $A = \mathcal{Q}(A) \cap B$, which is called a *generic dominant Krull pair*. This is related to invariant theory of normal varieties as follows: Let (X, G) be a regular action of affine algebraic group G on an affine normal variety over an algebraically closed field K. Then, putting $A = \mathcal{O}(X)^G$ and $B = \mathcal{O}(X)$, we obtain a generic dominant Krull pair (A, B), where $\mathcal{O}(X)$ denotes the affine algebra of regular functions on X.

We now introduce the notations which are used throughout in this paper (cf. [1] for a general reference). For a Krull domain R, let $\mathrm{Ht}_1(R) := \{\mathfrak{P} \in \mathrm{Spec}\ R \mid \mathrm{ht}(\mathfrak{P}) = 1\}$, $\mathrm{Div}(R) :=$ the divisor group of R, $\mathrm{Prin}(R) :=$ the group of principal divisors of R and $\mathrm{Cl}(R) :=$ the divisor class group of R. Let $I_R(D)$ denote the divisorial fractional ideal of R defined by a divisor $D \in \mathrm{Div}(R)$. For a non-empty $Y \subseteq \mathcal{Q}(R)$ such that $R \cdot Y$ is a fractional ideal of R, let $(R \cdot Y)^{\sim}$ denote the divisorialization of $R \cdot Y$ in R and put $\mathrm{div}_R(Y) :=$ the divisor defined by $(R \cdot Y)^{\sim}$. Let $\mathrm{v}_{R,\mathrm{P}}$ stand for the discrete valuation defined by $\mathfrak{P} \in \mathrm{Ht}_1(R)$.

At first we review A. Magid's descent (cf. [4, 5]) of a generic dominant Krull pair (A, B). Let $X_{\mathrm{q}}(B)$ be the set $\{\mathfrak{P} \in \mathrm{Ht}_1(B) \mid \mathfrak{P} \cap A = \mathfrak{q}\}$ which is non-empty for $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

The detailed version of this paper will be submitted for publication elsewhere.

Moreover put

$$x_{\mathfrak{q}} := \sum_{\mathfrak{P} \in X_{\mathfrak{q}}(B)} \mathrm{e}(\mathfrak{P}, \mathfrak{q}) \cdot \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Div}(B)$$

where $\mathrm{e}(\mathfrak{P}, \mathfrak{q})$ denotes the reduced ramification index of $\mathfrak{P}$ over $\mathfrak{q}$. Set $\mathrm{Ht}_1(A, B) := \{\mathfrak{P} \in \mathrm{Ht}_1(B) \mid \mathfrak{P} \cap A \in \mathrm{Ht}_1(A)\}$. We define the subgroup

$$E^*(A, B) := \left(\bigoplus_{\mathfrak{q} \in \mathrm{Ht}_1(A)} \boldsymbol{Z} \cdot x_{\mathfrak{q}} \right) \oplus \left(\bigoplus_{\mathfrak{P} \in \mathrm{Ht}_1(B), \mathrm{ht}(\mathfrak{P} \cap A) = 2} \boldsymbol{Z} \cdot \mathrm{div}_B(\mathfrak{P}) \right)$$

of $\mathrm{Div}(B)$ and the homomorphism

$$\Phi^*_{A,B} : E^*(A, B) \xrightarrow{\text{pr.}} \bigoplus_{\mathfrak{q} \in \mathrm{Ht}_1(A)} \boldsymbol{Z} \cdot x_{\mathfrak{q}} \longrightarrow \mathrm{Div}(A)$$

induced by $\Phi^*_{A,B}(x_{\mathfrak{q}}) = \mathrm{div}_A(\mathfrak{q}) \in \mathrm{Div}(A)$. Moreover we set

$$F(A, B) := (\mathrm{Prin}(B) \cap E^*(A, B))/\mathrm{div}_B(\mathrm{U}(\mathcal{Q}(A))),$$

$$E(A, B) := E^*(A, B))/\mathrm{div}_B(\mathrm{U}(\mathcal{Q}(A))$$

respectively. Then one has the following commutative diagram with exact rows and columns (e.g., [5]) which is called the Magid diagram of (A, B):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(A,B) & \longrightarrow & E(A,B) & \longrightarrow & \mathrm{Cl}(B) & & \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi_{A,B}} & & & & \\
 & & W(A,B) & \longrightarrow & \mathrm{Cl}(A) & \longrightarrow & Y(A,B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where $\Phi_{A,B}$ is the homomorphism induced by $\Phi^*_{A,B}$ and the groups $W(A, B)$ and $Y(A, B)$ are naturally defined.

Definition 1.1. A generic dominant Krull pair (A, B) has the *(MDP)*, if the Magid diagram induces the following diagram with exact rows

$$\begin{array}{ccccc}
E(A,B) & \xrightarrow{\text{can.}} & \mathrm{Cl}(B) & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle \cong}\downarrow & & \\
E(A,B) & \xrightarrow{\Phi_{A,B}} & \mathrm{Cl}(A) & \longrightarrow & 0
\end{array}.$$

On the other hand, define $\phi_{B,A} : \mathrm{Div}(B) \longrightarrow \mathrm{Div}(A)$ by

$$\phi_{B,A}(D) = \sum_{\mathfrak{q} \in \mathrm{Ht}_1(A)} \left(\max_{\mathfrak{P} \in X_{\mathfrak{q}}(B)} \left(- \left[- \frac{a_{\mathfrak{P}}}{\mathrm{e}(\mathfrak{P}, \mathfrak{q})} \right] \right) \right) \cdot \mathrm{div}_A(\mathfrak{q}) \in \mathrm{Div}(A)$$

where $D = \sum_{\mathfrak{P} \in \mathrm{Ht}_1(B)} a_{\mathfrak{P}} \cdot \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Div}(B)$ and $[\ \cdot\ ]$ denotes the Gauss symbol.

Put $\mathrm{BU}(A, B) := \bigoplus_{\mathrm{ht}(\mathfrak{P} \cap A) = 2} \boldsymbol{Z} \cdot \mathrm{div}_B(\mathfrak{P})$ and $\mathrm{Div}(A, B) := \bigoplus_{\mathfrak{P} \cap A \neq \{0\}} \boldsymbol{Z} \cdot \mathrm{div}_B(\mathfrak{P})$.

Definition 1.2. The following conditions are considered for the pair (A, B):

(1) $\mathrm{BU}(A, B) \subseteq \ker(E^*(A, B) \xrightarrow{\text{can.}} \mathrm{Cl}(B))$

(2) For $\mathrm{Div}(A, B) \ni D_0 \geqq 0$ s.t.

$$\mathrm{supp}_B(D_0) := \{\mathfrak{P} \mid \mathrm{v}_{B,\mathrm{P}}(I_B(D_0)) \neq 0\} \not\ni \mathfrak{P}\text{: } \forall \text{ principal prime}$$

and for $E^*(A, B) \ni D \geqq 0$ s.t. $D + D_0$: principal, we require that $\phi_{B,A}(D + D_0)$ is principal.

(3) The canonical morphism $E^*(A, B) \longrightarrow \mathrm{Cl}(B)$ is surjective.

We say that a generic dominant Krull pair (A, B) has the *(TDP)*, if these three conditions hold for (A, B).

In the next section we summarize our results on the (MDP) and (TDP) for generic dominant Krull pairs. In Sect. 3 we apply the results to the case where A is obtained as a subring of invariants in B under the action of an algebraic torus and characterize freeness of monomials of prime relative invariants. In Sect. 4 we study on the relation of certain class groups and freeness of a class of modules of relative invariants. Consequently we see a numerical criterion of obstructions of an algebraic torus of equidimensional actions. The detailed account of this part can be found in [8].

2. Descent property in abstract case

At first we point out the elementary relation between (MDP) and (TDP) in a general situation.

Proposition 2.1. For a generic dominant Krull pair (A, B), the (TDP) holds if and only if the (MDP) and (2) of Definition 1.2 hold.

Then we must have the following criterion that (TDP) holds for (A, B) which is useful in invariant theory of algebraic tori.

Theorem 2.2. For a generic dominant Krull pair (A, B), the following conditions (i) and (ii) are equivalent:

(i) (A, B) has the (TDP).

(ii) The following three conditions hold:

- *(a)* (A, B) has the (MDP).
- *(b)* $|\{\mathfrak{P} \in X_{\mathrm{q}}(B) \mid \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(B)\}| \geqq |X_{\mathrm{q}}(B)| - 1$ for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.
- *(c)* $\mathfrak{P} \in X_{\mathrm{q}}(B)$ s.t. $\mathrm{e}(\mathfrak{P}, \mathfrak{q}) > 1 \Longrightarrow \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(R)$, for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

The next result is another version of Theorem 2.2 which is useful in showing the ladder type induction of descents of class groups of a sequence of generic dominant Krull pairs.

Theorem 2.3. The following conditions (i), (ii) are equivalent:

(i) (A, B) has the (TDP).

(ii) The following four conditions hold:

- *(a)* $E^*(A, B) \longrightarrow \mathrm{Cl}(B)$ is surjective.
- *(b)* $\mathrm{BU}(A, B) \subseteq \ker(E^*(A, B) \longrightarrow \mathrm{Cl}(B))$.
- *(c)* $|\{\mathfrak{P} \in X_{\mathrm{q}}(B) \mid \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(B)\}| \geqq |X_{\mathrm{q}}(B)| - 1$ for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.
- *(d)* $\mathfrak{P} \in X_{\mathrm{q}}(B)$ s.t. $\mathrm{e}(\mathfrak{P}, \mathfrak{q}) > 1 \Longrightarrow \mathrm{div}_B(\mathfrak{P}) \in \mathrm{Prin}(R)$, for any $\mathfrak{q} \in \mathrm{Ht}_1(A)$.

We now consider an intermediate subring M of the extension B/A of rings as follows.

Notation 2.4. Let M be a subring of B containing A as a subring such that $M = \mathcal{Q}(M) \cap B$. Then there exist the Krull pairs as follows; i.e., (A, B), (M, B) and (A, M).

From now on to the end of this section, we use Notation 2.4 and describe how the descent properties of (A, M) and (M, B) induce one of (A, B).

Proposition 2.5. Suppose $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. Then

$$\phi_{B,M}(E^*(A, B)) \subseteq E^*(A, M)$$

and the following diagram is commutative:

$$\begin{array}{ccc} E^*(A,B) & \xrightarrow{\phi_{B,M}} & E^*(A,M) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \Phi^*_{A,M}} \\ E^*(A,B) & \xrightarrow{\Phi^*_{A,B}} & \mathrm{Div}(A) \end{array} .$$

This proposition is only a technical assertion, however from this we deduce the next two propositions.

Proposition 2.6. Suppose that $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. If (A, M) and (M, B) have the (MDP), then the canonical morphism $E^*(A, B) \longrightarrow \mathrm{Cl}(B)$ is surjective.

Proposition 2.7. Suppose that $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. If (A, M) has the (MDP) and (M, B) has the (TDP), then (A, B) has the (MDP).

Consequently we must have the following theorem which gives an inductive examination on the descent properties of a sequence of generic dominant Krull pairs. In fact consider a descending chain of normal series of subgroups

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_n = \{e\}$$

and a homomorphism $G \to \mathrm{Aut}(B)$. We have a chain of generic dominant Krull pairs $(B^{G_{n-1}}, B^{G_n})$, $(B^{G_{n-2}}, B^{G_{n-1}}), \ldots, (B^{G_0}, B^{G_1})$ and the study on the descent property of (B^G, B) can be reduced to the one on the sequence.

Theorem 2.8. Suppose that $\mathrm{Ht}_1(A, B) \subseteq \mathrm{Ht}_1(M, B)$. If both (A, M) and (M, B) have the (TDP), then (A, B) has the (TDP).

We have studied on the implication concerning the descent property of $(A, B) \Longrightarrow$ ones of (A, M) and (M, B) under some conditions which is the converse of the assertions in the results as above, however we omit to state the results in this paper.

3. Free monomials of prime semi-invariants and descent property

Let R be a Krull domain on which a group G acts as automorphisms and let $Z^1(G, \mathrm{U}(R))$ denote the (additive) group of 1-cocycles of G on $\mathrm{U}(R)$. For any $\chi \in Z^1(G, \mathrm{U}(R))$, put

$$R_\chi := \{a \in R \mid \sigma(a) = \chi(\sigma) \cdot a\}$$

whose elements are known as invariants of G in R relative to χ and is regarded as an R^G-module.

Since (R^G, R) is a generic Krull pair, we immediately have its Magid diagram with P. Samuel's diagram (cf. [2]) in the Galois descent method

$$
\begin{array}{ccccccccc}
 & & \mathrm{Prin}(R)^G & \longrightarrow & \mathrm{Div}(R)^G & \longrightarrow & \mathrm{Cl}(R) & & \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle =} & & \\
 & & \mathrm{Prin}(R) \cap E^*(R^G, R) & \longrightarrow & E^*(R^G, R) & \longrightarrow & \mathrm{Cl}(R) & & \\
 & & \downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & \\
0 & \longrightarrow & F(R^G, R) & \longrightarrow & E(R^G, R) & \longrightarrow & \mathrm{Cl}(R) & & \\
 & & \downarrow & & {\scriptstyle \Phi_{R^G,R}}\downarrow & & & & \\
 & & W(R^G, R) & \longrightarrow & \mathrm{Cl}(R^G) & \longrightarrow & Y(R^G, R) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $F(R^G, R)$ can be regarded as a subgroup of the first cohomology group $H^1(G, \mathrm{U}(R))$.

In this section we apply the results in Sect. 2 to the generic Krull pair induced by the action of an algebraic torus defined over an algebraically closed field K of characteristic zero. Let $\mathfrak{X}(H)$ be the rational character (additive) group of an algebraic group H.

Notation 3.1. Let G be an affine algebraic group over K whose identity component G^0 is an algebraic torus and let (X, G) be a *faithful* regular action of G on an affine normal variety X over K. Put $R := \mathcal{O}(X)$ on which G acts naturally.

Recall that (X, G) is said to be stable, if X contains a non-empty open set consisting of closed G-orbits.

Definition 3.2. For $\{f_1, \ldots, f_n\} \subseteq R$ such that f_i are prime in R; the set $\{f_1, \ldots, f_n\}$ is defined to be (R, G)-*free*, if there exist rational characters $\chi_k \in \mathfrak{X}(G)$ $(1 \leq k \leq n)$ such that

$$R_{\sum_{k=1}^n i_k \cdot \chi_k} = R^G \cdot \prod_{k=1}^n f_k^{i_k} \quad (\forall i_k \in \boldsymbol{Z}_0)$$

where $\boldsymbol{Z}_0$ denote the additive monoid of all nonnegative integers.

As in the statement preceding to the ladder property in Sect. 2, from Theorem 2.2, Theorem 2.3 and Theorem 2.8 we deduce the following characterization of (R, G)-freeness of prime relative invariants on X in the sense of the descent property defined in this paper:

Theorem 3.3. Under the circumstances as in Notation 3.1, suppose that $Z_G(G^0) = G$, H is a closed normal subgroup such that the induced action $(X/\!\!/H, G/H)$ is stable. Suppose that one can choose prime semi-invariants f_i $(1 \leqq i \leqq n)$ of G on R in such a way that $H = \cap_{i=1}^n G_{f_i}$. If $\mathrm{rank}(G/H) = n$, then the following conditions are equivalent:

(i) The generic Krull pair (R^G, R^H) has the (TDP).

(ii) There exists a finite normal subgroup N of G generated by a part of the union of inertia groups at principal ideals in $\mathrm{Ht}_1(R^G, R)$ under the action of G such that there exists an (R, G)-free prime set $\{g_1, \ldots, g_n\}$ contained in R^N satisfying $HN = \cap_{i=1}^n G_{g_i}$.

Remark 3.4. The equivalence in Theorem 3.3 does not hold without assumption that $\{f_1, \dots, f_n\}$ consists of prime elements. There are counter-examples for a set $\{f_1, \dots, f_n\}$ containing a non-prime element. One might generalize this in the case where f_i's may not be prime, although the conditions should be complicated.

4. Subgroups of class groups and modules of relative invariants

We now return to the general case where R is a Krull domain acted by a group G as automorphisms which is treated in the former half in Sect. 3 and introduce some subsets of the group of the 1-cocyles of G. From now on to the end of Proposition 4.12, without specifying we suppose that the equality $\mathcal{Q}(R^G) = \mathcal{Q}(R)^G$ holds.

Definition 4.1. Put $Z^1(G, \mathrm{U}(R))^R := \{\chi \in Z^1(G, \mathrm{U}(R)) \mid R_\chi \neq \{0\}\}$ and

$$Z^1_R(G, \mathrm{U}(R))_{\mathrm{e}} := \{\chi \in Z^1(G, \mathrm{U}(R))^R \mid \exists f_{\mathrm{P}} \in R_\chi \backslash \{0\} \text{ such that } \mathrm{v}_{R,\mathrm{P}}(f_{\mathrm{P}}) \equiv 0 \bmod (\mathrm{e}(\mathfrak{P}, \mathfrak{P} \cap \mathfrak{q}))\ (\forall \mathfrak{P} \in \mathrm{Ht}_1(R^G, R))\}.$$

Let $Z^1_R(G, \mathrm{U}(R))_{(2)}$ denote the set of all $\chi \in Z^1(G, \mathrm{U}(R))$ such that $\{0\} \neq R_{-\chi} \not\subseteq \mathfrak{P}$ for all $\mathfrak{P} \in \mathrm{Ht}_1(R)$ satisfying $\mathrm{ht}(\mathfrak{P} \cap R^G) \geqq 2$ and put

$$\widetilde{Z^1_R}(G, \mathrm{U}(R)) := Z^1_R(G, \mathrm{U}(R))_{(2)} \cap (-Z^1_R(G, \mathrm{U}(R))_{(2)}).$$

Definition 4.2. An effective divisor $D \in \mathrm{Div}(R)$ is said to be *minimal effective relative to* (R^G, R), if D has a decomposition $D = D_1 + D_2$ for $0 \leqq D_1 \in E^*(R^G, R)$ and $0 \leqq D_2 \in \mathrm{Div}(R)$, then the divisor D_1 must be equal to zero.

With each $\chi \in Z^1_R(G, \mathrm{U}(R))^R$ we can associate the divisor $D(\chi)$ minimal effective relative to (R^G, R) as follows:

Lemma 4.3. Let χ be a cocycle in $Z^1_R(G, \mathrm{U}(R))^R$. Then:

(i) There exists a unique minimal effective divisor $D(\chi)$ on R relative to (R^G, R) such that, for a nonzero element $f \in R_\chi$,

$$E^*(R^G, R) \ni \mathrm{div}_R(f) - D(\chi) \geqq 0.$$

Moreover $D(\chi)$ does not depend on the choice of a nonzero element $f \in R_\chi$.

(ii) If $\chi \in Z^1_R(G, \mathrm{U}(R))_{\mathrm{e}}$, then the divisor $D(\chi)$ and $D(m\chi)$ defined in (i) for χ and $m\chi$ satisfy $m \cdot D(\chi) = D(m\chi)$ in $\mathrm{Div}(R)$ for any $m \in \boldsymbol{N}$.

The next criterion for the individual R^G-module R_χ to be R^G-free can be easily shown in [7].

Proposition 4.4 ([7])**.** Without the assumption that $\mathcal{Q}(R^G) = \mathcal{Q}(R)^G$, for any cocycle $\chi \in Z^1(G, \mathrm{U}(R))^R$, R_χ is R^G-free of rank one if and only if the following conditions are satisfied:

(i) $\dim \mathcal{Q}(R^G) \otimes_{R^G} R_\chi = 1$.

(ii) There exists a nonzero element $f \in R_\chi$ satisfying

$$\forall \mathfrak{q} \in \mathrm{Ht}_1(R^G) \ \Rightarrow\ \exists \mathfrak{P} \in X_{\mathcal{Q}}(R) \text{ such that } \mathrm{v}_{R,\mathrm{P}}(f) < \mathrm{e}(\mathfrak{P}, \mathfrak{q}) \tag{4.1}$$

If these equivalent conditions are satisfied, $R_\chi = R^G \cdot f$ for any nonzero element $f \in R_\chi$ such that (4.1) holds.

We apply Proposition 4.4 to some restricted χ and obtain the corollary which shall be needed.

Corollary 4.5. Let χ be a cocycle in $Z^1(G, \mathrm{U}(R))^R$. Then R_χ is R^G-free if and only if $D(\chi) + \mathrm{BU}(R^G, R) \ni \mathrm{div}_R(f)$ for some nonzero $f \in R_\chi$. In the case where $\chi \in (-Z_R^1(G, \mathrm{U}(R))_{(2)})$, $R_\chi \cong R^G$ as R^G-modules if and only if $D(\chi) = \mathrm{div}_R(f)$ for some nonzero $f \in R_\chi$.

Moreover the equality $R_\chi = R^G \cdot f$ holds, in both the cases where these equivalent conditions are satisfied.

By the choice of χ, Lemma 4.3 and Corollary 4.5, we see

Proposition 4.6. Let $\chi \in Z_R^1(G, \mathrm{U}(R))_{\mathrm{e}} \cap (-Z_R^1(G, \mathrm{U}(R))_{(2)})$. Suppose that there exists a nonzero element $g \in R_\chi$ satisfying the condition as follows; for any $l \in \boldsymbol{N}$ and G-invariant principal ideal $R \cdot h$ in R containing g^l such that $\mathrm{div}_R(h) \in E^*(R^G, R)$,

$$\exists n \in \boldsymbol{N} \text{ such that } (h^n \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset \Rightarrow (h \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset.$$

Then the following conditions are equivalent:

(i) $D(\chi)$ is a principal divisor and there exists a number $m \in \boldsymbol{N}$ such that $R_{m\chi} \cong R^G$ as R^G-modules.

(ii) For any $m \in \boldsymbol{N}$, $R_{m\chi} \cong R^G$ as R^G-modules.

(iii) $R_\chi \cong R^G$ as R^G-modules.

Corollary 4.7. Under the same circumstances as in Proposition 4.6, suppose that there is a number $m \in \boldsymbol{N}$ satisfying $R_{m\chi} \cong R^G$ as R^G-modules. Then the divisor class $[D(\chi)]$ in $\mathrm{Cl}(R)$ has a finite order and the following equality holds;

$$\mathrm{ord}([D(\chi)]) = \min\{q \in \boldsymbol{N} \mid R_{q\chi} \cong R^G \text{ as } R^G\text{-modules}\}.$$

Proof. Since $p \cdot [D(\chi)] = [D(p\chi)]$ in $\mathrm{Cl}(R)$ for any $p \in \boldsymbol{N}$ as in Proposition 4.6 and $n \cdot [D(\chi)] = 0$ (cf. Corollary 4.5), the former assertion is obvious and the latter one follows from Proposition 4.6. □

Definition 4.8. For $\chi \in Z^1(G, \mathrm{U}(R))^R$, the R^G-module R_χ is R^G-isomorphic to a nonzero integral ideal I of R^G and the divisor class of the divisorialization $\tilde{I}$ in $\mathrm{Cl}(R^G)$ is denoted to $[R_\chi]$.

Proposition 4.9. Let χ be a cocycle in $Z_R^1(G, \mathrm{U}(R))_{\mathrm{e}} \cap Z_R^1(G, \mathrm{U}(R))_{(2)}$. Then, for a number $n \in \boldsymbol{N}$, $R_{n\chi} \cong R^G$ as R^G-modules if and only if $n \cdot [R_\chi] = 0$ in $\mathrm{Cl}(R^G)$.

Combining Corollary 4.7 with Proposition 4.9, we immediately have

Theorem 4.10. Let χ be a cocycle in $Z_R^1(G, \mathrm{U}(R))_{\mathrm{e}} \cap \widetilde{Z_R^1}(G, \mathrm{U}(R))$ and suppose that there exists a nonzero element $g \in R_\chi$ satisfying the condition as follows; for any $l \in \boldsymbol{N}$ and G-invariant principal ideal $R \cdot h$ in R containing g^l such that $\mathrm{div}_R(h) \in E^*(G, R)$,

$$\exists n \in \boldsymbol{N} \text{ such that } (h^n \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset \Rightarrow (h \cdot \mathrm{U}(R)) \cap R^G \neq \emptyset.$$

If $[R_\chi] \in \mathrm{tor}(\mathrm{Cl}(R^G))$, then

$$\mathrm{ord}([R_\chi]) \text{ in } \mathrm{Cl}(R^G) = \mathrm{ord}([D(\chi)]) \text{ in } \mathrm{Cl}(R),$$

which is equal to $\min\{q \in \boldsymbol{N} \mid R_{q\chi} \cong R^G \text{ as } R^G\text{-modules}\}$. □

Definition 4.11. Let $\mathrm{UrCl}(R,G)$ denote the subgroup of $\mathrm{CL}(R)$ generated by

$$\{[D(\chi)] \mid \chi \in Z^1_R(G,\mathrm{U}(R))_{\mathrm{e}} \cap \widetilde{Z^1_R}(G,\mathrm{U}(R))\},$$

where $[D(\chi)]$ denotes the divisor class of $D(\chi) \in \mathrm{Div}(R)$. Define $\widetilde{\mathrm{Cl}}(R,G)$ to be the subgroup $\left\langle \{[R_\chi] \mid \chi \in Z^1_R(G,\mathrm{U}(R))_{\mathrm{e}} \cap \widetilde{Z^1_R}(G,\mathrm{U}(R))\}\right\rangle$ of $\mathrm{Cl}(R^G)$.

The next result follows easily from Theorem 4.10.

Proposition 4.12. Suppose that the canonical image of the semigroup $Z^1_R(G,\mathrm{U}(R))_{\mathrm{e}} \cap (-(Z^1_R(G,\mathrm{U}(R))_{(2)})$ in $H^1(G,\mathrm{U}(R))$ does not contain a non-trivial torsion element. Suppose that $\widetilde{\mathrm{Cl}}(R,G)$ is a torsion group. If one of $\exp(\mathrm{UrCl}(R,G))$ and $\exp(\widetilde{\mathrm{Cl}}(R,G))$ is finite, then

$$\exp(\mathrm{UrCl}(R,G)) = \exp(\widetilde{\mathrm{Cl}}(R,G)),$$

which are equal to

$$\max\left\{\min\{q \in \boldsymbol{N} \mid R_{q\chi} \cong R^G\} \;\middle|\; \chi \in Z^1_R(G,\mathrm{U}(R))_{\mathrm{e}} \cap \widetilde{Z^1_R}(G,\mathrm{U}(R))\right\}. \quad \square$$

Hereafter let (X,G) be a regular faithful stable action of an algebraic torus G on an affine normal variety X defined over an algebraically closed field K of characteristic zero whose coordinate ring $\mathcal{O}(X)$ denoted to R. We have the canonical pairing $G \times \mathfrak{X}(G) \to \mathrm{U}(K)$.

Definition 4.13. Let $\widetilde{\mathfrak{R}}(R,G)$ be the subgroup of G generated by the set consisting of $\mathcal{I}_{\mathrm{P}}(G)$'s for all $\mathfrak{P} \in \mathrm{Ht}_1(R^G,R)$ such that $\mathfrak{P}$ are not principal which is called the *maximal non-principal pseudo-reflection subgroups of the action* (X,G). Here $\mathcal{I}_{\mathrm{P}}(G)$ stands for the inertia group of $\mathfrak{P}$ under the action of G. Put $\mathfrak{R}(R,G) := \langle \cup_{\mathrm{P}\in \mathrm{Ht}_1(R^G,R)} \mathcal{I}_{\mathrm{P}}(G)\rangle$. Clearly both $\mathfrak{R}(R,G)$ and $\widetilde{\mathfrak{R}}(R,G)$ are finite (normal) subgroups of G. In the case where $\exp(\mathrm{UrCl}(R,G))$ is finite, define

$$\mathrm{Obs}(R,G) := \{\sigma \in G \mid \sigma^{\exp(\mathrm{UrCl}(R,G))} \in \widetilde{\mathfrak{R}}(R,G)\},$$

which is called the *obstruction subgroup for cofreeness of* (X,G).

Lemma 4.14. We have $\mathfrak{X}(G)^{\perp \mathrm{R}(R,G)} = Z^1_R(G,\mathrm{U}(R))_{\mathrm{e}}$.

With the aid of [10], the following proposition is shown in [6].

Proposition 4.15. Suppose that both X and (X,G) are conical. If the action (X,G) is equidimensional, then $\widetilde{\mathrm{Cl}}(R,G)$ is a torsion group.

Applying Theorem 4.10 to this, we must have

Theorem 4.16. Suppose that both X and (X,G) are conical. Then the following conditions are equivalent:

(i) The action (X,G) is equidimensional.
(ii) The exponent $\exp(\mathrm{UrCl}(R,G))$ is finite and the action $(X/\!/\mathrm{Obs}(R,G), G/\mathrm{Obs}(R,G))$ induced naturally is cofree.

Especially if R is factorial, the obstruction subgroup $\mathrm{Obs}(R, G)$ should be a trivial group by its definition. It is not hard to formally generalize Theorem 4.16 to in the case where (X, G) may not be stable. For linear representations of connected algebraic groups with affine rings of invariants, V. G. Kac and V. L. Popov have conjectured that equidimensionality of these actions implies cofreeness, which is known as the Russian conjecture (cf. [3, 9]) and is partially related to this theorem.

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