ALMOST COMPARABILITY AND RELATED COMPARABILITIES IN VON NEUMANN REGULAR RINGS

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ABSTRACT. There are many comparabilities in von Neumann regular rings: general comparability, the comparability axiom, s-comparability, weak comparability, almost comparability etc.. In the article, we mainly investigate von Neumann regular rings satisfying almost comparability, comparing with other comparabilities.

Key Words: Von Neumann regular rings, Comparability.

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1. INTRODUCTION

In the article, we mainly study regular rings satisfying almost comparability, comparing with other related comparabilities: general comparability, the comparability axiom, s-comparability, weak comparability. In section 1, we give definitions and histories of the above related comparabilities. We begin with some notations and elementary definitions which will be needed in the article. For details, we can refee Goodearl's book [5].

Throughout this article, R is a ring with identity and R-modules are unitary right R-modules.

Notation 1. For two R-modules M, N, we use $M \lesssim N$ (resp. $M \lesssim_{\oplus} N$, $M \prec N$, $M \prec_{\oplus} N$) to mean that there exists an isomorphism from M to a submodule of N (resp. a direct summand of N, a proper submodule of N, a proper direct summand of N). For a submodule M of an R-module N, $M \leq_{\oplus} N$ (resp. $M < N, M <_{\oplus} N$) means that M is a direct summand of N (resp. a proper submodule of N, a proper direct summand of N). For a cardinal number k and an R-module M, kM denotes the direct sum of k-copies of M.

Definition 2. A ring R is said to be (von Neumann) *regular* if, for each $x \in R$, there exists an element y of R such that $xyx = x$, and a ring R is said to be *unit-regular* if, for each $x \in R$, there exists a unit element (i.e. an invertible element) u of R such that $xux = x$. It is well-known that a regular ring R is unit-regular if and only if $A \oplus B \cong A \oplus C$ implies $B \cong C$ for any finitely generated projective R-modules A, B, C. An R-module M is *directly finite* provided that M is not isomorphic to a proper direct summand of itself. A ring R is *directly finite* if the R-module R_R is directly finite, and R is said to be *stably finite* if the ring $M_n(R)$ of $n \times n$ matrices over R is directly finite for all positive

This paper is based on the author's talk and the detailed proof of some results in this paper will be submitted for publication elsewhere.

integers n. It is known that a ring R is stably finite if and only if every finitely generated projective R-module is directly finite.

Now, we recall definitions and histories of the related comparabilities.

Definition 3. A regular ring R satisfies *general comparability* if, for each $x, y \in R$, there exists a central idempotent $e \in R$ such that $e(xR) \lesssim e(yR)$ and $(1-e)(yR) \lesssim (1-e)(xR)$.

General comparability is the typical and oldest comparability, which evolved from operator algebras and Baer rings. All right self-injective regular rings are typical examples of regular rings with this comparability, which worked usefully to study these regular rings.

Definition 4. A regular ring R is said to satisfy *the comparability axiom* if, for each $x, y \in R$, either $xR \lesssim yR$ or $yR \lesssim xR$.

The comparability axiom is a special case of general comparability, which means that "each two principal right ideals are comparable". The notion was given by K.R. Goodearl and D. Handelman in 1975. All prime right self-injective regular rings are typical examples of regular rings with this comparability, and they investigated these regular rings using the comparability axiom.

Definition 5. Let s be a positive integer. A regular ring R is said to satisfy *s-comparability* if, for each $x, y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$. Note that 1-comparability means the comparability axiom above. It is well-known in [4] that s is either 1 or 2 only for any regular rings with s-comparability.

Connecting with the comparability axiom, s-comparability was also given by K.R. Goodearl and D. Handelman in 1976 to characterize uniqueness of rank functions on certain simple regular rings. But, the detailed study of regular rings with s-comparability became after one of regular rings with weak comparability. In the study of regular rings, there is a famous outstanding Open Problem: Is every directly finite simple regular ring always unit-regular? To solve the problem, K.C. O'Meara gave the notion of weak comparability and some interesting result, as follows.

Definition 6 ([11])**.** A regular ring R satisfies *weak comparability* if, for each nonzero $x \in R$, there exists a positive integer n such that $n(yR) \lesssim R_R$ implies $yR \lesssim xR$ for all $y \in R$, where the n depends on x.

Theorem 7 ([11])**.** *Every directly finite simple regular rings with weak comparability are unit-regular.*

After that a criterion of weak comparability for simple regular rings was given, as follows.

Theorem 8 ([3])**.** *For a simple regular ring R, the following are equivalent*:

(a) *R has weak comparability.*

(b) *$nA \prec nB$ implies $A \prec B$ for any finitely generated projective R-modules A, B and any positive integer n.*

2. Almost comparability

In Section 2, we give some fundamental results of almost comparability for finitely generated projective modules over regular rings satisfying almost comparability. We begin with the history for almost comparability. The notion of almost comparability for regular rings was first introduced by Ara and Goodearl [1], for giving an alternative proof of O'Meara's Theorem that every directly finite simple regular rings with weak comparability are unit-regular (Theorem 7). After that the study of almost comparability for simple regular rings was continued by Ara et al. [3], who showed that, for simple regular rings, s-comparability for some positive integer s is equivalent to the ring satisfying almost comparability. Thus directly finite simple regular rings satisfying almost comparability are unit-regular, from a result in O'Meara [11]. Also, Ara et al. [4] studied regular rings with s-comparability, and fixed the relation between s-comparability and almost comparability giving some examples. Here we give the definition of almost comparability, as follows.

Definition 9 ([1])**.** A regular ring R satisfies *almost comparability* if, for each $x, y \in R$, either $xR \lesssim_a yR$ or $yR \lesssim_a xR$, where $xR \lesssim_a yR$ (called "almost subisomorphic") means that $xR \lesssim yR \oplus C$ for all nonzero principal right ideals C of R.

From the definition of almost comparability, we see that "1-comparability $\Rightarrow$ almost comparability $\Rightarrow$ 2-comparability" obviously. Thus almost comparability is a middle condition between 1-comparability and 2-comparability. But the converse implications do not hold from the following examples.

Example 10 ([4])**.**
(1) There exists a non-simple stably finite regular ring satisfying almost comparability which is not unit-regular. Hence the ring does not satisfy 1-comparability.
(2) There exists a unit-regular ring with 2-comparability but not almost comparability.

Now we investigate the properties for regular rings satisfying almost comparability, comparing with 1-comparability or 2-comparability. First we ask if almost comparability for regular rings is Morita invariant. To see this, we need the definition of almost comparability for finitely generated projective modules, as follows.

Definition 11. Let R be a regular ring, and P be a finitely generated projective R-module. Then P satisfies *almost comparability* if, for each direct summands A, B of P, either $A \lesssim_a B$ or $B \lesssim_a A$, where $A \lesssim_a B$ means that $A \lesssim B \oplus C$ for all nonzero principal right ideals C of R. Also, P satisfies *strictly almost comparability* if, for each direct summands A, B of P, either $A \prec_a B$ or $B \prec_a A$, where $A \prec_a B$ means that $A \prec B \oplus C$ for all nonzero principal right ideals C of R.

For the above definitions, we can tell that the notion of almost comparability is the just same as one of strictly almost comparability as below, which result can be used as a criterion for almost comparability.

Lemma 12. *Let R be a regular ring, and P be a finitely generated projective R-module. Then the following are equivalent:*
(a) *P satisfies almost comparability.*

(b) *P satisfies strictly almost comparability.*

Lemma 13. *Let R be a regular ring satisfying almost comparability. For every nonzero finitely generated projective R-modules A, B, there exists a nonzero principal right ideal X of R such that both $X \lesssim A$ and $X \lesssim B$. In particular, if S is a simple right ideal of R, then $S \lesssim M$ for all nonzero finitely generated projective R-modules M.*

Here, we recall the definition of separativity for a ring and its criterion, which were born in the study of s-comparability and will be used in proofs of the results after.

Definition 14 ([2])**.** A ring R is *separative* if $A \oplus A \cong A \oplus B \cong B \oplus B$ implies $A \cong B$ for any finitely generated projective R-modules A, B.

Lemma 15 ([2])**.** *For a ring R, the following are equivalent*:
(a) *R is separative.*
(b) *For any finitely generated projective R-modules A, B, C, if $A \oplus C \cong B \oplus C$ with $C \lesssim_{\oplus} mA$ and $C \lesssim_{\oplus} nB$ for some positive integers m, n, then $A \cong B$.*

We also recall the definition of exchange rings.

Definition 16. A ring R is said to be *an exchange ring* if the R-module R satisfies the exchange property, where an R-module M *satisfies the exchange property* if for every R-module A and any decompositions $A = M' \oplus N = \oplus_{i \in I} A_i$ with $M' \cong M$, there exist submodules $A'_i \leq A_i$ such that $A = M' \oplus (\oplus_{i \in I} A'_i)$. It is known that regular rings are typical examples of exchange rings.

For exchange rings with s-comparability, we recall an interesting result as follows.

Lemma 17 ([12])**.** *Any exchange ring with s-comparability is separative. Thus any regular ring satisfying almost comparability is separative.*

Using Lemmas 12,13,15,17 above, we can obtain the following result.

Proposition 18. *Let R be a regular ring, and assume that nR satisfies almost comparability for some positive integer n. Let A, B, C, D be finitely generated projective R-modules, all which are subisomorphic to nR.*
(1) *If $A \prec_a C$ and $B \prec_a D$, then $A \oplus B \prec_a C \oplus D$.*
(2) *If $A \prec_a C$ and $B \prec_a D$, then either $A \oplus D \prec_a B \oplus C$ or $B \oplus C \prec_a A \oplus D$.*

Almost comparability is inherited by direct summands. Hence, using Proposition 18 above and the mathematical induction, we have the following result.

Proposition 19. *Let R be a regular ring. Then the following are equivalent*:
(a) *R satisfies almost comparability.*
(b) *Any finitely generated projective R-module satisfies almost comparability.*
(c) *nR satisfies almost comparability for all positive integers n.*
(d) *There exists a positive integer n such that nR satisfies almost comparability.*

By the way, we have almost subisomorphic relations of the family of all finitely generated submodules between a finitely generated projective R-module over a regular ring R and its endomorphism ring, as Lemma 20 below shows. To see this, for an R-module M_R, we put $add(M_R) = \{$an R-module $N \mid N \lesssim_{\oplus} nM$ for some positive integer $n\}$. Then we see that the lemma follows from equivalences of the Hom and Tensor functors by $Hom_R({}_SM_R, -)$ and $- \otimes_S {}_SM_R$ between the categories $add(M_R)$ and $add(S_S)$, where $S = End_R(M)$.

Lemma 20. *Let R be a regular ring, and P be a finitely generated projective R-module. Set a ring $T = End_R(P)$. Then the following are equivalent:*
(a) *P satisfies almost comparability.*
(b) *T satisfies almost comparability.*

Combining Proposition 19 with Lemma 20, we can answer whether almost comparability for regular rings is Morita invariant, as follows.

Theorem 21. *Let R be a regular ring. Then the following are equivalent*:
(a) *R satisfies almost comparability.*
(b) *For any finitely generated projective R-module P, $End_R(P)$ satisfies almost comparability.*
(c) *Any ring S which is Morita equivalent to R satisfies almost comparability.*
(d) *For all positive integers n, $M_n(R)$ satisfies almost comparability.*
(e) *There exists a positive integer n such that $M_n(R)$ satisfies almost comparability.*

Also we can show Theorem 22 below, from Proposition 19.

Theorem 22. *Let R be a regular ring satisfying almost comparability. Then the family of all finitely generated projective R-modules satisfies almost comparability, which means that either $A \prec_a B$ or $B \prec_a A$ for any finitely generated projective R-modules A, B.*

In addtion, more generally, we can extend almost comparability for the family of all finitely generated projectives to the family of all countably generated projectives, as follows.

Theorem 23. *Let R be a regular ring satisfying almost comparability. Then the family of all countably generated projective R-modules satisfies almost comparability, which means that either $P \prec_a Q$ or $Q \prec_a P$ for any countably generated projective R-modules P, Q.*

The results in Theorems 22,23 above can be used in §3.

3. Cancellation and unperforation

In Section 3, we treat the cancellation and unperforation properties for regular rings satisfying almost comparability. As we mentioned in §2, any directly finite simple regular rings satisfying almost comparability are unit-regular, but there exists a non-simple stably finite regular ring R satisfying almost comparability but not unit-regularity, from which $A \oplus C \lesssim B \oplus C$ does not imply $A \lesssim B$ for some finitely generated projective R-modules A, B, C. Thus, instead of the above property, we consider *the strict cancellation property* for a regular ring R which means that $A \oplus C \prec B \oplus C$ implies $A \prec B$ for any finitely generated projective R-modules A, B, C. Obviously, any unit-regular rings always have the strict cancellation property. First, we ask if any directly finite regular rings satisfying almost comparability have the strict cancellation property. Then we can show the following result.

Theorem 24. *Let R be a regular ring satisfying almost comparability, and A, B, C be directly finite and finitely generated projective R-modules. If $A \oplus C \prec B \oplus C$, then $A \prec B$.*

From the above, we have the following Corollary 25 as desired.

Corollary 25. *Let R be a directly finite regular ring satisfying almost comparability, and A, B, C be finitely generated projective R-modules. If $A \oplus C \prec B \oplus C$, then $A \prec B$.*

We note that Corollary 25 remembers the result in [8] that every directly finite regular ring with weak comparability has the strict cancellation property. By the way, we can give a more general result in Theorem 29 below, by using Theorem 24. To see this, we need the definition and a well-known result for stable range of a ring.

Definition 26. A row $(a_1, \ldots, a_r)$ of elements from a ring R is said to be *right unimodular* if $\sum_{i=1}^{r} a_i R = R$. Given a positive integer n, a ring R is said to *have n in the stable range* provided that for any right unimodular row $(a_1, \ldots, a_r)$ of $r \geq n+1$ elements of R, there exist elements $b_1, \ldots, b_{r-1} \in R$ such that the row $(a_1 + a_r b_1, \ldots, a_{r-1} + a_r b_{r-1})$ is right unimodular. If n is the least positive integer such that R has n in the stable range, then R is said to *have stable range n*. It is well-known that a regular ring R has stable range 1 if and only if R is unit-regular.

We notice that the stable range for a ring nearly relates with the cancellation property, as follows.

Lemma 27 ([13, 14])**.** *Let R be a ring, and A be an R-module such that $End_R(A)$ has n in the stable range for some positive integer n. If B and C are any R-modules such that $A \oplus B \cong A \oplus C$ and $nA \lesssim_{\oplus} B$, then $B \cong C$.*

Here we recall the following interesting result on the stable range for regular rings with 2-comparability.

Lemma 28 ([4])**.** *Let R be a regular ring with 2-comparability, and A be directly finite and finitely generated projective R-module. Then $End_R(A)$ has stable range at most 2.*

Using Theorem 24 and Lemmas 27,28 above, we can show one of main results, as follows.

Theorem 29. *Let R be a regular ring satisfying almost comparability. Let A, B be projective R-modules, and C be directly finite and finitely generated projective R-module. If $A \oplus C \prec_{\oplus} B \oplus C$, then $A \prec_{\oplus} B$.*

Next, we treat the following properties.

Definition 30. A ring R has *the unperforation property* provided that $nA \lesssim nB$ implies $A \lesssim B$ for any finitely generated projective R-modules A, B and any positive integer n. Also, a ring R has *the strict unperforation property* provided that $nA \prec nB$ implies $A \prec B$ for any finitely generated projective R-modules A, B and any positive integer n.

For the above properties, we can recall some interesting results. Goodearl [6] ensured the existence of a simple unit-regular ring R with weak comparability which does not have the unperforation property, and the ring R satisfies 2-comparability too (hence satisfies almost comparability). Thus, simple directly finite regular rings satisfying either almost comparability or weak comparability do not have the unperforation property in general.

On the other hand, it was shown in [9] that every regular ring with weak comparability has the strict unperforation property. But Ara et al. [4] showed that unit-regular rings with 2-comparability do not have the strict unperforation property in general. In spite of the above result, we can show that every regular ring satisfying almost comparability has the strict unperforation property, as follows.

Theorem 31. *Let R be a regular ring satisfying almost comparability, and A, B be finitely generated projective R-modules. If $nA \prec nB$ for some positive integer n, then $A \prec B$.*

Moreover, we can generalize Theorem 31 by using Lemmas below.

Lemma 32. *Let R be a regular ring satisfying almost comparability.*

(1) *Let X_1, X_2, X_3 be finitely generated projective R-modules. If $X_1 \prec_a X_2$, $X_2 \prec_a X_3$, then $X_1 \prec_a X_3$.*

(2) *Let $X_1, \cdots, X_n$ be finitely generated projective R-modules. Then there exists a positive integer k $(1 \leq k \leq n)$ such that $X_i \prec_a X_k$ for all $i = 1, \ldots, n$.*

Lemma 33 ([7])**.** *Let R be a regular ring with 2-comparability. Then every directly finite projective R-module is countably generated, and every finite direct sum of directly finite projective R-modules is directly finite.*

Using Lemmas 32,33 and the strict cancellation property (Theorem 29) effectively, we have the following result.

Theorem 34. *Let R be a regular ring satisfying almost comparability, and A, B be projective R-modules such that A is either finitely generated or directly finite. If $nA \prec_{\oplus} nB$ for some positive integer n, then $A \prec B$.*

We also can show the following result, by using Theorem 34 above.

Theorem 35. *Let R be a regular ring satisfying almost comparability, and A, B be projective R-modules such that A is either finitely generated or directly finite. Then the following are equivalent*:

(a) *$A \prec_a B$.*

(b) *$nA \prec_a nB$ for some positive integer n.*

(c) *$nA \prec_a nB$ for all positive integers n.*

Finally, we inform some interesting problems concerned with the above results. By Corollary 25 and Theorem 31, every directly finite regular ring satisfying almost comparability has the strict cancellation property and every regular ring satisfying almost comparability has the strict unperforation property. Also, any regular rings with weak comparability have similar results, from talks after Corollary 25 and Definition 30. But, there exists a unit-regular ring with 2-comparability which does not have the strict unperforation property, from the talk before Theorem 31. Also we can construct a directly finite regular ring R which does not have the strict cancellation property. For example, we may take $R = S \times T$, where S, T are nonzero stably finite regular rings such that S is not unit-regular (see Example 10(1)). Thus we have the problems:

(A) Which directly finite regular rings have the strict cancellation property?

(B) Which regular rings have the strict unperforation property?

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References

[1] P. Ara and K.R. Goodearl, *The almost isomorphism relation for simple regular rings*, Publ. Mat. UAB **36** (1992), 369–388.
[2] P. Ara, K.R. Goodearl, K.C. O'Meara and E. Pardo, *Separative cancellation for projective modules over exchange rings*, Israel J. Math. **105** (1998), 105–137.
[3] P. Ara, K.R. Goodearl, E. Pardo and D.V. Tyukavkin, *K-theoretically simple von Neumann regular rings*, J. Algebra **174** (1995), 659–677.
[4] P. Ara, K.C. O'Meara and D.V. Tyukavkin, *Cancellation of projective modules over regular rings with comparability*, J. Pure Appl. Algebra **107** (1996), 19–38.
[5] K.R. Goodearl, *Von Neumann Regular Rings*, Second Ed. Krieger, Malabar, Florida, 1991.
[6] K.R. Goodearl, *Torsion in* K_0 *of unit-regular rings*, Proc. Edinburgh Math. Soc. **38(2)** (1995), 331–341.
[7] M. Kutami, *On regular rings with* s*-comparability*, Comm. Algebra **27(6)** (1999), 2917–2933.
[8] M. Kutami, *On von Neumann regular rings with weak comparability*, J. Algebra **265** (2003), 285–298.
[9] M. Kutami, *On von Neumann regular rings with weak comparability* II, Comm. Algebra **33(9)** (2005), 3137–3147.
[10] M. Kutami, *On regular rings satisfying almost comparability*, Comm. Algebra **35(7)** (2007), 2171–2182.
[11] K.C. O'Meara, *Simple regular rings satisfying weak comparability*, J. Algebra **141** (1991), 162–186.
[12] E. Pardo, *Comparability, separability, and exchange rings*, Comm. Algebra **24(9)** (1996), 2915–2929.
[13] R.B. Warfield, Jr., *Notes on cancellation, stable range, and related topics*, Univ. of Washington (August 1975).
[14] R.B. Warfield, Jr., *Cancellation of modules and groups and stable range of endomorphism rings*, Pacific J. Math. **91** (1980), 457–485.

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