REPRESENTATION RINGS OF STRING ALGEBRAS

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Abstract. String algebras are a class of algebras given by certain quivers with monomial relations. Thus the category of finite dimensional left modules over a string algebra is equipped with a tensor product defined point-wise and arrow-wise on the level of quiver representations. We describe the corresponding representation ring for any string algebra.

1. Introduction

The category of finite dimensional representations of a group G is equipped with a tensor product defined by diagonal action. Thus the set of isoclasses of such representations has the structure of a semi-ring, where addition is given by the direct sum and multiplication by the tensor product. From this semi-ring one constructs the representation ring $R(G)$ by including formal additive inverses.

It would be interesting to generalise this procedure to the category of left modules over an associative algebra A instead of group representations. However, in general there is no know way of defining a tensor product on this category. Now assume that A is given as the path algebra of a quiver Q with monomial relations, i.e. $A = kQ/\langle X\rangle$ for some set X of paths in Q and a field k. Then finite dimensional left A-modules are given by finite dimensional representations of Q satisfying the relations X (we call such representations (Q, X)-representations). Thus we can define a tensor product point-wise and arrow-wise. Moreover, as in the case of group representations we obtain a representation ring $R(Q, X)$, which we denote simply by $R(Q)$ in case X is empty. Our aim is to describe this ring.

By the Krull-Schmidt Theorem $R(Q, X)$ has a $\mathbb{Z}$-basis consisting of the isoclasses of indecomposable (Q, X)-representations. Thus, describing the multiplicative structure of $R(Q, X)$ amounts to solving the following problem: given to indecomposable (Q, X)-representations V, W decompose $V \otimes W$ into indecomposables. This problem is called the Clebsch-Gordan problem and has its origin in the study of binary algebraic forms by Clebsch and Gordan [2].

The most classical case is when Q is the loop quiver. For k algebraically closed of characteristic zero, the solution to the Clebsch-Gordan problem for the loop was found by Aitken [1]. The case when k is algebraically closed of positive characteristic was solved by Iima-Iwamatsu [11] and the case when k is perfect was treated in [3].

For Q a Dynkin quiver, $R(Q)$ was described for type $\mathbb{A}$ and $\mathbb{D}$ in [10] and for type $\mathbb{E}_6$ in [9]. The remaining cases $\mathbb{E}_7$ and $\mathbb{E}_8$ are still unsolved to my knowledge.

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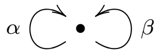
For extended Dynkin quivers of type $\tilde{\mathbb{A}}$ the solution to the Clebsch-Gordan problem was found in [7]. There is also a solution in case Q is the double loop quiver



with relations $\alpha^n = \beta^n = \alpha\beta = \beta\alpha = 0$, found in [8]. These two cases are instances of string algebras. In the present article we shall describe the representation ring for each quiver with relations corresponding to a string algebra.

Gelfand and Ponomarev classified the indecomposable representations of the double loop quiver appearing above in [6], as part of their classification of Harish-Chandra modules over the Lorentz group. The indecomposables in this case fall into two classes called strings and bands. This type of classification was later used in other settings by Ringel [13] and Donovan-Freislich [4]. A well-rounded setting to which it applies is that of string algebras.

2. Preliminaries

Let us recall some definitions and set notation. More detail can be found in [5]. Throughout fix a perfect field k. A quiver Q consists of a set of vertices Q_0 and a set of arrows Q_1. Moreover, it is equipped with two maps $t, h : Q_1 \to Q_0$ mapping each arrow α to its tail $t\alpha$ and head $h\alpha$ respectively. We depict this by $t\alpha \xrightarrow{\alpha} h\alpha$.

A representation V of Q consists of a collection of finite dimensional k-vector spaces V_x, where $x \in Q_0$ and linear maps $V(\alpha) : V_x \to V_y$ where $x \xrightarrow{\alpha} y \in Q_1$. Let X be a set of paths in Q. We call V a (Q, X)-representation if for every path $\alpha_1 \cdots \alpha_n \in X$ the equality $V(\alpha_1) \cdots V(\alpha_n) = 0$ holds. The category of (Q, X)-representations is denoted $\operatorname{rep}_k(Q, X)$.

Given two (Q, X)-representation their tensor product $V \otimes W$ is defined as follows. For each $x \in Q_0$, $\alpha \in Q_1$ set

$$(V \otimes W)_x = V_x \otimes W_x \text{ and } (V \otimes W)(\alpha) = V(\alpha) \otimes W(\alpha).$$

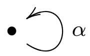
It is routine to check that $V \otimes W$ is a (Q, X)-representation.

Let $S(Q, X)$ be the set of isoclasses of (Q, X)-representations. For all $[V], [W] \in S(Q, X)$ set

$$[V] + [W] = [V \oplus W] \text{ and } [V][W] = [V \otimes W].$$

This endows $S(Q, X)$ with the structure of a semi-ring. Let $R(Q, X)$ be the corresponding Grothendieck ring [12].

Our aim is to describe $R(Q, X)$ in case (Q, X) corresponds to a string algebra. Of particular importance is the case when Q is the loop quiver:



In this case there is an equivalence of categories

$$\operatorname{rep}_k Q \xrightarrow{\sim} \operatorname{mod} k[x]$$

defined for each representation V by letting x act on $V_\bullet$ by $V(\alpha)$. The tensor product induced by this equivalence on $\operatorname{mod} k[x]$ comes from the coproduct $k[x] \to k[x] \otimes k[x]$, $x \mapsto x \otimes x$.

Let V_n correspond to the indecomposable $k[x]/x^n$ under the above equivalence for every $n > 0$. We have the following result from [3].

Proposition 1. *Let Q be the loop quiver and V a Q-representation such that $V(\alpha)$ is an invertible linear operator. Then the following statements hold.*

(1) $V_n \otimes V_m \xrightarrow{\sim} (m-n+1)V_n \oplus \bigoplus_{i=1}^{n-1} 2V_i$ *for* $n \leq m$.
(2) $V \otimes V_n = (\dim V)V_n$ *for all* n.

Let $I_s \subset R(Q)$ be the $\mathbb{Z}$-span of $\{[V_n] \mid n > 0\}$. By Proposition 1, I_s is an ideal in $R(Q)$ and $R(Q)/I_s \xrightarrow{\sim} R(k[x, x^{-1}])$. The structure of $R(k[x, x^{-1}])$ depends heavily on the field k. Let us recall its description from [3]. We need to construct another ring which we denote by R'.

If $\operatorname{char} k = 0$, then set $R' = \mathbb{Z}[T]$.

If $\operatorname{char} k = p > 0$, then R' is constructed as follows. For each $i \in \mathbb{N}$ let C_{p^i} be the cyclic group of order p^i and set $R_i = R(kC_{p^i})$. There are canonical inclusions $R_i \subset R_{i+1}$, and we set $R' = \bigcup_{i \in \mathbb{N}} R_i$.

Let $\overline{k}^{\iota}$ be the group of invertible elements in the algebraic closure of k and $\mathbb{Z}\overline{k}^{\iota}$ the corresponding group ring. The absolute Galois group $G = \operatorname{Gal}(\overline{k}/k)$ acts on $\overline{k}^{\iota}$ and consequently on $\mathbb{Z}\overline{k}^{\iota}$. Denote by $(\mathbb{Z}\overline{k}^{\iota})^G$, the ring of invariants. The following Theorem is from [3].

Theorem 2. *There is an isomorphim*

$$R(k[x, x^{-1}]) \xrightarrow{\sim} (\mathbb{Z}\overline{k}^{\iota})^G \otimes_{\mathbb{Z}} R'.$$

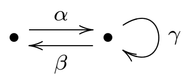
3. String algebras

As before fix a quiver Q and a set X of paths in Q. Set $I = \langle X \rangle$ and $A = kQ/I$.

Definition 3. The algebra A is called a string algebra if it is finite dimensional and satisfies the following conditions.

(1) Each $x \in Q_0$ is the tail, respectively head, of at most two arrows.
(2) For each $\alpha \in Q_1$ there is at most one $\beta \in Q_1$ and at most one $\gamma \in Q_1$ such that $\beta\alpha \notin I$ and $\alpha\gamma \notin I$.

Example 4. The following quiver with relations defines a string algebra for every $n > 0$.



$$\beta\alpha = \alpha\beta = (\beta\alpha\gamma)^n = 0$$

We proceed to describe the indecomposable modules over string algebras.

Definition 5. A *quiver morphism* $F : P \to Q$ consists of two maps $F : P_0 \to Q_0$, $F : P_1 \to Q_1$ such that for any arrow $x \xrightarrow{\alpha} y$ we get $Fx \xrightarrow{F\alpha} Fy$.

We call $\mathbf{F} = (F, P)$ a *shape* over Q if for any two distinct arrows $x_1 \xrightarrow{\alpha_1} y_1, x_2 \xrightarrow{\alpha_2} y_2 \in Q_1$ we have that $F\alpha_1 = F\alpha_2$ implies $x_1 \neq y_1$ and $x_2 \neq y_2$.

A *morphism of shapes* $(F, P) \to (F', P')$ is a quiver morphism $G : P \to P'$ such that $F = F'G$. Denote by $|\mathbf{F}' : \mathbf{F}|$, the number of morphisms $\mathbf{F} \to \mathbf{F}'$.

We only consider shapes (F,P) and such that for any path $\alpha_1\cdots\alpha_n$ in P we have that $F\alpha_1\cdots F\alpha_n \notin I$.

With each shape (F,P) we associate two functors

$$\begin{array}{c} \operatorname{rep}_k P \\ {\scriptstyle F^*}\big\uparrow \big\downarrow{\scriptstyle F_*} \\ \operatorname{rep}_k(Q,X) \end{array}$$

defined as follows.

For each $V \in \operatorname{rep}_k(Q,X)$, $x \in P_0$ and $\alpha \in P_1$ set $(F^*V)_x = V_{Fx}$ and $(F^*V)(\alpha) = V(Fx)$. For each $W \in \operatorname{rep}_k P$, and $x' \in Q_0$ set

$$(F_*W)_x = \bigoplus_{Fx=x'} W_x.$$

Let $x' \xrightarrow{\alpha'} y' \in Q_1$. Write the linear map

$$(F_*W)(\alpha') : \bigoplus_{Fx=x'} W_x \to \bigoplus_{Fy=y'} W_y$$

as a matrix A with elements

$$A_{yx} = \begin{cases} W(\alpha) & \text{if there is } x \xrightarrow{\alpha} y \text{ such that } F\alpha = \alpha', \\ 0 & \text{else.} \end{cases}$$

Definition 6. A shape $\mathbf{F} = (F,L)$ is called *linear* if L is Dynkin of type $\mathbb{A}$, i.e. if its underlying graph is

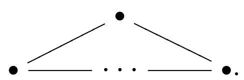
$$\bullet \text{ ——— } \cdots \text{ ——— } \bullet.$$

We define the L-representation V by

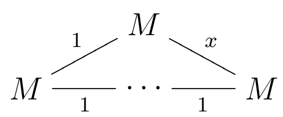
$$k \overset{1}{\text{———}} \cdots \overset{1}{\text{———}} k.$$

The *string* associated to $\mathbf{F}$ is the (Q,X)-module $S_{\mathbf{F}} := F_*V$. It is always indecomposable.

A shape $\mathbf{G} = (G,Z)$ is called *cyclic* if it has trivial automorphism group and Z is extended Dynkin of type $\tilde{\mathbb{A}}$, i.e. if its underlying graph is



Now let M be a $k[x,x^{-1}]$-module and $\gamma \in Z_1$. We define the Z-representation W by



where the arrow acting as x is γ. The *band* associated with $(\mathbf{G}, M, \gamma)$ is the (Q,X)-module $B_{\mathbf{G}}(M,\gamma) := G_*W$. It is indecomposable if and only if M is indecomposable. For $\gamma' \in Z_1$ we say that γ and γ' are oriented equally if when cycling through the vertices of Z we encounter $t\gamma$ and $h\gamma$ in the same order as we encounter $t\gamma'$ and $h\gamma'$. In that case

$B_{\mathbf{G}}(M,\gamma') \tilde{\to} B_{\mathbf{G}}(M,\gamma)$. Otherwise $B_{\mathbf{G}}(M,\gamma') \tilde{\to} B_{\mathbf{G}}(M^{-1},\gamma)$, where M^{-1} is obtained from M by inverting the action of x.

The following Theorem follows from [14].

Theorem 7. *Assume that A is a string algebra. Then strings and (indecomposable) bands classify all indecomposables, i.e.*

(1) *Each indecomposable A-module is isomorphic to either a string or band.*
(2) *No strings are isomorphic to bands.*
(3) *Two strings $S_{\mathbf{F}}$ and $S_{\mathbf{F}'}$ are isomorphic if and only if they have isomorphic shapes.*
(4) *Two bands $B_{\mathbf{G}}(M,\gamma)$ and $B_{\mathbf{G}'}(M',\gamma')$ are isomorphic if and only if their shapes are isomorphic via some H such that M' is isomorphic to M if $H(\gamma)$ and γ' are equally oriented and M' is isomorphic to M^{-1} otherwise.*

Let $\mathcal{L}$ be the set of isoclasses of linear shapes and $\mathcal{Z}$ be the set of isoclasses of cyclic shapes.

We need the following preliminary result.

Proposition 8. *Let (F,P) be a shape over Q, $V \in \operatorname{rep}_k(Q,X)$ and $W \in \operatorname{rep}_k P$. Then*

$$F_*W \otimes V \tilde{\to} F_*(W \otimes F^*V)$$

Let $I_s \subset R(Q,X)$ be the $\mathbb{Z}$-span of $\{[S_{\mathbf{F}}] \mid \mathbf{F} \in \mathcal{L}\}$. By Proposition 8, it is an ideal, since $[S_{\mathbf{F}}][V] = [F_*(F^*V)] \in I_s$.

The following Theorem completely describes the structure of $R(Q,X)$ in the case A is a string algebra.

Theorem 9. *Assume that A is a string algebra. Then the ideal I_s has a unique $\mathbb{Z}$-basis of pair-wise orthogonal idempotents $\{e_{\mathbf{F}} = e_{\overline{\mathbf{F}}}\}_{\overline{\mathbf{F}} \in \mathcal{L}}$, such that the following statements hold:*

(1) *For each linear shape $\mathbf{F}$*

$$[S_{\mathbf{F}}] = \sum_{\overline{\mathbf{F}'} \in \mathcal{L}} |\mathbf{F} : \mathbf{F}'| e_{\mathbf{F}'}.$$

(2) *For each cyclic shape $\mathbf{G} = (G,Z)$, $\gamma \in Z_1$ and $k[x,x^{-1}]$-module M*

$$[B_{\mathbf{G}}(M,\gamma)]e_{\mathbf{F}'} = \dim M |\mathbf{G} : \mathbf{F}'| e_{\mathbf{F}'}.$$

(3) *For each pair of non-isomorphic cyclic shapes $\mathbf{G_1} = (G_1, Z^1)$, $\mathbf{G_2} = (G_2, Z^2)$, $\gamma_1 \in Z_1^1$, $\gamma_2 \in Z_1^2$ and $k[x,x^{-1}]$-modules M, N*

$$[B_{\mathbf{G_1}}(M,\gamma_1)][B_{\mathbf{G_2}}(N,\gamma_2)] = \sum_{\overline{\mathbf{F}'} \in \mathcal{L}} \dim M \dim N |\mathbf{G_1} : \mathbf{F}'||\mathbf{G_2} : \mathbf{F}'| e_{\mathbf{F}'}.$$

Moreover,

$$[B_{\mathbf{G_1}}(M,\gamma_1)][B_{\mathbf{G_1}}(N,\gamma_1)] = [B_{\mathbf{G_1}}(M \otimes N,\gamma_1)] + \sum_{\overline{\mathbf{F}'} \in \mathcal{L}} \dim M \dim N |\mathbf{G_1} : \mathbf{F}'|(|\mathbf{G_1} : \mathbf{F}'| - 1) e_{\mathbf{F}'}.$$

We end with the following observations. As a (non-unital) subring $I_s \tilde{\to} \bigoplus_{\overline{\mathbf{F}} \in \mathcal{L}} \mathbb{Z}$. On the other hand, $R(Q,X)/I_s \tilde{\to} \bigoplus_{\overline{\mathbf{G}} \in \mathcal{Z}} R(k[x,x^{-1}])$.

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