# FULLY WEAKLY PRIME RINGS

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Abstract. Anderson and Smith studied weakly prime ideals for a commutative ring with identity. Blair and Tsutsui studied the structure of a ring in which every ideal is prime. In this paper we investigate the structure of rings, not necessarily commutative, in which all ideals are weakly prime.

## 1. Introduction

*Anderson-Smith* [1] defined a proper ideal $P$ of a commutative ring $R$ with identity to be weakly prime if $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. They proved that every proper ideal in a commutative ring $R$ with identity is weakly prime if and only if either $R$ is a quasilocal ring (possibly a field) whose maximal ideal is square zero, or $R$ is a direct sum of two fields [1, Theorem 8]. On the other hand, *Blair-Tsutsui* [2] studied the structure of a ring in which every ideal is prime. In this paper we first consider the structure of rings, not necessarily commutative nor with identity, in which all ideals are weakly prime. A necessary and sufficient condition for a ring to have such property is given and several examples to support given propositions are constructed. We then further investigate commutative rings in which every ideal is weakly prime and the structure of such rings under assumptions that generalize commutativity of rings. At the end, we consider the structure of rings in which every right ideal is weakly prime.

## 2. General results

We generalize the definition of a weakly prime ideal to arbitrary (not necessarily commutative) rings as follows.

**Definition.** A proper ideal $I$ of a ring $R$ is *weakly prime* if $0 \neq JK \subseteq I$ implies either $J \subseteq I$ or $K \subseteq I$ for any ideals $J$, $K$ of $R$.

Our first proposition is Theorem 1 of *Anderson-Smith* [1] in a more general setting.

**Proposition 1.** *If $P$ is weakly prime but not prime, then $P^2 = 0$.*

**Proof.** Since $P$ is weakly prime but not prime, there exist ideals $I \not\subseteq P$ and $J \not\subseteq P$ but $0 = IJ \subseteq P$. But if $P^2 \neq 0$, then $0 \neq P^2 \subseteq (I + P)(J + P) \subseteq P$, which implies $I \subseteq P$ or $J \subseteq P$, a contradiction.

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**Proposition 2.** *Let $P$ be an ideal in a ring $R$ with identity. The following statements are equivalent:*

(1) *$P$ is a weakly prime ideal.*
(2) *If $J$, $K$ are right (left) ideals of $R$ such that $0 \neq JK \subseteq P$, then $J \subseteq P$ or $K \subseteq P$.*
(3) *If $a, b \in R$ such that $0 \neq aRb \subseteq P$, then $a \in P$ or $b \in P$.*

**Proof.** The implications (1) => (2) and (2) => (3) are easy. The implication (3) => (1) can be verified by checking a number of cases.

Since weakly prime ideals are defined to be proper ideals, we shall say that every ideal of a ring $R$ is weakly prime when every proper ideal of $R$ is weakly prime. In this case we say that $R$ is fully weakly prime.

If $R^2 = 0$, then clearly every ideal of $R$ is weakly prime. In particular, if an ideal $I$ of a ring $R$ is weakly prime but not a prime ideal, then every ideal of $I$ as a ring is weakly prime by Proposition 1.

**Proposition 3**. *Every ideal of a ring $R$ is weakly prime if and only if for any ideals $I$ and $J$ of $R$, $IJ = I$,$IJ = J$, or $IJ = 0$.*

**Corollary 1.** *Let $R$ be a ring in which every ideal of $R$ is weakly prime. Then for any ideal $I$ of $R$, either $I^2 = I$ or $I^2 = 0$.*

**Example 1.** Let $F$ be a field and $R = F \oplus F \oplus F$. Then every ideal of $R$ is idempotent but the ideal $I = F \oplus 0 \oplus 0$ is evidently not weakly prime, showing that the converse of Corollary 1 is false.

Suppose that a ring $R$ with identity has a maximal ideal $M$ and $M^2 = 0$. One can readily check that $R$ is fully weakly prime, and $M$ is the only prime ideal of $R$.

Corollary 1 in particular yields that if a ring $R$ has the property that every ideal is weakly prime, then either $R^2 = R$, or $R^2 = 0$. Notice that $R^2$ is neither 0 nor $R$ in the example given below.

**Example 2.** Let $S$ be a ring such that $S^2 = 0$, and let $F$ be a field. Then the ring $R = F \oplus S \oplus S$ with component-wise addition and multiplication has a maximal ideal $M = 0 \oplus S \oplus S$ and $M^2 = 0$. However, $I = F \oplus 0 \oplus S$ is not weakly prime since $0 \neq (F \oplus S \oplus 0)^2 \subseteq I$.

If a ring $R$ satisfying $R^2 = R$ has a maximal ideal $M$ and $M^2 = 0$, then every proper ideal of $R$ is contained in $M$. However, it is possible that $MR \neq M$. Thus, such a ring does not necessarily have the property that every ideal is weakly prime as the following example shows.

**Example 3.** Let $F$ be a field and $S = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a, b, c, d \in F \right\}$. Then $S$ has a unique maximal ideal $L = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix} \middle| \; a, b, d \in F \right\}$.

Let $N = \left\{ \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| \, b \in F \right\}$. Consider the factor ring $R = S/N$. While $R^2 = R$ and $M = L/N$ is a maximal ideal whose square is zero, the proper ideals $RM$ and $MR$ are not weakly prime.

**Proposition 4.** *If every ideal of a ring $R$ is weakly prime and $R^2 = R$, then $R$ has at most two maximal ideals.*

**Proof.** By contradiction.

The following example shows that the condition $R^2 = R$ in Proposition 4 cannot be dropped.

**Example 4.** Let $R$ be the unique maximal ideal of $\mathbb{Z}_4$.Then $S = R \oplus R \oplus R$ is an example of a ring all of whose ideals are weakly prime and having more than 2 maximal ideals.

**Proposition 5.** *Suppose that every ideal of a ring $R$ is weakly prime. If $R$ has two maximal ideals $M_1$ and $M_2$, then their product is zero. Furthermore, if $R$ has an identity element, then $R$ is a direct sum of two simple rings.*

**Proof.** Note $M_1 M_2 \subseteq M_1 \cap M_2$. If $R$ has an identity, then $M_1 \cap M_2 = (M_1 \cap M_2)(M_1 + M_2) = 0$.

We denote the prime radical of $R$ by $P(R)$, and the sum of all ideals whose square is zero by $N(R)$.

**Theorem 1.** *Suppose that every ideal of a ring $R$ is weakly prime and $R^2 = R$. Then $P(R) = N(R)$ and $(P(R))^2 = (N(R))^2 = 0$.*

**Proof.** Any finite sum of square-zero ideals is nilpotent, and hence square-zero, so $N(R)^2 = 0$. Thus $N(R) \subseteq P(R)$.

Either $P(R)$ is not prime (in which case $P(R)^2 = 0$, so $P(R) \subseteq N(R)$), or $P(R)$ is prime, in which case $N(R)$ can be shown to be prime (apply Theorem 1.2 of Blair-Tsutsui [2] to $R/P(R)$).

**Corollary 2.** *Suppose that every ideal of a right Noetherian ring $R$ with identity is weakly prime and $R^2 = R$. Then $P(R) = N(R) = J(R)$ and $(J(R))^2 = 0$, where $J(R)$ is the Jacobson radical of $R$.*

**Proof.** If $(J(R))^2 = J(R)$, then $J(R) = 0 \subseteq P(R)$ by Nakayama's lemma. If $(J(R))^2 = 0$, then $J(R) \subseteq P(R)$.

Note that for a ring $R$ in which every ideal is weakly prime, in general it is possible that $P(R) = N(R) \neq J(R)$ [2, §5 *An Example*].

**Corollary 3.** *Suppose that every ideal of a ring $R$ is weakly prime. Then every nonzero ideal of $R/N(R)$ is prime.*

**Corollary 4.** *Suppose every ideal of a ring $R$ is weakly prime. Then $(N(R))^2 = 0$ and every prime ideal contains $N(R)$. There are three possibilities:*

*(a) $N(R) = R$.*

*(b) $N(R) = P(R)$ is the smallest prime ideal and all other prime ideals are idempotent and prime ideals are linearly ordered. If $N(R) \neq 0$, then it is the only non-idempotent prime ideal.*
*(c) $N(R) = P(R)$ is not a prime ideal. In this case, there exist two nonzero minimal prime ideals $J_1$ and $J_2$ with $N(R) = J_1 \cap J_2$ and $J_1J_2 = J_2J_1 = 0$. All other ideals containing $N(R)$ also contain $J_1 + J_2$ and they are linearly ordered.*

**Proof.** Use Theorem 1. If we are not in case (a) or (b), apply [2, Theorem 1.2] and [4, Theorem 2.1] to $R/N(R)$.

**Example 5.** Let $R$ be a ring and $M$ an $R$-bimodule. Define

$$R * M = \{(r,\, m) | r \in R,\, m \in M\}$$

with component-wise addition and multiplication

$$(r,\, m)(s, n) = (rs,\, rn + ms).$$

Then $R * M$ is a ring whose ideals are precisely of the form $I * N$ where $I$ is an ideal of $R$ and $N$ is a submodule (a bimodule) of $M$ containing $IM$ and $MI$.

(a) Let $R$ be a prime ring with exactly one nonzero proper ideal $P$. For example, the ring of linear transformations of a vector space $V$ over a field $F$ where $\dim_F V = \aleph_0$ has such a property. Then every ideal of $S_1 = R * P$ is weakly prime: the maximum ideal $P_1 = P * P$ is idempotent and the nonzero minimal ideal $P_2 = 0 * P$ is nilpotent, both of which are prime.

(b) Every ideal of $S_2 = S_1 * P_2$ is weakly prime: The maximum ideal $Q_1 = P_1 * P_2$ is idempotent and the three nonzero nilpotent ideals are $Q_2 = P_2 * P_2$, $Q_3 = 0 * P_2$, and $Q_4 = P_2 * 0$.

(c) If we redefine the multiplication above as

$$(r,\, m)(s, n) = (rs,\, rn + ms + mn),$$

then $S_1$ in (a) has an additional minimal ideal $P_3 = \{(p,\, -p) |\, p \in P\}$. In this case, $N(S_1) = P_3 \cap P_2 = 0$.

We don't know of an example of Corollary 4, case (c) where $N(R) \neq 0$.

## 3. Commutative Rings and Generalizations thereof

We now consider the structure of rings in which every ideal is weakly prime under the assumption of the ring being commutative or with commutative-like conditions.

**Proposition 6.** *Let $R$ be a commutative ring in which every ideal is weakly prime. If $R^2 = R$, then $R$ has a maximal ideal.*

**Proof.** If $N(R)$ is not maximal there exists a prime ideal $I$. Apply [2, Theorem 1.3] to $R/I$.

We note that a commutative ring $R$ with the property $R^2 = R$ does not necessarily have a maximal ideal. For example, if a commutative ring $S$ has a unique nonzero maximal ideal $M$ and $M^2 = M$, then $M$ as a ring cannot have a maximal ideal.

The next corollary follows from Propositions 4 and 6.

**Corollary 5.** *Let $R$ be a commutative ring all of whose ideals are weakly prime. Suppose that $R^2 = R$. Then $R$ has either a unique maximal ideal or exactly two maximal ideals.*

**Theorem 2.** *Let $R$ be a commutative ring all of whose ideals are weakly prime. Suppose that $R^2 = R$.*

*(1) If $R$ has a unique maximal ideal $M$, then $M^2 = 0$.*

*(2) If $R$ has two maximal ideals $M$ and $N$, then $MN = 0$.*

**Proof.** By contradiction.

**Proposition 7**. *Let $R$ be a commutative ring all of whose ideals are weakly prime. Suppose that $R^2 = R$. Then every proper ideal is contained in a maximal ideal.*

**Proof.** Use the preceding theorem.

**Corollary 6.** *Let $R$ be a commutative ring and suppose that every ideal of $R$ is weakly prime. If $R^2 = R$, then $R$ has an identity element.*

**Proof.** We show that if a commutative ring $R$ satisfies the following conditions, then $R$ has an identity element:
(a) $R^2 = R$,
(b) every proper ideal is contained in a maximal ideal, and
(c) $R$ has a finite number of maximal ideals $M_1, M_2, \ldots, M_n$.

Choose $x \in R$ such that $x \notin M_j$ for any $j$. Let $(x) = \{xR + nx | n \in \mathbb{Z}\}$. If $xR \subseteq M_j$, then $R = R^2 = (M_j + (x))^2 \subseteq M_j$, a contradiction. Hence $xR = R$ and consequently, $R$ has an identity element.

**Corollary 7.** *Let $R$ be a commutative ring all of whose ideals are weakly prime. Then one of the following holds:*
*(a) $R^2 = 0$,*
*(b) $R$ is a ring with identity and a square zero maximal ideal $M$, or*
*(c) $R$ is a direct sum of two fields.*

For the case (b) in Corollary 7, the following theorem further determines the structure of $R$.

**Theorem 3.** *Let $R$ be a commutative ring with a square-zero maximal ideal $M$ and $R^2 = R$. If $(ch(R/M) = 0$, then $R$ is isomorphic to $(R/M) * M$ (as defined in Example 5).*

**Proof.** Note that if $E$ is a subfield of $R/M$ and $\psi : E \to R$ is a homomorphism satisfying $\pi \circ \psi = id|_E$, then the map $\varphi : E * M \to R$ given by $\varphi((\bar{x}, m)) = \psi(\bar{x}) + m$ is a monomorphism. So, it suffices to show such a map $\psi$ exists for $E = R/M$ (in this case $\varphi$ is also onto); the proof proceeds by defining $\psi$ on successively larger subfields $E \subseteq R/M$.

Using the same idea, the result also holds for $ch(R/M) = p$ if $pR = 0$ and $R/M$ is separable over $F$. In general, however, the theorem is false if $ch(R/M) = p \neq 0$.

**Example 6.** **(a)** Let $R = Z_{p^2}$ where $p$ is prime. Then $R$ has maximal ideal $M = pR \neq 0$ but $p(R/M * M) = 0$.

**(b)** Let $R = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ where $F$ is a field. Then $R^2 = R \neq 0$, and every ideal of $R$ is weakly prime but $R$ does not contain an identity element.

As a natural generalization of commutative rings, we next consider polynomial identity (PI) rings.

**Theorem 4.** *Let $R$ be a PI-ring with identity. If every ideal of $R$ is weakly prime, then one of the following folds:*
*(a) $R/P(R)$ is a finite dimensional central simple algebra.*
*(b) $R$ is a direct sum of two finite dimensional central simple algebras.*

**Proof.** Use Corollary 4 and [2, Theorem 3.3].

More general than the class of PI-rings is the class of fully bounded rings. Using [2, Theorem 3.4] yields the following theorem.

**Theorem 5.** *Let $R$ be a ring with identity in which every ideal is weakly prime. If $R$ is a right fully bounded, right Noetherian ring, then one of the following holds:*

*(a) $R/P(R)$ is a simple Artinian ring.*
*(b) $R$ is a direct sum of two simple Artinian rings.*

## 4. Rings in which every right ideal is weakly prime

**Definition.** We define a proper right ideal $I$ of a ring $R$ to be weakly prime if $0 \neq JK \subseteq I$ implies either $J \subseteq I$ or $K \subseteq I$ for any right ideals $J$, $K$ of $R$.

For a ring $R$ that is not square zero, Koh[3] showed that $R$ is simple and $a \in aR$ for all $a \in R$ if and only if every right ideal of $R$ is prime. Now consider the structure of rings in which every right ideal is weakly prime. For the commutative case, it is evident that such rings need not be simple. Example 6 (b) gives an example of a ring $R = R^2$ in which every right and left ideal is weakly prime.

Unlike the case of weakly prime two sided ideals, there exists a nonzero idempotent weakly prime right ideal that is not prime. For example, if $R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| a, b, c \in F \right\}$, then $K = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & t \end{bmatrix} \middle| t \in F \right\}$ is a weakly prime right ideal and $K^2 = K \neq 0$. But $K$ is not a prime right ideal.

We conclude with the following generalization of Corollary 7.

**Theorem 6.** *Suppose that every right ideal of a ring $R$ is weakly prime. Then one of the following holds:*
*(a) $R^2 = 0$.*
*(b) $R$ has a square zero maximal ideal.*
*(c) $R$ is a direct sum of two division rings.*

Under an additional condition we can say more about case (b).

**Proposition 8.** *Let $R$ be a ring all of whose right ideals are weakly prime. Suppose $R$ has a square zero maximal ideal $N \neq 0$. If $NR = 0$, then $RN = N$ and either:*

*(a) $R/N$ is a simple dense ring of endomorphisms over the infinite-dimensional vector space $N$ (and every nonzero endomorphism is surjective), or*

*(b) $R/N$ is a division ring and $R$ is isomorphic to*

$$\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in R/N \right\}.$$

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