DERIVED EQUIVALENCES FOR ENDOMORPHISM RINGS

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Abstract. We provide derived equivalences for endomorphism rings associated with a certain exact sequences.

1. Notation

For a ring A, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of Mod-A consisting of finitely generated projective modules. For $M \in$ Mod-A, we denote by proj dim M_A (resp., inj dim M_A) the projective (resp., injective) dimension of M, where we use the notation M_A to stress that the module M considered is a right A-module, and by $\Omega^n M$ the nth syzygy of M. For a ring A, we denote by gl dim A the global dimension of A. For an object X in an additive category $\mathcal{B}$, we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{B}$ whose objects are direct summands of finite direct sums of copies of X.

2. Main result

In [1], we have shown the following.

Theorem 1 ([1, Lemma 1.1]). *Let $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ be an exact sequence in an abelian category $\mathcal{A}$ and P an object of $\mathcal{A}$. Assume that $E \in \mathrm{add}(P)$ and that both $\mathrm{Hom}_{\mathcal{A}}(P, \varepsilon)$ and $\mathrm{Hom}_{\mathcal{A}}(\mu, P)$ are epic. Then $\mathrm{End}_{\mathcal{A}}(X \oplus P)$ and $\mathrm{End}_{\mathcal{A}}(Y \oplus P)$ are derived equivalent to each other.*

The next two propositions are direct consequences of Theorem 1.

Proposition 2. *Let A be a right noetherian ring and $M \in$ mod-A. If $\mathrm{Ext}^i_A(M, A) = 0$ for $1 \le i \le n$, then $\mathrm{End}_A(M \oplus A)$ and $\mathrm{End}_A(\Omega^n M \oplus A)$ are derived equivalent to each other.*

Proposition 3. *Let A be an Artin algebra, $P \in$ mod-A and $0 \to Y \to E \to X \to 0$ an almost split sequence in mod-A. If $E \in \mathrm{add}(P)$ and $X, Y \notin \mathrm{add}(P)$, then $\mathrm{End}_A(X \oplus P)$ and $\mathrm{End}_A(Y \oplus P)$ are derived equivalent to each other.*

The propositions above enable us to construct many derived equivalences between endomorphism rings. For example, we obtain the following.

Example 4. Let k be a field, $R = k[X_1, \cdots, X_n]/\langle X_i^2 - X_j^2, X_i X_j \mid 1 \le i \ne j \le n\rangle$ with $n \ge 2$ and S the simple R-module. Then the following hold.

The detailed version of this paper will be submitted for publication elsewhere.

(1) $\mathrm{End}_R(\Omega^{-i-1}S \oplus \Omega^{-i}S)$ and $\mathrm{End}_R(\Omega^{-1}S \oplus S)$ are derived equivalent to each other for all $i \geq 1$, where the first algebra has global dimension three and the last algebra has global dimension two.
(2) $\mathrm{End}_R(\Omega^{i}S \oplus \Omega^{i+1}S)$ and $\mathrm{End}_R(S \oplus \Omega S)$ are derived equivalent to each other for all $i \geq 1$, where the first algebra has global dimension three and the last algebra has global dimension two.
(3) $\mathrm{End}_R(\Omega^{i}S \oplus \Omega^{i+1}S \oplus R)$ and $\mathrm{End}_R(S \oplus \Omega S \oplus R)$ are derived equivalent to each other for all $i \in \mathbb{Z}$, where these algebras have global dimension three.
(4) $\mathrm{End}_R(\Omega^{i}S \oplus \Omega^{i+1}S)$ is isomorphic to a trivial extension of $\begin{pmatrix} k & k^n \\ 0 & k \end{pmatrix}$ for all $i \in \mathbb{Z}$.

3. Auslander algebra

In this section, we apply the results of the previous section to Auslander algebras. We start by recalling the definition of Auslander algebras (see e.g. [3] for details).

Definition 5. Let Λ be an Artin algebra and $0 \to \Lambda \to I^0 \to I^1 \to \cdots$ a minimal injective resolution in mod-Λ. Set dom dim $\Lambda = \sup\{k \in \mathbb{Z} \mid I^i \in \mathcal{P}_\Lambda \text{ for } 0 \leq i \leq k-1\}$, which is called the dominant dimension of Λ. Then Λ is said to be an Auslander algebra provided gl dim $\Lambda \leq 2$ and dom dim $\Lambda \geq 2$.

Let A be a representation-finite Artin algerba and assume that A is basic and connected. Let $M_1, \cdots, M_m$ be a complete set of nonisomorphic indecomposable modules in mod-A and set $I = \{1, \cdots, m\}$. We assume that $m \geq 2$, i.e., A is not simple. Then, setting $M = \bigoplus_{i \in I} M_i$, we have an Auslander algebra $\Lambda = \mathrm{End}_A(M)$, which will be called the Auslander algebra of A. For each indecomposable module $X \in$ mod-A, since there exists a unique $i_X \in I$ such that $X \cong M_{i_X}$, we set $I(X) = I \setminus \{i_X\}$, $M_X = \bigoplus_{i \in I(X)} M_i$ and $\Lambda_X = \mathrm{End}_A(M_X)$. Then by Proposition 3 we have the following.

Proposition 6. *The following hold.*
(1) *If X is not projective then Λ_X is derived equivalent to $\Lambda_{\tau X}$, where τ denotes the Auslander-Reiten translation.*
(2) *If X is not injective then Λ_X is derived equivalent to $\Lambda_{\tau^{-1}X}$.*

We can calculate the global dimension and the dominant dimension of Λ_X.

Lemma 7. *Assume that X is not projective, not injective and $\tau X \cong X$. Then A is a local Nakayama algebra and the following hold.*
(1) *If $m = 2$, then $\Lambda_X \cong A$ as algebras.*
(2) *If $m > 2$, then* inj dim $\Lambda_X = 2$.

Proposition 8. *The following hold.*
(1) *If X is projective (resp., injective), then* gl dim $\Lambda_X \leq 2$.
(2) *If X is not projective, not injective and $\tau X \ncong X$, then* gl dim $\Lambda_X = 3$.
(3) *If X is not projective, not injective and $\tau X \cong X$, then* gl dim $\Lambda_X = \infty$.

Proposition 9. *The following hold.*
(1) *If X is projective (resp., injective), not injective (resp., not projective) and not simple, then* dom dim $\Lambda_X = 0$.

(2) *If X is projective (resp., injective), not injective (resp., not projective) and simple, then* dom dim $\Lambda_X = 1$.
(3) *If X is projective and injective, then* dom dim $\Lambda_X \geq 2$.
(4) *If X is not projective and not injective, then* dom dim $\Lambda_X \geq 2$.

It follows by the propositions above that Λ_X is an Auslander algebra if and only if X is projective and injective.

Consider next the case where X is a simple projective module with inj dim $X_A = 1$. Let $P_1, \cdots, P_n = X$ be a complete set of nonisomorphic indecomposable modules in $\mathcal{P}_A$ and set $T = (\bigoplus_{i=1}^{n-1} P_i) \oplus \tau^{-1}X$. Then T is a classical tilting module, i.e., a tilting module of projective dimension ≤ 1 (cf. [2]). Set $B = \mathrm{End}_A(T)$ and $Y = \mathrm{Ext}^1_A(T, X) \in$ mod-B. Then Y is a simple injective module with proj dim $Y_B = 1$. We set $N_Y = \mathrm{Hom}_A(T, M_X)$, $N = N_Y \oplus Y$, $\Gamma = \mathrm{End}_B(N)$ and $\Gamma_Y = \mathrm{End}_B(N_Y)$. Note that Γ is the Auslander algebra of B.

Proposition 10. *We have $\Gamma_Y \cong \Lambda_X$ as algebras and hence for any $i, j \geq 0$, if $\tau^i Y, \tau^{-j} X$ are nonzero, $\Gamma_{\tau^i Y}$ and $\Lambda_{\tau^{-j} X}$ are derived equivalent to Λ_X.*

Remark 11. Set $\tilde{T} = \mathrm{Hom}_A(M, M_X) \oplus \mathrm{Ext}^1_A(M, X) \in$ mod-Λ. Then the following hold.
(1) $\mathrm{End}_\Lambda(\tilde{T}) \cong \Gamma$ as algebras.
(2) proj dim $\tilde{T}_\Lambda = 2$.
(3) there exists an exact sequence $0 \to \Lambda \to T^0 \to T^1 \to T^2 \to 0$ in mod-Λ with the $T^i \in \mathrm{add}(\tilde{T})$.
(4) $\mathrm{Ext}^1_\Lambda(\tilde{T}, \tilde{T}) = 0$.
(5) $\mathrm{Ext}^2_\Lambda(\tilde{T}, \tilde{T}) = 0$ if and only if $A \cong \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ with $D = \mathrm{End}_A(X)$.

4. Tilting module

Finally, we point out that the exact sequence in Theorem 1 enables us to construct another tilting module from a given tilting module by exchanging direct summands.

Proposition 12. *Let A be a ring, $P \in$ Mod-A and $0 \to Y \xrightarrow{\mu} E \xrightarrow{\varepsilon} X \to 0$ an exact sequence in* Mod-A. *Assume that $E \in \mathrm{add}(P)$ and that both $\mathrm{Hom}_A(P, \varepsilon)$ and $\mathrm{Hom}_A(\mu, P)$ are epic. Then $X \oplus P$ is a tilting module if and only if so is $Y \oplus P$. In particular, if $X \oplus P$ is a classical tilting module, then so is $Y \oplus P$.*

Corollary 13. *Let A be a Noether algebra and $X \in$ mod-A. Assume that there exists $T \in$ mod-A such that $X \oplus T$ is a tilting module. Then the following hold.*
(1) *If there exists an epimorphism of the form $f : T^{(l)} \to X$, then there exists an epimorphism $\varepsilon : T^{(r)} \to X$ such that* Ker $\varepsilon \oplus T$ *is a tilting module. In particular, if $X \oplus T$ is a classical tilting module, then so is* Ker $\varepsilon \oplus T$.
(2) *If there exists a monomorphism of the form $g : X \to T^{(l)}$, then there exists a monomorphism $\mu : X \to T^{(r)}$ such that* Cok $\mu \oplus T$ *is a tilting module.*

References

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