ON GALOIS EXTENSIONS WITH AN INNER GALOIS GROUP AND A GALOIS COMMUTATOR SUBRING

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Abstract. Properties of a Galois ring extension with an inner Galois group are given, and equivalent conditions for a Galois extension with a Galois commutator subring are shown.

1. Introduction

In 1960's, Galois theory was developed for rings by M. Auslander-O.Goldman ([2]), S.U. Chase-D.K. Harrison-A. Rosenberg ([3]), F.R. DeMeyer ([4], [5]), M. Harada ([7]), Y. Miyashita ([13]), T. Nagahara ([14]), T. Kanzaki ([12]), K. Sugano ([15], [16]), and others. It was shown ([4], Theorem 6, [5], Theorem 3) that B is a central Galois algebra over its center C with an inner Galois group G if and only if it is an Azumaya projective group algebra CG_f where $f : G \times G \longrightarrow$ units of C is a factor set. In section 3, we shall generalize the above theorem to any Galois extension B with an inner Galois group G where $G = \{g \in G \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$. It is shown that B contains a projective group algebra CG_f. An equivalent condition for a central Galois algebra CG_f with Galois group induced by G is given, and characterizations for a Galois extension B with an inner Galois group G generated by $\{U_g \,|\, g \in G\}$ over B^G are obtained. When B is also an Azumaya algebra, in section 4, some properties are given for a Galois extension B with an inner Galois group G. We note that any Galois extension with an inner Galois group G is a Hirata separable extension of B^G ([17], Corollary 3). For a Hirata separable Galois extension B with Galois group G (not necessarily inner), in [17], Sugano investigated the Galois commutator subring $V_B(B^G)$ of B^G in B. We shall study when $V_B(B^G)$ is a Galois extension with Galois group induced by G for any Galois extension B with Galois group G in section 5. Equivalent conditions are given in terms of a composition Galois extensions: $B \supset B^G \cdot V_B(B^G) \supset B^G$ and crossed products respectively. Some examples are also given to demonstrate the results.

2. Basic Definitions and Notations

Let B be a ring with identity 1, C the center of B, G a finite automorphism group of B, B^G the set of elements in B fixed under each element in G. Following the definitions as given in the references, we call B a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum_{i=1}^{m} a_i g(b_i) = \delta_{1,g}$ for each $g \in G$ ([4]). Such a set $\{a_i, b_i\}$ is called a G-Galois system for B. A Galois extension B of B^G is called a Galois algebra if B^G is contained in C ([21]), and a central Galois algebra if $B^G = C$ ([20]). We call B a center Galois extension with Galois group

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G if C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$. Let A be a subring of B with the same identity 1. We denote $V_B(A)$ the commutator (also called centralizer) subring of A in B, that is, $V_B(A) = \{b \in B | bx = xb \text{ for all } x \in A\}$. We call B a separable extension of A if there exist $\{a_i, b_i$ in B, $i = 1, 2, ..., m$ for some integer $m\}$ such that $\sum a_i b_i = 1$, and $\sum ba_i \otimes b_i = \sum a_i \otimes b_i b$ for all b in B where $\otimes$ is over A. An Azumaya algebra is a separable extension of its center. A Galois extension B of B^G with Galois group G is called an Azumaya Galois extension if B^G is an Azumaya C^G-algebra ([1]). A Galois extension B of B^G with Galois group G is called a DeMeyer-Kanzaki Galois extension if B is an Azumaya algebra over C which is a Galois algebra over C^G with Galois group $G|_C \cong G$. A ring B is called a Hirata separable extension of A if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of B as a B-bimodule, and B is called a Hirata separable Galois extension of B^G if it is a Galois and a Hirata separable extension of B^G. Let R be a commutative ring with 1 and $U(R)$ the set of units of R. As given in [4], for a factor set f: $G \times G \longrightarrow U(R)$ (that is, $f(g,h)f(gh,k) = f(h,k)f(g,hk)$ for all g, h, and k in G), $RG_f = \sum_{g \in G} RU_g$ is called a projective group algebra over R if RG_f is an algebra with a free basis $\{U_g \mid g \in G\}$ over R where U_g is an invertible element for each $g \in G$, the multiplications are given by $(r_g U_g)(r_h U_h) = r_g r_h U_g U_h$ and $U_g U_h = f(g,h) U_{gh}$ for $r_g, r_h \in R$ and $g, h \in G$; that is, $f(g,h) = U_g U_h U_{gh}^{-1}$.

3. Galois Extensions with an Inner Galois Group

Let B be a Galois extension of B^G with an inner Galois group G whose order $|G|$ is invertible in B where $G = \{g \in G \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$. We shall show that B contains a projective group algebra CG_f where C is the center of B. An equivalent condition is given for a central Galois algebra CG_f. Thus several characterizations are obtained for B generated by $\{U_g \,|\, g \in G\}$ over B^G. These characterizations generalize the results for a central Galois algebra with an inner Galois group ([4], Theorem 6).

Theorem 3.1. ([23], Theorem 2.1) *Let B be a Galois extension of B^G with an inner Galois group G, $G = \{g \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$, and C the center of B. Then B contains a projective group algebra CG_f of G over C with a factor set $f : G \times G \longrightarrow$ units of C.*

Proof. We first claim that $\{U_g \,|\, g \in G\}$ are linearly independent over C. Let $\{x_i, y_i \in B \,|\, i = 1, 2, ..., m$ for some integer $m\}$ be a G-Galois system such that $\sum_{i=1}^m x_i g(y_i) = \delta_{1,g}$ for each $g \in G$. Let $\sum_{g \in G} a_g U_g = 0$ for some $a_g \in C$. Then

$$\sum_{i=1}^{m} x_i \sum_{g \in G} a_g U_g h^{-1}(y_i) = 0 \text{ for each } h \in G \text{ and}$$

$$\sum_{g \in G} a_g \sum_{i=1}^{m} x_i g h^{-1}(y_i) U_g = \sum_{g \in G} a_g \delta_{1,gh^{-1}} U_g = a_h U_h.$$

Noting that $a_g \in C$ and $U_g h^{-1}(y_i) = gh^{-1}(y_i)U_g$, we have that

$$\sum_{i=1}^{m} x_i \sum_{g\in G} a_g U_g h^{-1}(y_i) = \sum_{g\in G} a_g \sum_{i=1}^{m} x_i g h^{-1}(y_i) U_g;$$

and so $a_h U_h = 0$. But U_h is invertible in B, so $a_h = 0$ for each $h \in G$. Also, noting that $U_{gh}^{-1} U_g U_h$ is a unit in C, we have a factor set $f : G \times G \longrightarrow$ units of C by $f(g,h) = U_{gh}^{-1} U_g U_h$. Thus $\sum_{g\in G} CU_g = CG_f \subset B$.

Let Z be the center of G and $\overline{G}$ the restriction of G to CG_f. Then $\overline{G} \cong G/K$ where $K = \{g \in Z \,|\, f(g,h) = f(h,g) \text{ for all } h \in G\}$. Next is necessary and sufficient condition for a central Galois algebra CG_f with an inner Galois group $\overline{G}$.

Theorem 3.2. ([23], Theorem 2.2) *Let B be a Galois extension of B^G with an inner Galois group G of order n invertible in B and CG_f as given in Theorem 3.1. Then CG_f is a central Galois algebra over its center S with an inner Galois group $\overline{G}$ if and only if $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S where $U_{\overline{g}} = U_g$ for each $g \in G$.*

Proof. ($\Longrightarrow$) Since CG_f is a central Galois algebra with an inner Galois group $\overline{G}$, $CG_f = S\overline{G}_{\overline{f}}$ ([4], Theorem 6). Thus $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S.

($\Longleftarrow$) Since $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S, $S\overline{G}_{\overline{f}} = \oplus_{\overline{g}\in\overline{G}} SU_{\overline{g}}$ is a projective group algebra of $\overline{G}$ over S with factor set $f : \overline{G} \times \overline{G} \longrightarrow$ units of S induced by $f : G \times G \longrightarrow$ units of C. Noting that $\{U_g \,|\, g \in K\} \subset S$, we have that $CG_f = \oplus_{\overline{g}\in\overline{G}} SU_{\overline{g}} = S\overline{G}_{\overline{f}}$. But CG_f is an Azumaya S-algebra (for n is a unit in C), so $S\overline{G}_f$ is an Azumaya S-algebra. Thus $S\overline{G}_f$ is a central Galois S-algebra with an inner Galois group $\overline{G}$ ([5], Theorem 3). Therefore CG_f is a central Galois algebra over S with an inner Galois group $\overline{G}$.

Theorem 3.2 can be generalized to a projective group ring RG_f of a group G over a ring R (not necessarily commutative) with a factor set $f : G \times G \longrightarrow$ units of the center of R.

Theorem 3.3. ([22], Theorem 3.2) *Let RG_f be a Galois projective group ring of G over a ring R, C the center of RG_f, and R_0 the center of R. Then the following are equivalent: (1) RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$ induced by $\{U_g \,|\, g \in G\}$. (2) $C\overline{G}_{\overline{f}}$ is a central Galois projective group algebra of $\overline{G}$ over C with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C induced by $f : G \times G \longrightarrow$ units of R_0. (3) $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are free over RC and $RC = \oplus \sum_{g\in K} RU_g$ where $U_{\overline{g}} = U_g$ for each $g \in G$ and $K = \{g \in$ the center of $G \,|\, f(g,g') = f(g',g)$ for all $g' \in G\}$.*

Proof. Let Z be the center of G. We first note that $\overline{G} \cong G/K$ where $K = \{g \in Z \,|\, f(g,g') = f(g',g) \text{ for all } g' \in G\}$ and that $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are free over C where $U_{\overline{g}} = U_g$ for each $g \in G$ by the argument used in the proof of Theorem 3.1. Next we prove (1) $\Longrightarrow$ (2) and leave other implications (2) $\Longrightarrow$ (1) and (2) $\Longrightarrow$ (3) $\Longrightarrow$ (2) to readers.

Since RG_f is a Galois extension of $(RG_f)^{\overline{G}}$ with an inner Galois group $\overline{G}$, $\{U_{\overline{g}}\,|\,\overline{g}\in\overline{G}\}$ are free over RC. Noting that $\overline{f}:\overline{G}\times\overline{G}\longrightarrow$ units of R_0 contained in C, we have that $C\overline{G}_{\overline{f}}$ is a projective group algebra of $\overline{G}$ over C with factor set $\overline{f}:\overline{G}\times\overline{G}\longrightarrow$ units of C where $\overline{f}$ is induced by $f:G\times G\longrightarrow$ units of R_0. Moreover, since $R_0K_f\subset C$, $\sum_{\overline{g}\in\overline{G}}(R_0K_f)U_{\overline{g}}\subset C\overline{G}_{\overline{f}}$. But $\overline{G}=G/K$, so

$$RG_f=\sum_{g\in G}RU_g=R(R_0G_f)\subset R(\sum_{\overline{g}\in\overline{G}}CU_{\overline{g}})=R(C\overline{G}_{\overline{f}})\subset RG_f.$$

Hence $RG_f=R(C\overline{G}_{\overline{f}})$. Thus $\overline{G}|_{C\overline{G}_{\overline{f}}}\cong\overline{G}$. Next we claim that C is also the center of $\sum_{\overline{g}\in\overline{G}}CU_{\overline{g}}$ $(=C\overline{G}_{\overline{f}})$. In fact, clearly, C is contained in the center of $C\overline{G}_{\overline{f}}$. Conversely, for any $x\in$ the center of $C\overline{G}_{\overline{f}}$, x is in the center of $\sum_{\overline{g}\in\overline{G}}CU_{\overline{g}}$. Also, for any $r\in R$, $rx=xr$, so x is in the center of $R(\sum_{\overline{g}\in\overline{G}}CU_{\overline{g}})$ which is RG_f. Thus $x\in C$. Therefore $C\overline{G}_f$ is an Azumaya C-algebra; and so $C\overline{G}_f$ is a central Galois C-algebra with an inner Galois group $\overline{G}|_{C\overline{G}_{\overline{f}}}\cong\overline{G}$ ([4], Theorem 6).

We give two examples of Galois extensions with an inner Galois group G.

Example 1. Let $R[i,j,k]$ be the real quaternion algebra over real field R with inner automorphism group $G=\{1,\overline{i},\overline{j},\overline{k}\}$ where $\overline{i}(x)=ixi^{-1}$, $\overline{j}(x)=jxj^{-1}$, and $\overline{k}(x)=kxk^{-1}$ for $x\in R[i,j,k]$. Then $R[i,j,k]=R\oplus Ri\oplus Rj\oplus Rk$, a projective group algebra RG_f with center R; and so it is a central Galois algebra over R with an inner Galois group G.

Example 2. Let $T=R[i]\subset R[i,j,k]$ as given in Example 1 and $H_i=\{1,\overline{i}\}\subset G$. Then $(R[i,j,k])^{H_i}=R[i]$ and $R[i,j,k]$ is a noncommutative Galois extension of $R[i]$ with a cyclic Galois group H_i. We note that any Galois algebra with a cyclic Galois group is commutative ([4], Theorem 11).

By using Theorem 3.2, we derive some characterizations for a Galois extension B as given in Theorem 3.2 which is generated by $\{U_g\,|\,g\in G\}$ over B^G. We recall that C is the center of B, S the center of CG_f, Z the center of G, and $K=\{g\in Z\,|\,f(g,h)=f(h,g)$ for all $h\in G\}$.

Theorem 3.4. ([23], Theorem 2.3) *Let B be a Galois extension of B^G with an inner Galois group G of order n invertible in B. Then the following are equivalent:*

(1) $B=\sum_{g\in G}B^GU_g$, i.e., B is generated by $\{U_g\,|\,g\in G\}$ over B^G;

(2) $B=B^GG_f$, a projective group ring of G over B^G with factor set $f:G\times G\longrightarrow$ units of C;

(3) $C=S$;

(4) $\sum_{g\in G}CU_g$, the subring of B generated by $\{U_g\,|\,g\in G\}$ over C, is a central Galois C-algebra with Galois group $\overline{G}\cong G$;

(5) $\sum_{g\in G}CU_g$ is an Azumaya C-algebra;

(6) $K = \langle 1 \rangle$ and $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S.

4. The Azumaya Algebra

Let B be a Galois extension of B^G with an inner Galois group G whose order n is invertible in B as given in Theorem 3.2, $G = \{g \in G \,|\, g(x) = U_g x U_g^{-1}$ for some $U_g \in B$ and for all $x \in B\}$, C the center of B, Z the center of G, and $K = \{g \in Z \,|\, f(g,h) = f(h,g)$ for all $h \in G\}$. Assume that B is an Azumaya C-algebra. We shall show an equivalent condition for a central Galois algebra CG_f in terms of the Galois extension B^K of B^G with Galois group G/K.

Theorem 4.1. ([23], Theorem 3.1) *Let B be given in Theorem 3.2. If B is an Azumaya C-algebra, then $V_B(B^G) = CG_f$.*

Proof. Since n is invertible in B, CG_f is a separable subalgebra of the Azumaya C-algebra B. Hence $V_B(V_B(CG_f)) = CG_f$. Noting that $V_B(CG_f) = B^G$, we have that $V_B(B^G) = CG_f$.

Theorem 4.2. ([23], Theorem 3.2) *Let B be given in Theorem 3.2. Assume B is an Azumaya C-algebra. Then CG_f is a central Galois algebra over its center S with Galois group $\overline{G}$ $(= G/K)$ if and only if $B^K = B^G \cdot (CG_f)$.*

Proof. ($\Longrightarrow$) Since CG_f is a central Galois algebra with Galois group $\overline{G}$ $(= G/K)$, CG_f has a $\overline{G}$-Galois system. Clearly, $CG_f \subset B^G \cdot (CG_f) \subset B^K$ and $(B^G \cdot (CG_f))^G = (B^K)^G = B^G$, so $B^G \cdot (CG_f)$ and B^K are also Galois extensions with the same Galois system as CG_f by noting that the restrictions of G to $B^G \cdot (CG_f)$ and B^K are isomorphic with $\overline{G}$ $(= G/K)$. Thus $B^K = B^G \cdot (CG_f)$.

($\Longleftarrow$) By hypothesis, B is a Galois extension of B^G with an inner Galois group G of order n invertible in B, so B^K is a Galois extension of B^G with an inner Galois group G/K. Let S be the center of CG_f. Since CG_f is a separable C-subalgebra of the Azumaya C-algebra B, $V_B(V_B(CG_f)) = CG_f$. Hence CG_f, B^G $(= V_B(CG_f))$, and $B^G \cdot (CG_f)$ have the same center S. By hypothesis, $B^K = B^G \cdot (CG_f)$. Thus S is the center of B^K. But B^K is a Galois extension of B^G with an inner Galois group $\overline{G}$ $(= G/K)$, so B^K contains the separable projective group algebra $S\overline{G}_{\overline{f}}$ where $f : \overline{G} \times \overline{G} \longrightarrow$ units of S induced by $f : G \times G \longrightarrow$ units of C by Theorem 3.1. Thus $\{U_{\overline{g}} \,|\, \overline{g} \in \overline{G}\}$ are linearly independent over S. Therefore CG_f is a central Galois algebra with Galois group $\overline{G}$ by Theorem 3.2.

Corollary 4.3. ([23], Corollary 3.1) *Let B be given in Theorem 4.2. Then B^K is a Galois projective group ring of $\overline{G}$ over $B^G S$ with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C.*

Proof. By Theorem 4.2, $B^K = B^G \cdot (CG_f)$ and $CG_f = S\overline{G}_{\overline{f}}$, so $B^K = B^G \cdot (CG_f) = B^G(S\overline{G}_{\overline{f}}) = (B^G S)\overline{G}_{\overline{f}}$ which is a Galois projective group ring of $\overline{G}$ over $B^G S$ with factor set $\overline{f} : \overline{G} \times \overline{G} \longrightarrow$ units of C.

5. The Galois Commutator Subring

We note that a Galois extension with an inner Galois group G is a Hirata separable extension of B^G ([17], Corollary 3). In [17], let B be a Hirata separable Galois extension of B^G with Galois group G and $\Delta = V_B(B^G) = \{b \in G \mid ba = ab$ for each element $a \in B^G\}$, the commutator subring of B^G in B. A sufficient condition was given for Δ being a Galois algebra with Galois group G/N where $N = \{g \in G \mid g(x) = x$ for all $x \in \Delta\}$. We shall study the problem for a Galois extension B of B^G with Galois group G such that Δ is a Galois extension with Galois group G/N. Such a Galois extension B with Galois group G will be characterized in terms of a composition of two Galois extensions: $B \supset B^G \cdot V_B(B^G) \supset B^G$ and in terms of crossed products respectively.

We begin with two lemmas whose proofs are straightforward.

Lemma 5.1. ([24], Lemma 3.1) *Let T be a ring and G an automorphism group of T. Then (1) $V_T(T^G)$ is a G-invariant subring of T and (2) $(V_T(T^G))^G$ is contained in the center of $V_T(T^G)$ (hence $V_T(T^G)$ is an algebra over $(V_T(T^G))^G$).*

Lemma 5.2. ([24], Lemma 3.2) *Let B be a Galois extension of B^G with Galois group G and A a G-invariant subring of B under the action of G. If A is a Galois extension of B^G with Galois group induced by and isomorphic with G, then $A = B$.*

Theorem 5.3. ([24], Theorem 3.3) *Let B be a Galois extension of B^G with Galois group G, $\Delta = V_B(B^G)$, and $D = \Delta^G$. Then the following statements are equivalent: (1) Δ is a Galois algebra over D with Galois group induced by and isomorphic with G/N where $N = \{g \in G \mid g(x) = x$ for all $x \in \Delta\}$. (2) $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with G/N and Δ is a finitely generated and projective module over D. (3) B is a composition of two Galois extensions: $B \supset B^G\Delta$ with Galois group N and $B^G\Delta \supset B^G$ with Galois group induced by and isomorphic with G/N such that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over D for each $\overline{g} \in G/N$ where $J_{\overline{g}}^{(\Delta)} = \{b \in \Delta \mid bx = g(x)b$ for all $x \in \Delta\}$.*

Proof. (1) $\Longrightarrow$ (2) Since the automorphism groups induced by G/N on $B^G\Delta$ and Δ are isomorphic and Δ is a Galois algebra over D where $D = \Delta^G$, $B^G\Delta$ is a Galois extension of $(B^G\Delta)^G$ $(= B^G)$ with Galois group induced by and isomorphic with G/N.

(2) $\Longrightarrow$ (1) Since $B^G\Delta \supset B^G$ is a Galois extension with Galois group induced by and isomorphic with G/N, the crossed product

$$(B^G\Delta) * (G/N) \cong \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta).$$

Denoting G/N by $\overline{G}$, we have that

$$\alpha : (B^G\Delta) * \overline{G} \cong \mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)$$

by $(\alpha(\sum_{\overline{g}\in\overline{G}} a_{\overline{g}}\overline{g}))(x) = \sum_{\overline{g}\in\overline{G}} a_{\overline{g}}\overline{g}(x)$ for each $x \in B^G\Delta$. Then

$$\Delta * \overline{G} = V_{B^G\Delta*\overline{G}}(B^G) \cong V_{\mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)}(\alpha(B^G)).$$

It can be verified that $V_{\mathrm{Hom}_{B^G}(B^G\Delta, B^G\Delta)}(\alpha(B^G)) = \mathrm{Hom}_D(\Delta, \Delta)$ where $D = \Delta^{\overline{G}} = \Delta^G$. But Δ is a finitely generated and projective module over D, so Δ is a Galois algebra over D with Galois group isomorphic with $\overline{G}$.

(2) $\Longrightarrow$ (3) Since $B^G\Delta \subset B^N$ such that $(B^G\Delta)^G = B^G = (B^N)^G$ and $B^G\Delta$ is a Galois extension of B^G with Galois group induced by and isomorphic with $\overline{G}$ $(= G/N)$, $B^N = B^G\Delta$ by Lemma 5.2. Moreover, noting that $V_{B^G\Delta}(B^G) = \Delta = \oplus \sum_{\overline{g}\in\overline{G}} J_{\overline{g}}^{(\Delta)}$ ([12], Proposition 1 and Theorem 1), we conclude that $J_{\overline{g}}^{(\Delta)}$ is a finitely generated projective module over D for each $\overline{g} \in G/N$.

(3) $\Longrightarrow$ (2) is clear.

By Theorem 5.3, we shall derive some consequences for several well known classes of Galois extensions. We recall that B is a center Galois extension with Galois group G if its center C is a Galois algebra over C^G with Galois group $G|_C \cong G$, and B is a commutator Galois extension of B^G with Galois group G if $V_B(B^G)$ is a Galois extension of $(V_B(B^G))^G$ with Galois group $G|_{V_B(B^G)} \cong G$.

Corollary 5.4. *Let B be a Galois extension of B^G with Galois group G. If $B = B^GC$ such that C is finitely generated and projective over C^G, then B a center Galois extension with Galois group G.*

Corollary 5.5. *Let B be a Galois extension of B^G with Galois group G. If $B = B^G\Delta$ such that Δ is finitely generated and projective over Δ^G, then B a commutator Galois extension with Galois group G.*

Remark. Since a DeMeyer-Kanzaki Galois extension is also a center Galois extension ([4], Lemma 2) and an Azumaya Galois extension is a commutator Galois extension ([1], Theorem 2), Corollary 5.4 and Corollary 5.5 hold for the classes of DeMeyer-Kanzaki Galois extensions and Azumaya Galois extensions.

Corollary 5.6. *Let B be a Hirata separable Galois extension of B^G with Galois group G. If $B = B^G\Delta$, then Δ is a Galois algebra with Galois group induced by and isomorphic with G/N.*

Proof. Since B is a Hirata separable Galois extension of B^G with Galois group G, J_g is a finitely generated and projective rank one module over C^G for each $g \in G$ ([17], Theorem 2). The corollary holds by Theorem 5.3.

We continue to characterize a Galois commutator subring Δ in terms of crossed products.

Theorem 5.7. *Keeping the notations of Theorem 5.3, the following statements are equivalent: (1) Δ is a Galois algebra over Δ^G with Galois group induced by and isomorphic with G/N where $N = \{g \in G \mid g(x) = x$ for all $x \in \Delta\}$. (2) Let $\Delta * (G/N)$ be the crossed* *product of G/N over Δ with trivial factor set. Then $\Delta * (G/N)$ is an Azumaya algebra over Δ^G. (3) Let $(B^G\Delta) * (G/N)$ be the crossed product of G/N over $B^G\Delta$ with trivial factor set. Then $(B^G\Delta) * (G/N)$ is a Hirata separable extension of B^G such that B^G is a direct summand of $(B^G\Delta) * (G/N)$ as a B^G-bimodule.*

Proof. (1) $\Longrightarrow$ (2) Since Δ is a Galois algebra over Δ^G with Galois group $\overline{G}$ induced by and isomorphic with G/N, $\Delta * \overline{G} \cong \mathrm{Hom}_{\Delta^G}(\Delta, \Delta)$ where Δ is a finitely generated and projective module over Δ^G. Noting that Δ is an algebra with 1 over Δ^G, we have that $\mathrm{Hom}_{\Delta^G}(\Delta, \Delta)$ is an Azumaya algebra over Δ^G. Hence $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G.

(2) $\Longrightarrow$ (1) By hypothesis, $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G, so $\Delta * \overline{G}$ is a Hirata separable extension of Δ ([8], Theorem 1). Since Δ is a progenerator of Δ, Δ is a progenerator of $\Delta * \overline{G}$. Thus Δ is a Galois algebra over Δ^G with Galois group isomorphic with $\overline{G}$.

(2) $\Longrightarrow$ (3) Since $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G, $B^G \otimes_{\Delta^G} (\Delta * \overline{G})$ is a Hirata separable extension of B^G; and so, as a homomorphism image of $(B^G \otimes_{\Delta^G} \Delta) * \overline{G}$, $(B^G\Delta) * \overline{G}$ is also a Hirata separable extension of B^G. Since $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G again, Δ is a Galois algebra over Δ^G with Galois group $\overline{G}$ by (2) $\Longrightarrow$ (1). Hence there exists an element $d \in \Delta$ such that $\mathrm{tr}_{\overline{G}}(d) = 1$ ([12], proof of Proposition 5) where $\mathrm{tr}_{\overline{G}}(\ ) = \sum_{\overline{g} \in \overline{G}} \overline{g}(\ )$. Thus $\mathrm{tr}_{\overline{G}}(\ ) : B^G\Delta \longrightarrow B^G \longrightarrow 0$ is exact as B^G-bimodule homomorphism, and so B^G is a direct summand of $B^G\Delta$ as B^G-bimodule homomorphism. Noting that $B^G\Delta$ a direct summand of $(B^G\Delta) * \overline{G}$ as a B^G-bimodule, we conclude that so is B^G.

(3) $\Longrightarrow$ (2) Since $(B^G\Delta) * \overline{G}$ is a Hirata separable extension of B^G such that B^G is a direct summand of $(B^G\Delta) * \overline{G}$ as a B^G-bimodule, $V_{(B^G\Delta) * \overline{G}}(B^G)$ is a separable algebra over the center of $(B^G\Delta) * \overline{G}$ ([16], Theorem 1). But $V_{(B^G\Delta) * \overline{G}}(B^G) = \Delta * \overline{G}$, so $\Delta * \overline{G}$ is a separable algebra over the center of $(B^G\Delta) * \overline{G}$. We claim that the centers of $\Delta * \overline{G}$ and $(B^G\Delta) * \overline{G}$ are Δ^G. In fact, by hypothesis, $(B^G\Delta) * \overline{G}$ is a Hirata separable extension of B^G such that B^G is a direct summand of $(B^G\Delta) * \overline{G}$ as a B^G-bimodule again, $V_{(B^G\Delta) * \overline{G}}(V_{(B^G\Delta) * \overline{G}}(B^G)) = B^G$ ([16], Theorem 1). Hence the center of $(B^G\Delta) * \overline{G}$ is contained in B^G; and so it is contained in the center of B^G. Conversely, the center of B^G is clearly contained in the center of $(B^G\Delta) * \overline{G}$. Thus, the center of $(B^G\Delta) * \overline{G}$ is equal to the center of B^G. Moreover, since the center of B^G is Δ^G, the center of $(B^G\Delta) * \overline{G}$ is Δ^G. But the centers of $\Delta * \overline{G}$ and $(B^G\Delta) * \overline{G}$ are the same, so the center of $(B^G\Delta) * \overline{G}$ is Δ^G. Therefore, $\Delta * \overline{G}$ is an Azumaya algebra over Δ^G.

Corollary 5.8. *Let B satisfy the equivalent conditions of Theorem 5.7. Then $N = \langle 1 \rangle$ if and only if $B = B^G\Delta$ such that $\Delta^G = C^G$ where C is the center of B.*

Corollary 5.9. *Let B satisfy the equivalent conditions of Theorem 5.7. If $\bar{N}$ is a maximal subgroup of G, then Δ is a commutative Galois algebra over Δ^G with a cyclic Galois group G/N ([4], Theorem 11).*

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