# THE BUCHSBAUM-RIM FUNCTION OF A PARAMETER MODULE

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Abstract. This note is basically a summary of a part of the paper [11] with Eero Hyry (University of Tampere). In this note we prove that the Buchsbaum-Rim function $\ell_A(\mathcal{S}_{\nu+1}(F)/N^{\nu+1})$ of a parameter module $N$ in $F$ is bounded above by $e(F/N)\binom{\nu+d+r-1}{d+r-1}$ for every integer $\nu \geq 0$. Moreover, it turns out that the base ring $A$ is Cohen-Macaulay once the equality holds for some integer $\nu$. As a direct consequence, we observe that the first Buchsbaum-Rim coefficient $e_1(F/N)$ of a parameter module $N$ is always non-positive.

## 1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d$. Let $F = A^r$ be a free module of rank $r > 0$, and let $S = \mathcal{S}_A(F)$ be the symmetric algebra of $F$, which is a polynomial ring over $A$. For a submodule $M$ of $F$, let $\mathcal{R}(M)$ denote the image of the natural homomorphism $\mathcal{S}_A(M) \to \mathcal{S}_A(F)$, which is a standard graded subalgebra of $S$. Assume that the quotient $F/M$ has finite length and $M \subseteq \mathfrak{m}F$. Then we can consider the function

$$\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \ ; \quad \nu \mapsto \ell_A(S_\nu/M^\nu)$$

where $S_\nu$ and $M^\nu$ denote the homogeneous components of degree $\nu$ of $S$ and $\mathcal{R}(M)$, respectively. Buchsbaum and Rim studied this function in [4] in order to generalize the notion of the usual Hilbert-Samuel multiplicity of an $\mathfrak{m}$-primary ideal. They proved that $\lambda(\nu)$ eventually coincides with a polynomial $P(\nu)$ of degree $d + r - 1$. This polynomial can then be written in the form

$$P(\nu) = \sum_{i=0}^{d+r-1} (-1)^i e_i(F/M) \binom{\nu + d + r - 2 - i}{d + r - 1 - i}$$

with integer coefficients $e_i(F/M)$. The coefficients $e_i(F/M)$ are called the *Buchsbaum-Rim coefficients* of $F/M$. The *Buchsbaum-Rim multiplicity of* $F/M$, denoted by $e(F/M)$, is now defined to be the leading coefficient $e_0(F/M)$.

In their article Buchsbaum and Rim also introduced the notion of a parameter module (matrix), which generalizes the notion of a parameter ideal (system of parameters). The module $N$ in $F$ is said to be *a parameter module in* $F$, if the following three conditions are satisfied: (i) $F/N$ has finite length, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d + r - 1$, where $\mu_A(N)$ is the minimal number of generators of $N$.

A starting point of this note is the characterization of the Cohen-Macaulay property of $A$ given in [4, Corollary 4.5] by means of the equality $\ell_A(F/N) = e(F/N)$ for every

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parameter module $N$ of rank $r$ in $F = A^r$. Brennan, Ulrich and Vasconcelos observed in [1, Theorem 3.4] that if $A$ is Cohen-Macaulay, then in fact

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

for all integers $\nu \geq 0$. Our main result is now as follows:

**Theorem 1.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$.*

(1) *For any rank $r > 0$, the inequality*

$$\ell_A(S_{\nu+1}/N^{\nu+1}) \geq e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

*always holds true for every parameter module $N$ in $F = A^r$ and for every integer $\nu \geq 0$.*

(2) *The following statements are equivalent:*

(*i*) *$A$ is a Cohen-Macaulay local ring;*

(*ii*) *There exists an integer $r > 0$ and a parameter module $N$ of rank $r$ in $F = A^r$ such that the equality*

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N)\binom{\nu+d+r-1}{d+r-1}$$

*holds true for some integer $\nu \geq 0$.*

This generalizes our previous result [10, Theorem 1.3] where we assumed that $\nu = 0$. The equivalence of (i) and (ii) in (2) seems to contain some new information even in the ideal case. Indeed, it improves a recent observation that the ring $A$ is Cohen-Macaulay if there exists a parameter ideal $Q$ in $A$ such that $\ell_A(A/Q^{\nu+1}) = e(A/Q)\binom{\nu+d}{d}$ for all $\nu \gg 0$ (see [8, 12]). Moreover, as a direct consequence of (1), we have the non-positivity of the first Buchsbaum-Rim coefficient of a parameter module.

**Corollary 2.** *For any rank $r > 0$, the inequality*

$$e_1(F/N) \leq 0$$

*always holds true for every parameter module $N$ in $F = A^r$.*

Mandal and Verma have recently proved that $e_1(A/Q) \leq 0$ for any parameter ideal $Q$ in $A$ (see [15], and also [8]). Corollary 2 can be viewed as the module version of this fact. However, our proof based on the inequality in Theorem 1 (1) is completely different from theirs and is considerably more simpler.

## 2. Preliminaries

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension $d$. Let $F = A^r$ be a free module of rank $r > 0$. Let $S = \mathcal{S}_A(F)$ be the symmetric algebra of $F$. Let $N$ be a parameter module in $F$, that is, $N$ is a submodule of $F$ satisfying the conditions: (i) $\ell_A(F/N) < \infty$, (ii) $N \subseteq \mathfrak{m}F$, and (iii) $\mu_A(N) = d+r-1$. We put $n = d+r-1$. Let $N^\nu$ be the homogeneous component of degree $\nu$ of the standard graded subalgebra $\mathcal{R}(N) = \mathrm{Im}(\mathcal{S}_A(N) \to S)$ of $S$. Let $\tilde{N} = (c_{ij})$ be the matrix associated to a minimal free presentation

$$A^n \xrightarrow{\tilde{N}} F \to F/N \to 0$$

of $F/N$. Let $X = (X_{ij})$ be a generic matrix of the same size $r \times n$. We denote by $I_s(X)$ the ideal in the polynomial ring $A[X] = A[X_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n]$ generated by the $s$-minors of $X$. Let $B = A[X]_{(\mathfrak{m},X)}$ be the ring localized at the graded maximal ideal $(\mathfrak{m}, X)$ of $A[X]$. The substitution map $A[X] \to A$ where $X_{ij} \mapsto c_{ij}$ now induces a map $\varphi : B \to A$. We consider the ring $A$ as a $B$-algebra via the map $\varphi$. Let

$$\mathfrak{b} = \mathrm{Ker}\, \varphi = (X_{ij} - c_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n)B.$$

Set $G = B^r$, and let $L$ denote the submodule $\mathrm{Im}(B^n \xrightarrow{X} G)$ of $G$. Let $G_\nu$ and $L^\nu$ be the homogeneous components of degree $\nu$ of the graded algebras $\mathcal{S}_B(G)$ and $\mathcal{R}(L)$, respectively. Then one can check the following.

**Lemma 3.** *For any integers $\nu \geq 0$, we have the following:*

(1) $(G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b}) \cong S_{\nu+1}/N^{\nu+1}$;
(2) $\mathrm{Supp}_B(G_{\nu+1}/L^{\nu+1}) = \mathrm{Supp}_B(B/I_r(X)B)$;
(3) *The ideal $\mathfrak{b}$ is generated by a system of parameters of the module $G_{\nu+1}/L^{\nu+1}$.*

The following fact concerning $G_{\nu+1}/L^{\nu+1}$ is known by [3, Corollary 3.2] (see also [13, Proposition 3.3]).

**Lemma 4.** *For any integer $\nu \geq 0$, we have $G_{\nu+1}/L^{\nu+1}$ is a perfect $B$-module of grade $d$.*

The following plays a key role in the proof of Theorem 1. See [11, Proposition 2.4] for the proof.

**Proposition 5.** *For any $\mathfrak{p} \in \mathrm{Min}_B(B/I_r(X)B)$, the equality*

$$\ell_{B_\mathfrak{p}}\left((G_{\nu+1}/L^{\nu+1})_\mathfrak{p}\right) = \ell_{B_\mathfrak{p}}\left((B/I_r(X)B)_\mathfrak{p}\right)\binom{\nu+d+r-1}{d+r-1}$$

*holds true for all integers $\nu \geq 0$.*

## 3. Proof of Theorem 1

In order to prove Theorem 1, we need to introduce more notation. For any matrix $\mathfrak{a}$ of size $r \times n$ over an arbitrary ring, we denote by $K_\bullet(\mathfrak{a})$ its Eagon-Northcott complex [6]. When $r = 1$, the complex $K_\bullet(\mathfrak{a})$ is just the ordinary Koszul complex of the sequence $\mathfrak{a}$. See [7, Appendix A2] for the definition and more details of complexes of this type. Recall in particular that if $N$ is a parameter module in a free module $F$ as in section 2, then

$$e(F/N) = \chi(K_\bullet(\tilde{N})),$$

where $\chi(K_\bullet(\tilde{N}))$ denotes the Euler-Poincaré characteristic of the complex $K_\bullet(\tilde{N})$ (see [4] and [14]). Moreover, one can check the following by computing $\mathrm{Tor}_p^B(B/IB, A)$ for any $p \geq 0$ (see [5]).

**Lemma 6.** *Using the setting and notation of section 2, we have*

$$\chi(K_\bullet(\mathfrak{b}) \otimes_B (B/I_r(X)B)) = \chi(K_\bullet(\tilde{N})).$$

Now we can give the proof of Theorem 1.

*Proof of Theorem 1.* We use the same notation as in section 2. Put $I = I_r(X)$.

(1): Fix integers $\nu \geq 0$. The ideal $\mathfrak{b}$ being generated by a system of parameters of the module $G_{\nu+1}/L^{\nu+1}$, we get

$$\begin{aligned}
& \ell_A(S_{\nu+1}/N^{\nu+1}) \\
= \; & \ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b})) \\
\geq \; & e(\mathfrak{b}; G_{\nu+1}/L^{\nu+1}) \\
= \; & \sum_{\mathfrak{p} \in \operatorname{Assh}_B(G_{\nu+1}/L^{\nu+1})} e(\mathfrak{b}; B/\mathfrak{p}) \cdot \ell_{B_\mathfrak{p}}((G_{\nu+1}/L^{\nu+1})_\mathfrak{p}) \\
= \; & \sum_{\mathfrak{p} \in \operatorname{Assh}_B(B/IB)} e(\mathfrak{b}; B/\mathfrak{p}) \cdot \ell_{B_\mathfrak{p}}((B/IB)_\mathfrak{p}) \binom{\nu+d+r-1}{d+r-1} \\
= \; & e(\mathfrak{b}; B/IB) \binom{\nu+d+r-1}{d+r-1} \\
= \; & \chi(K_\bullet(\mathfrak{b}) \otimes_B (B/IB)) \binom{\nu+d+r-1}{d+r-1} \\
= \; & \chi(K_\bullet(\tilde{N})) \binom{\nu+d+r-1}{d+r-1} \\
= \; & e(F/N) \binom{\nu+d+r-1}{d+r-1}
\end{aligned}$$

as desired, where $e(\mathfrak{b}; *)$ denotes the multiplicity of $*$ with respect to $\mathfrak{b}$.

(2): The other implication being clear, by the ideal case, for example, it is enough to show that (ii) implies (i). Assume thus that

$$\ell_A(S_{\nu+1}/N^{\nu+1}) = e(F/N) \binom{\nu+d+r-1}{d+r-1}$$

for some $\nu \geq 0$. The above argument then gives

$$\ell_B((G_{\nu+1}/L^{\nu+1}) \otimes_B (B/\mathfrak{b})) = e(\mathfrak{b}; G_{\nu+1}/L^{\nu+1}).$$

It follows that $G_{\nu+1}/L^{\nu+1}$ is a Cohen-Macaulay $B$-module of dimension $rn$ ([2, (5.12) Corollary]). By Lemma 4, $G_{\nu+1}/L^{\nu+1}$ is a perfect $B$-module of grade $d$. Thus, by the Auslander-Buchsbaum formula,

$$\begin{aligned}
\operatorname{depth} B &= \operatorname{depth}_B(G_{\nu+1}/L^{\nu+1}) + \operatorname{pd}_B(G_{\nu+1}/L^{\nu+1}) \\
&= \dim_B(G_{\nu+1}/L^{\nu+1}) + \operatorname{grade}_B(G_{\nu+1}/L^{\nu+1}) \\
&= rn + d \\
&= \dim B.
\end{aligned}$$

Therefore $B$ is Cohen-Macaulay so that $A$ is Cohen-Macaulay, too. □

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