EXTENSION OF THE MATLIS DUALITY TO A FILTERED NOETHERIAN RING

N.KAMEYAMA*

Abstract. A ring theoretic investigation of the Iwasawa algebra is accomplished. Therefore, we look at a filtered pseudocompact algebra (abbreviation:FPC algebra) which is a reasonable generalization of the Iwasawa algebra (1.1). It is shown that an FPC algebra has the Matlis duality between suitable categories. When an FPC algebra is Auslander regular and with homogeneity condition, we study the local cohomology and local dualty.

Key Words: Iwasawa algebra, pseudocompact algebra, local cohomology, local duality.

2000 *Mathematics Subject Classification*: 13D45, 16E30, 16E65, 16P40

1. Introduction

This paper is a summary of [12]. A class of (non)commutative Iwasawa algebras, studied as main objects in Iwasawa theory, occupies quite interesting position in that of noncommutative Noetherian rings. Moreover, they possess a filtered ring structure which is an algebraic device of topological notion. In the present paper, we study ring theoretic properties of Iwasawa algebras, through a filtered pseudocompact algebra, FPC algebra, for short.

Let us explain essential properties of Iwasawa algebras shortly. Let p be a prime number, and G a compact p-adic analytic group. The Iwasawa algebra is defined by

$$\Lambda(G) := \varprojlim \mathbb{Z}_p[G/U],$$

where U ranges over all open normal subgroups of G. A key fact for us is the following. Assume that G is a uniform pro-p group. Then $\Lambda(G)$ is a right and left Noetherian ring ([6], Corollary 7.25) and local Auslander-regular domain ([2], 4.1, 4.3, 5.1, 5.2). It has a J-adic filtration $F\Lambda(G)$, where $J = \mathrm{rad}\Lambda(G)$, with $F_i\Lambda(G) := J^{-i}$ $(i < 0)$, $= \Lambda(G)$ $(i \geq 0)$. $\Lambda(G)$ is complete with respect to this filtration ([2], 3.5). Suppose that G is a p-valued compact p-adic Lie group or a uniform extra-powerful pro-p group, then the filtration $F\Lambda(G)$ is Zariskian ([5], §7 or [20], Theorem 3.22, see also [9], Chapter II §2, 2.1.2 Theorem (4)). For these cases, $\Lambda(G)$ is a typical example of an FPC algebra.

We can say that a reasonably generalized algebra of the Iwasawa algebra is a pseudocompact algebra due, for example, to [3], [19]. There is a duality between the category of pseudocompact Γ-modules and that of discrete Γ-modules for a pseudocompact algebra Γ. This is a basic result for homological study of such algebras. To begin with, we make this duality over for the suitable categories over an FPC algebra (see 1.4). Then we study the local cohomology and local duality over such algebras. This provides a generalization

The detailed version of this paper will be submitted for publication elsewhere.

of [20], §§5,6. Further, we expect that the homological properties such that Bass number, Gorensteiness etc. are within view as module-finite algebras [8].

2. Duality over a filtered pseudocompact algebra

2.1. **Assumption.** Let Λ be a left and right Noetherian filtered ring with a Zariskian filtration $F\Lambda = \{F_i\Lambda\}_{i\in\mathbb{Z}}$ ([9], Chapter II, §2) such that

(a1) $H_i = F_i\Lambda$ is an ideal of Λ for every $i \in \mathbb{Z}$,
(a2) Λ is complete with respect to $F\Lambda$,
(a3) Λ/H_i is of finite length as a right and left Λ-module for every $i \in \mathbb{Z}$.

For further use, it is desirable that Λ is an algebra over a commutative ring. Let $(R, \mathfrak{m}, k)$ be a commutative local Noetherian ring and Λ an R-algebra. We consider that R is a subring of Λ via a structure map $R \to \Lambda$.

Put $I_i := R \cap H_i (i \in \mathbb{Z})$ and $FR = \{I_i\}(i \in \mathbb{Z})$. Then FR is a filtration of R. We assume that

(b1) R is complete with respect to FR,
(b2) R/I_i is a finite length R-module for every $i \in \mathbb{Z}$,
(b3) $\mathfrak{m}^n$ is open for all $n > 0$, i.e., $\mathfrak{m}^n \supset I_i$ for some $i \in \mathbb{Z}$,
(b4) Λ/H_i is a module-finite R/I_i-algebra for every $i \in \mathbb{Z}$, i.e., Λ/H_i is a finitely generated R/I_i-module.

We call an R-algebra satisfying all above assumptions a *filtered pseudocompact algebra* and FPC algebra for short. Moreover, Λ/H_i is a finite length R-module for every i. Therefore all finite length Λ-modules are finite length R-modules.We sometimes consider a filtered Λ-module (M, FM) as a filtered R-module with the same filtration FM, but regard as an R-module. We assume that all filtrations are separated.

Let $E := E_R(k)$ be an injective hull of k as an R-module. It follows that E is an injective cogenerator of Mod$R$. Put $E_i := \{x \in E | I_{-i}x = 0\}$ an R-submodule of E for every $i \in \mathbb{Z}$. The assumption (b3) and [13], Theorem 18.4 implies $E = \cup E_i$, so E is a filtered R-module with a filtration $FE = \{E_i\}_{i\in\mathbb{Z}}$.

2.2. **Filtration and filtration topology.** Let R be a filtered ring and M, N filtered R-modules. Let $F_p\mathrm{HOM}_R(M, N) = \{f \in \mathrm{Hom}_R(M, N) | f(F_iM) \subset F_{i+p}N$ for all $i \in \mathbb{Z}\}$. Put $\mathrm{HOM}_R(M, N) := \bigcup_{p\in\mathbb{Z}} F_p\mathrm{HOM}_R(M, N)$.

In some cases, all homomorphisms are of finite degree. In particular, the following will be used frequently.

Proposition 1. *Let M be a filtered R-module with a filtration $FM = \{F_iM\}_{i\in\mathbb{Z}}$. Assume that M is of finite length. Then* $\mathrm{Hom}_R(M, E) = \mathrm{HOM}_R(M, E)$.

2.3. **Pseudocompact modules and copseudocompact modules.** We put the category $\mathcal{F}_\Lambda$ as follows,

Objects: all filtered Λ-modules,
Morphisms: all Λ-homomorphisms of finite degree, i.e.,
the elements of $\mathrm{HOM}_\Lambda(M, N)$ for M, $N \in \mathcal{F}_\Lambda$.

We put $M^\vee := \mathrm{HOM}_R(M,E)$ by regarding M as a filtered R-module with a filtration $\{F_iM\}_{i\in\mathbb{Z}}$. Then $(-)^\vee = \mathrm{HOM}_R(-,E)$ turns out to be a contravariant functor between $\mathcal{F}$ and $\mathcal{F}^{op}$. We put $(-)' := \mathrm{Hom}_R(-,E)$, which induces usual Matlis Duality. Let M be a filtered Λ-module with a filtration FM. We call M *pseudocompact*, if $M \cong \varprojlim M/F_iM$, that is, M is complete ([9], Chapter I, §3, 3.5) and $H_iM \subset F_iM$ for every $i \in \mathbb{Z}$ (cf. [3], [19]). Dually, a filtered Λ-module N with a filtration FN is called *copseudocompact*, if $N \cong \varinjlim F_iN$ and $H_{-i}F_iN = 0$ for every $i \in \mathbb{Z}$.

Proposition 2. *Let $M, N \in \mathcal{F}_\Lambda$. Then*
(1) If M is pseudocompact, then $M^\vee \cong \varinjlim (M/F_iM)^\vee$.
(2) If N is copseudocompact, then $N^\vee \cong \varprojlim (F_iN)^\vee$.

2.4. **Duality.** Let $\mathcal{C}$ be a full subcategory of $\mathcal{F}_\Lambda$ consisting of all finitely generated pseudocompact Λ-modules, and $\mathcal{D}$ a full subcategory of $\mathcal{F}_\Lambda$ consisting of all finitely cogenerated copseudocompact Λ-modules. Here, a module is finitely cogenerated if and only if its socle is essential and finitely generated (cf. [1], Proposition 10.7).

Theorem 3. *Let M, $N \in \mathcal{F}_\Lambda$. Then*
(1) If M is pseudocompact, then $M^\vee$ is copseudocompact.
(2) If N is copseudocompact, then $N^\vee$ is pseudocompact.
(3) $\Lambda \cong \Lambda^{\vee\vee}$ and $\Lambda^\vee$ is Artinian.

Theorem 4. *Let M, $N \in \mathcal{F}_\Lambda$. Then*
(1) If $M \in \mathcal{C}$ then $M^\vee \in \mathcal{D}$ and $M^{\vee\vee} \cong M$.
(2) If $N \in \mathcal{D}$ then $N^\vee \in \mathcal{C}$ and $N^{\vee\vee} \cong N$.

Proof. (1): Since M is finitely generated, there is an epimorphism $f : \Lambda^n \to M$. Dualizing it, we have a monomorphism $f^\vee : M^\vee \to \Lambda^{\vee n}$, so $M^\vee$ is Artinian, and $M^\vee \in \mathcal{D}$. We see

$$M^{\vee\vee} \cong \varprojlim (M/F_iM)^{\vee\vee} \cong \varprojlim (M/F_iM)'' \cong \varprojlim M/F_iM \cong M.$$

(2)similarly. □

3. Local cohomology

3.1. **Depth and Auslander-Buchsbaum Formula.** We assume that a FPC algebra Λ is
1) Auslander regularity with gl.dimΛ=d ([9], Chapter Ⅲ, §2, 2.1.7),
2) the homogeneity condition,
where the homogeneity condition (cf. [7], (hc13) and (hc14), p.326) is that every simple left (respectively, right) Λ-module is contained in E^d (respectively, E'^d), where $0 \to \Lambda \to E^0 \to E^1 \to \cdots \to E^d \to 0$ (respectively, $0 \to \Lambda \to E'^0 \to E'^1 \to \cdots \to E'^d \to 0$) is a minimal injective resolution of Λ as a left (respectively, right) Λ-module. Let $J = \mathrm{rad}\Lambda$ be a Jacobson radical of Λ.

Definition 5. The *depth* of a finitely generated Λ-module M is defined by

$$\mathrm{depth}M := \min\{i \geq 0 | \mathrm{Ext}^i_\Lambda(\Lambda/J, M) \neq 0\}.$$

It equals ∞, whenever $\mathrm{Ext}^i_\Lambda(\Lambda/J, M) = 0$ for all $i \geq 0$.

The direct consequence of homogeneity condition is the following determination of depthΛ.

Proposition 6. *It holds that* depth$\Lambda = d$.

Theorem 7. *Let* $M \in \mathrm{mod}\Lambda$. *Then*

$$\mathrm{pd}M + \mathrm{depth}M = \mathrm{depth}\Lambda = d.$$

3.2. **Local cohomology and local duality.** Let M be a Λ-module. Put $\Gamma(M) := \{x \in M | H_{-i}x = 0 \text{ for some } i \geq 0\}$. Then Γ is a left exact additive functor: $\mathrm{Mod}\Lambda \to \mathrm{Mod}\Lambda$ such that $\Gamma(M) \cong \varinjlim \mathrm{Hom}_\Lambda(\Lambda/H_{-i}, M)$.

Definition 8. The local cohomology functors, denoted by $H^i(-)$, are the right derived functors of $\Gamma(-)$.

The following lemma is indispensable for proving the important property of local cohomology modules.

We can determine the structure of the last term E^d of a minimal injective resolution of Λ.

Proposition 9. *Let* $M \in \mathrm{Mod}\Lambda$. *Then, for any* $i \geq 1$,
(1) $H^i(M)$ *is a copseudocompact module for some filtration. Moreover, if* M *is finitely generated, then* $H^i(M) \in \mathcal{D}$,
(2) $H^i(M) \cong \varinjlim \mathrm{Ext}^i_\Lambda(\Lambda/H_{-p}, M)$.

As is usually done, we describe depth using the local cohomology modules.

Theorem 10. *Let* M *be a finitely generated* Λ*-module. Then*

$$\mathrm{depth}\, M = \min\{i \geq 0 : H^i(M) \neq 0\}.$$

Proof. The proof is done by modifying that of [20], Lemma 5.5. Note that $H_{-p} \subset J$ for every $p > 0$. □

We also observe that $\Lambda^\vee$ is copseudocompact. We write $X|Y$, When X is isomorphic to a direct summand of copies of Y. Then by the above corollary, we see $E^d|\Lambda^\vee$ and $\Lambda^\vee|E^d$. As concerns the local cohomology module of Λ, we see

Proposition 11. *There is an isomorphism* $H^d(\Lambda) \cong E^d$.

Proposition 12. *Assume that* Λ *is basic. Then there is the isomorphisms* $\Lambda^\vee \cong E^d \cong H^d(\Lambda)$.

Proposition 13. *Assume that* Λ *is basic. Let a* Λ*-module* M *be in* $\mathcal{C}$ *or* $\mathcal{D}$. *Then* $\mathrm{Hom}_\Lambda(M, H^d(\Lambda)) \cong M^\vee$.

We establish the local duality theorem using the above results. All the assumptions for Λ given before are preserved, that is, 1.1, 2.1 and to be basic.

Theorem 14. *(Local duality) Let* M *be an arbitrary finitely generated* Λ*-module. Then, for all integers* i, *there are natural isomorphisms*

$$H^i(M) \cong \mathrm{Ext}^{d-i}_\Lambda(M, \Lambda)^\vee \quad \textit{and}$$
$$\mathrm{Ext}^i_\Lambda(M, \Lambda) \cong H^{d-i}(M)^\vee.$$

Proof. Using the above preparation, we can show the statement by the similar way to the commutative case. □

References

1. F.W. Anderson and K.R. Fuller, *Rings and categories of modules*, GTM13, Springer,1992.
2. K. Ardakov and K.A. Brown, *Ring-theoretic properties of Iwasawa algebras: a survey*, Documenta Math. Extra vol. Coates, 7-33(2006).
3. A. Brumer, *Pseudocompact algebras, profinite groups and class formations*, J. Alg. 4, 442-470(1966).
4. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge stud. Adv. Math. 39, Cambridge Univ. Press,1993.
5. J. Coates, P. Schneider and R. Sujatha, *Modules over Iwasawa algebras*, J. Inst. Math. Jussieu 2, 73-108(2003).
6. J.D. Dixon, M.P.F. Du Sautoy, A. Mann and D. Segal, *Analytic pro-p groups*, Cambridge stud. Adv. Math. 61, Cambridge Univ. Press,1999.
7. S. Goto and K. Nishida, *Finite modules of finite injective dimension over a Noetherian algebra*, J.London Math. Soc, 63, 319-335(2001)
8. S. Goto and K. Nishida, *Towards a theory of Bass numbers with application to Gorenstein algebras*, Colloq. Math. 91(2), 191-253(2002).
9. L. Huishi and F. Van Oystaeyen, *Zariskian filtrations*, K-Monograph in Mathematics, 2, 1996.
10. Y. Iwanaga, *Duality over Auslander-Gorenstein rings*, Math. Scand. 81(2), 184-190(1997).
11. Y. Iwanaga and H. Sato, *On Auslander's n-Gorenstein rings*, J. Pure Appl. Alegbra, 106, 61-76(1996).
12. N. Kameyama, H. Miyahara and K, Nishida, *Local Duarity for noncommutative Iwasawa algebra*, preprint.
13. H. Matsumura, *Commutative ring theory*, Cambridge stud. Adv. Math. 8, Cambridge Univ. Pres, 1989.
14. C. Năstăsescu, F. Van Oystaeyen, *Graded ring theory*, Math. Library, 28, North Holland, Amsterdam, 1982.
15. J. Neukirch, S. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren math. Wiss., vol. 323, Springer, Berlin, Heidelberg, New York, 2000
16. A. Neumann, *Completed group algebras without zero divisors*, Arch. Math. (Basel) 51, 496-499(1988).
17. J. Rotman, *An introduction to homological algebra*, Academic Press, 1979, *second edition*, Springer, 2009.
18. Bo Stenström, *Rings of quotients*, Springer, 1975.
19. M. Van Gastel and M. Van den Bergh, *Graded modules of Gelfand-Kirillov dimension one over three-dimensional Artin-Schelter regular algebras*, J. Alg. 196, 251-282(1997).
20. O. Venjakob, *On the structure theory of the Iwasawa algebra of a p-adic Lie group*, J. Eur. Math. Soc.4, 271-311(2002).

(*) Graduate School of Science and Technology, Shinshu University, Matsumoto, Japan