AMPLENESS OF TWO-SIDED TILTING COMPLEXES AND FANO ALGEBRAS

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Abstract. From the view point of noncommutative algebraic geometry (NCAG), a two-sided tilting complex is an analog of a line bundle. In this paper we define the notion of ampleness for two-sided tilting complexes over finite dimensional algebras of finite global dimension, and prove its basic properties, which justify the name "ampleness". From the view point of NCAG, Serre functors are considered to be shifted canonical bundles. A finite dimensional algebra A of finite global dimension is called Fano if the shifted Serre functor $A^*[-d]$ is anti-ample. Some classes of algebras studies before are Fano. We show by an example that the property of $A^*[-d]$ from the view point of NCAG captures some representation theoretic property of the algebra A.

From our view point, we give a structure theorem of AS-regular algebras. AS-regular algebras are defined to extract a good homological property of polynomial algebras. Our theorem shows that AS-regular algebra is polynomial algebra in some sense.

1. Introduction

The notion of two-sided tilting complex is introduced independently by Rouquier-Zimmermann [RZ] and Yekutieli [Y] based on the Rickard's derived Morita theory [Ric1]. Let A be a ring. A two-sided tilting complex σ is, by definition, the bounded above complex of A-bimodule such that the derived tensor product $-\otimes^{\mathbf{L}}_A \sigma$ gives an autoequivalence of the derived category $D(\text{Mod-}A)$. If algebra A is noetherian and has finite global dimension, then a complex σ of A-bimodules is a two-sided tilting complex if and only if $-\otimes^{\mathbf{L}}_A \sigma$ gives an autoequivalence of $D^b(\text{mod-}A)$.

$$-\otimes^{\mathbf{L}}_A \sigma : D^b(\text{mod-}A) \xrightarrow{\sim} D^b(\text{mod-}A).$$

From the view point of noncommutative algebraic geometry(NCAG), one thinks of a triangulated category $\mathcal{T}$ as the derived category $D^b(\operatorname{coh} X)$ of coherent sheaves of some "space" X. ¿From this view point, a two-sided tilting complex is an analog of a line bundle. In algebraic geometry, for line bundles ampleness is an important notion. In this paper we define the notion of ampleness of tilting complexes over finite dimensional k-algebras.

We justify this definition by using the theory of noncommutative projective schemes due to Artin-Zhang [AZ] and Polishchuk [Po]. In the theory of noncommutative projective schemes, for a graded coherent ring R over k, we attach an imaginary geometric object $\operatorname{proj} R = \big(\operatorname{cohproj} R, \overline{R}, (1)\big)$. An abelian category $\operatorname{cohproj} R$ is considered as the category of coherent sheaves on $\operatorname{proj} R$.

The final version of this paper will be submitted for publication elsewhere.

In [Le] and [GL], geometric notions are introduced to study certain class of algebras. These works are the inspiration for this work.

Notation and convention. Throughout this paper k denotes a field. For a ring A we denote by Mod-A (resp. mod-A) the abelian category of right A-modules (resp. the abelian category of finite right A-modules). For a k-vector space M, we denote by M^* its k-dual vector space. Let $D^b(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$. We denote the standard t-structure in $D^b(\mathcal{A})$ by $(D^{\geq 0}(\mathcal{A}), D^{\leq 0}(\mathcal{A}))$, i.e., $D^{\geq 0}(\mathcal{A})$ (resp. $D^{\leq 0}(\mathcal{A})$) is the full subcategory of $D^b(\mathcal{A})$ with objects $\mathcal{F}^{\cdot}$ such that $\mathrm{H}^i(\mathcal{F}^{\cdot}) = 0$ for $i < 0$ (resp. $i > 0$). If there is no danger of confusion, we identify the two-side tilting complex σ with the autoequivalence $- \otimes^{\mathbf{L}}_A \sigma$. For example $\sigma M := M \otimes^{\mathbf{L}}_A \sigma$ for $M \in D(\text{Mod-}A)$ and $\sigma^n := \sigma \otimes^{\mathbf{L}}_A \cdots \otimes^{\mathbf{L}}_A \sigma$ (n times) for $n \in \mathbb{N}$.

2. Ampleness of two-sided tilting complexes

We start by reformulating the Serre's criteria of ampleness in the theory of derived categories. Let X be a variety over k and let $\mathcal{T} := D^b(\operatorname{coh} X)$ be the bounded derived category of coherent sheaves on X.

Definition 1. Let $\mathcal{L}$ be a line bundle on X. The full subcategory $\mathcal{T}^{\mathcal{L},\geq 0}$ (resp. $\mathcal{T}^{\mathcal{L},\leq 0}$) of $D^b(\operatorname{coh} X)$ consists of objects $\mathcal{F}^{\cdot}$ which satisfy

$$\mathbb{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_X, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{L}^n) \in D^{\geq 0}(k\text{-}vect) \quad \text{for } n \gg 0$$
$$(\text{resp.} \quad \mathbb{R}\operatorname{Hom}^{\cdot}(\mathcal{O}_X, \mathcal{F}^{\cdot} \otimes^{\mathbf{L}} \mathcal{L}^n) \in D^{\leq 0}(k\text{-}vect). \quad \text{for } n \gg 0)$$

We define $\mathcal{T}^{\mathcal{L}} := (\mathcal{T}^{\mathcal{L},\geq 0}, \mathcal{T}^{\mathcal{L},\leq 0})$.

Theorem 2 (Serre's criteria of ampleness [Har, Propsition III.5.3])**.** *Suppose that X is proper. Then a line bundle $\mathcal{L}$ on X is ample if and only if $\mathcal{T}^{\mathcal{L}}$ is the standard t-structure in $D^b(\operatorname{coh} X)$.*

Reversing this observation, to formulate ampleness in the study of derived categories, we define the following.

Definition 3. Let A be a k-algebra and let σ be a two-sided tilting complex over A. The full subcategory $D^{\sigma,\geq 0}$ (resp. $D^{\sigma,\leq 0}$) of $D^b(\text{mod-}A)$ consists of objects $M^{\cdot}$ which satisfy

$$\sigma^n M \in D^{\geq 0}(\text{Mod-}A) \quad \text{for } n \gg 0$$
$$(\text{resp.} \quad \sigma^n M \in D^{\leq 0}(\text{Mod-}A) \quad \text{for } n \gg 0).$$

We define $D^{\sigma} := (D^{\sigma,\geq 0}, D^{\sigma,\leq 0})$.

Since $\sigma^n M \simeq \mathbb{R}\operatorname{Hom}(A, \sigma^n M)$, we think of A as the "structure sheaf" in Definition 3.

A two-sided tilting complex σ is called *pure* if $\mathrm{H}^i(\sigma) = 0$ for $i \neq 0$. We give the definition of ampleness of two-sided tilting complexes.

Definition 4. Let A be a finite dimensional k-algebra and let σ be a two-sided tilting complex over A.

(1) The two-sided tilting complex σ is called *ample* if σ^n is pure for $n \gg 0$ and D^{σ} is a t-structure in $D^b(\text{mod-}A)$.

(2) The two-sided tilting complex σ is called *very ample* if $\mathrm{H}^i(\sigma) = 0$ for $i \geq 1$ and σ is ample.
(3) The two-sided tilting complex σ is called *extremely ample* if σ^n is pure for $n \geq 0$ and D^σ is a t-structure in $D^b(\text{mod-}A)$.

To give a justification of this terminology. we need a bit of notations. Let $S = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ be a $\mathbb{N}$-graded ring. We denote by $\mathrm{Gr}\, S$ the abelian category of graded right S-modules. An element x of a graded right S-module M is called a *torsion element* if $xS_{\geq n} = 0$ for some $n \in \mathbb{N}$. We define $\mathrm{Tor}\, S$ to be the full subcategory of $\mathrm{Gr}\, S$ consisting of those objects M such that each element $x \in M$ is a torsion element. Note that if S is finitely generated over S_0 then $\mathrm{Tor}\, S$ is a localizing subcategory of $\mathrm{Gr}\, S$. In the case when our graded ring S is finitely generated over S_0 we define $\mathrm{QGr}\, S$ to be the quotient category $\mathrm{Gr}\, S/\, \mathrm{Tor}\, S$.

Definition 5. A right (resp. left) graded S-module M called right (resp. left) coherent if it satisfies the following two conditions:
(a) M is finitely generated;
(b) for every homomorphism $f : \oplus_{i=1}^n S(s_i) \to M$ of right S-modules, the kernel $\ker(f)$ is finitely generated.

A graded ring S is called right (resp. left) coherent if S and $S/S_{\geq 1}$ are right (resp. left) graded coherent as a right (resp. left) graded S-module. A graded ring S is called coherent if S is both right and left coherent.

We denote by $\mathrm{coh}\, S$ the full subcategory of $\mathrm{Gr}\, S$ consisting of right coherent S modules. We define $\mathrm{tor}\, S$ to be the intersection between $\mathrm{Tor}\, S$ and $\mathrm{coh}\, S$. In the case when S is right coherent we define $\mathrm{cohproj}\, S$ to be the quotient category $\mathrm{coh}\, S/\, \mathrm{tor}\, S$.

The following is the one of our main theorem.

Theorem 6. *Let A be a finite dimensional algebra of finite global dimension. Let σ be a two-sided tilting complex over A such that $H^i(\sigma) = 0$ for $i \geq 1$ and σ^n is pure for $n \gg 0$. Then the followings holds.*

(1) *There is the following equivalence of k-linear triangulated categories:*
$$D(\text{QGr-}T) \simeq D(\text{Mod-}A).$$
(2) *The following conditions are equivalent.*
   (a) *T is a right graded coherent algebra.*
   (b) *$\mathcal{D}^\sigma$ is a t-structure in $D^b(\text{mod-}A)$.*
(3) *If the conditions (a) or (b) holds, then there is the following equivalence of k-linear triangulated categories:*
$$D^b(\mathrm{cohproj}\, T) \simeq D^b(\text{mod-}A).$$

As a corollary we prove the following.

Corollary 7. *Let A be a finite dimensional algebra of finite global dimension and let σ be a very ample two-sided tilting complex. Then there is a natural equivalence of triangulated categories*
$$D^b\,(\text{mod-}A) \xrightarrow{\sim} D^b\,(\mathrm{cohproj}\, T)\,.$$
where $T := T_A(\mathrm{H}^0(\sigma))$ is the tensor algebra of $\mathrm{H}^0(\sigma)$ over A.

In [Be] Beilinson showed that $\mathbb{P}^n$ is derived equivalent to a finite dimensional algebra. This result has been generalized to other varieties. The above corollary gives a partial converse.

3. Fano algebras

Let A be a finite dimensional k-algebra of finite global dimension. The k-dual A^* has the natural A-bimodule structure. It is known that $- \otimes^{\mathbf{L}}_A A^* : D^b(\text{mod-}A) \longrightarrow D^b(\text{mod-}A)$ is the Serre functor ([Hap, I.4.6]). For a nonsingular projective variety X over k, the $[\dim X]$-shifted derived tensor $- \otimes^{\mathbf{L}}_X \omega_X[\dim X]$ of the canonical bundle ω_X is the Serre functor of $D^b(\operatorname{coh} X)$. ¿From a view point of noncommutative algebraic geometry A^* is thought as "shifted canonical bundle". For example, if $(A^*)^m \simeq [n]$ for some positive integers m, n, then A is called *fractional Calabi-Yau of CY dimension* $\dfrac{n}{m}$, which is named after analogy to the property of the derived category of a Calabi-Yau variety.

Definition 8. Let A be a finite dimensional k-algebra of finite global dimension, let d be a integer, and set $\omega := (A^*[-d])$. A is said to be a Fano algebra of Fano dimension d if the two-sided tilting complex ω^{-1} is ample. A is said to be an extremely Fano algebra of Fano dimension d if ω^{-1} is extremely ample.

Let X be a Fano variety or a variety with ample canonical bundle. Then the celebrated Bondal-Orlov's Theorem [BO, Theorem 3.1] state that the k-linear triangulated autoequivalence group (up to natural isomorphisms) is described by the term of algebraic geometry of X. A weaker version holds for Fano algebras and algebras with ample "canonical bundle".

Theorem 9. *Let A be a finite dimensional k-algebra of finite global dimension and let d a natural number. Set $\omega := A^*[-d]$. If the two-sided tilting complex ω or ω^{-1} is ample, then every k-linear triangulated autoequivalence F is standard, i.e. there exist a two-sided tilting complex σ such that there is a natural isomorphism of functors $F \simeq - \otimes^{\mathbf{L}}_A \sigma$.*

Remark 10. It is known that same property holds for hereditary algebras [MY, Theorem 1.8].

The following Theorem gives examples of Fano algebras.

Theorem 11. *Let A be a finite dimensional k-algebra of finite global dimension. Set $\omega := A^*[-1]$. If ω^n (resp. ω^{-n}) is pure for $n \gg 0$. Then D^{ω} (resp. $D^{\omega^{-1}}$) is a t-structure in $D(\text{mod-}A)$.*

4. A noncommutative algebro-geometric characterization of representation type of a quiver

Let Q be a connected finite acyclic quiver, i.e., a connected quiver with finitely many vertexes and finitely many arrows without loops and oriented cycles. Then the path algebra $A = kQ$ of Q is a finite dimensional k-algebra of global dimension 1. Set $\omega_Q := A^*[-1]$. Note that $- \otimes^{\mathbf{L}} \omega_Q^{-1}$ is the inverse of the Auslander-Reiten translation in Happel's derived version of Auslander-Reiten theory [Hap]. By [Hap, II.4.7] if the quiver Q has infinite representation type, then ω_Q^{-n} is pure for any $n \geq 0$. Therefor by theorem 11 we prove the following proposition.

Proposition 12. *Let Q be a connected finite acyclic quiver of infinite representation type. Then the path algebra kQ of Q is a Fano algebra of Fano dimension* 1.

If a finite acyclic quiver Q has finite representation type, then its path algebra kQ is fractional Calabi-Yau. (This fact has been known by specialists. See [MY] for the precise CY dimension of these algebras.)

Now we have the following characterization of representation type of quivers from the view point of noncommutative algebraic geometry.

Theorem 13. *A finite acyclic quiver has finite representation type if and only if its path algebra is fractional Calabi-Yau, and a finite acyclic quiver has infinite representation type if and only if its path algebra is Fano.*

Remark 14. For canonical algebras in the sense of Ringel [Rin] the same type characterization holds.

By Theorem 7 and Theorem 12 we obtain the following corollary.

Corollary 15. *Let Q be a finite acyclic quiver of infinite representation type. Then there is a natural equivalence of triangulated categories*

$$D^b(\text{mod-}kQ) \xrightarrow{\sim} D^b(\operatorname{cohproj}\Pi(Q))$$

where $\Pi(Q)$ is the preprojective algebra of Q.

Remark 16. The similar result is proved in [Le].

5. A structure of AS-regular algebras (joint work with I.Mori.)

Definition 17. A connected graded algebra R is called AS-regular if it has finite global dimension d and satisfies the following Gorenstein property:

$$\underline{\operatorname{Ext}}^q_{\operatorname{Gr} R}(k_R, R) \cong \begin{cases} k(e) \text{ for some } e \in \mathbb{Z} & \text{if } q = d \\ 0 & \text{otherwise} \end{cases}$$

The integer e is called Gorenstein parameter.

Remark 18. In some paper these algebras are called regular algebra. In Artin-Schelter's original definition [AS], (AS-)regular algebras are defined by three conditions: above two conditions and finiteness of Gelfand-Kirillov dimension.

We use the r-th quasi-Veronese algebra introduced by I.Mori.

Definition 19 ([Mo])**.** Let $r \geq 1$ be a natural number. The r-th quasi-Veronese algebra $R^{[r]}$ of R is a graded algebra defined by

$$R^{[r]} := \bigoplus_{n \geq 0} \begin{pmatrix} R_{nr} & R_{nr+1} & \cdots & R_{(n+1)r-1} \\ R_{nr-1} & R_{nr} & \cdots & R_{(n+1)r-2} \\ \cdots & \cdots & \cdots & \cdots \\ R_{(n-1)r+1} & R_{(n-1)r+2} & \cdots & R_{nr} \end{pmatrix}$$

with the multiplication defined as follows: for $(a_{i,j}) \in (R^{[r]})_p$, $(b_{i,j}) \in (R^{[r]})_q$ where $a_{i,j} \in R_{pr+j-i}$, $b_{i,j} \in R_{qr+j-i}$,

$$(a_{i,j})(b_{i,j}) := \left(\sum_{k=0}^{r-1} a_{k,j} b_{i,k} \right) \in (R^{[r]}).$$

We define F to be the degree 0 part $(R^{[e]})$ of e-th quasi Veronese algebra $R^{[e]}$. Note that F is a finite dimensional algebra of finite global dimension. Set $\omega := F^*[-(d-1)]$.

Theorem 20. (1) *ω^{-n} is pure for $n \geq 1$.*

(2) *Let T be the tensor algebra $T_F(\omega^{-1})$ of ω^{-1} over F. There is an automorphism ϕ of T as graded algebras. such that the e-th quasi Veronese algebra $R^{[e]}$ is isomorphic to the twisted algebra T^{ϕ} as graded algebras.*

$$R^{[e]} \cong T_F(\omega^{-1})^{\phi}$$

Artin and Schelter gave the definition of AS-regular algebras to extract good homological property of polynomial algebras. Our structure theorem shows that AS-regular algebras are polynomial algebra in some sense. The point is that we do not consider connected graded algebras over a field k, but consider connected graded algebras over a finite dimensional algebra F.

As an application we reprove the following statement.

Corollary 21 (Piontkovski [Pi]). *An AS-regular algebra R of global dimension 2 is coherent.*

Remark 22. This corollary is already proved by D. Piontkovski [Pi]. He used the description of AS-regular algebras of global dimension 2 obtained by J. J. Zhang [Z2].

In the case when $d = \mathrm{gl.dim}\, R = 0$ the above statement is trivial. Since AS-regular algebra of global dimension 1 is isomorphic to polynomial ring $k[x]$ in one variable, the case $d = 1$ is also trivial.

A.Bondal conjectured that all AS-regular algebras are coherent.

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