NONCOMMUTATIVE ALGEBRAIC GEOMETRY : A SURVEY OF THE APPROACH VIA SHEAVES ON NONCOMMUTATIVE SPACES

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0. Introduction

To me noncommutative algebraic geometry came from the consideration of noncommutative spaces defined in terms of notions like : noncommutative valuations and pseudovaluations, primes in algebras, prime ideals of Noetherian rings or prime torsion theories for rings or categories. The root of the theory was in the theory of the Brauer group of a field via suitable subrings of central simple algebras, therefore at first rings satisfying polynomial identities played a dominating role. For such a ring R the noncommutative space prompting itself is Spec$R$, the prime ideal spectrum with its Zariski topology; in [93], a structure sheaf over Spec$R$ for a noncommutative ring R had been first constructed. In the case of rings with polynomial identities this could be tied to arithmetical pseudo-valuation theory and a corresponding divisor theory leading to a noncommutative version of a Riemann-Roch theorem for central simple algebras over curves (see [130], [137] which turned out to be an extension of some idea of E. Witt (see the book by M. Deuring, *Algebra*), This combined in the concept of noncommutative geometry in the P.I. case, the subject being first called that in the publication [137]. Also this theory connected well with maximal orders and Azumaya algebras and it developed into a branch related to the Brauer group of schemes and varieties. Now the localization theory was well established for abelian categories, see P. Gabriel [47], while on the other hand a result of Van Oystaeyen, Verschoren stated that $Br\mathrm{Proj}C = Br(C, K_+)$-gr where C is a commutative positively graded ring and (C, K_+)-gr is the quotient category of finitely generated graded C-modules for the torsion theory κ_+ associated to the positive cone $C_+ = C_1 \oplus, \ldots, \oplus C_n \oplus, \ldots$ of C. Deleting Br in the formula suggests that Proj$C$ is "identified" with that quotient category. The J. P. Serre's global section theorem does relate the quasi-coherent sheaves over Proj$C$ to that quotient category, in fact when $C_o = k$, a field, and $C = C_0[C_1]$, then the quotient category is just finitely generated graded C-modules modulo finite length modules. So assuming that a noncommutative version of J.P. Serre's result exists, the noncommutative geometry of Proj$R$ should be approachable via the homological algebraic theory of the category (R, κ_+)-gr. It turned out that a noncommutative versions of the global section theorem is available only in case one introduces a noncommutative topology on the localizations spectrum allowing compositions of localizations that are not again localizations. This leads to the definition of schematic algebras and it was checked that a very large class of noncommutative rings are schematic, inducing all interesting quantized algebras and other rings appearing in recent literature.

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Rings appearing in nature were given by generators and relations and as such they inherited the filtration defined by the grading filtrations on the free algebra. The definition of Zariskian filtration, introduced in [79] and the use of the Rees ring (blow-up ring) then allowed the interplay between algebraic geometry and its projective version much as in the commutative case. The filter-graded transfer of homological properties and of the schematic condition provided for a fruitful technical framework to study many interesting examples, e.g. generalized Weyl algebras, generalized gauge algebra containing E. Witten's gauge algebra for gauge theory of slU_2, etc... Using Auslander's regularity condition it was possible to extend regularity from Azumaya algebras over regular center to more general noncommutative rings, not necessarily finite over the center; the filter-graded transfer for Auslander regularity provided many interesting examples of noncommutative regular algebras (schemes). The study of regular algebras and their classification in low dimension became a fruitful research direction, recently developing into the direction of Calabi-Yau algebras (see [26]) etc...

Let us point out that a good version of geometric product may be found in the general twisted product of algebras, cf. [83]; its good behaviour with respect to connections provides a link with the work of A. Connes. The noncommutative geometry developed by A. Connes after the 1980s was more based in operator theory and C^*-algebras, one could call it noncommutative differential geometry. The space in this geometry remains virtual and one imagines the noncommutative algebra as a ring of "functions" defined on the virtual variety. There are several contact points between both versions of noncommutative geometry, it remains to be seen whether the phrasing in terms of noncommutative spaces in algebraic geometry is feasable and useful in the other case.

Contents

In an arbitrary ring S a couple (P, S') where S' is a subring of S and P is a prime ideal of S' is called a **prime of** S if $xS'y \subset P$ with $x, y \in S$ yields that x of y is in P. The map $S' \to S'/P$ is a **pseudoplace** of S. A couple (P, S') **dominates** (P_1, S_1) if $S' \supset S_1$ and $P \cap S_1 = P_1$, a **dominating prime** is one that is maximal with respect to domination. The set $\mathrm{Prim}(S)$ of all primes of S has a topology with basis $D(F) = \{P \in \mathrm{Prim}(S), P \cap F = \emptyset\}$ for F a finite subset of S. For example if K is a field and A a K- central simple algebra, O_v a valuation ring of K then any maximal O_v-order Λ in A yields a dominating prime $(J(\Lambda), \Lambda)$ where $J(\Lambda)$ is the Jacobson radical of Λ.

Consider a prime P.I.-ring S with quotient ring $(Q(S)$, which is then a central simple algebra. A **fractional ideal** I of S is a twosided S-submodule of $Q(S)$ such that $cI \subset S$ for some nonzero $c \in Z(S)$ (the centre of S). S is an arithmetical ring when fractional ideals commute for the product in $Q(S)$. Let $F(S)$ be the set of fractional ideals of S. Consider a totally ordered semigroup Γ, a **pseudovaluation** v on $F(S)$ is a function $v : F(S) \to \Gamma$, satisfying :

i) $v(IJ) \geq v(I)v(J)$ for $I, JvF(S)$
ii) $v(I + J) \geq \inf\{v(I), v(J)\}$ for $I, J \in F(S)$
iii) $v(S) = 0$ and $v(o) = \infty$
iv) If $I \subset J$ then $v(I) \geq v(J)$ for $I, J \in F(J)$

If moreover we have : (v)$v(IJ) = v(I) + v(J)$, we say v is an **arithmetical pseudovaluation** (a.p.v.).

Any a.p.v. on Q defines a prime (P, Q^P) where Q^P is the idealizer of P in Q and $P = \{q \in Q, v(SqS) \subset 0\}$. Conversely any prime (P, Q^P) where $S \subset Q^P$ defines an a.p.v., v say, such that $P = \{q \in Q, v(SqS) \supset o\}$.

If the value semi-group of an a.v.p. is a group then the corresponding prime is dominating. Any prime (P, Q^P) of a central simple algebra is said to be **discrete** if Q^P contains an arithmetical ring S and satisfies the a.c.c. on ideals while P is the unique maximal ideal of Q such that $P = \pi Q^P$ for some invertible π in Q. In the discrete case $\Gamma \cong \mathbb{Z}$ and Q^P is itself arithmetical. In particular any maximal order in Q over a dicrete valuation ring of $K = Z(Q)$ is a discrete prime. A set of discrete primes inducing inequivalent valuations on K is said to be **divisorial** if for $q \in Q$ we have $v(q) = 0$ for almost all a.p.v. associated to the discrete primes in the set, this condition has to be checked only for $q \in Z(Q)$! The elements of a divisorial set Q are called **prime divisors**. A divisorial set Q is associated to be chosen fixed : a **divisor** δ of Q associated to Q is a formal product $\prod_{v \in Q} v^{\tau_v}$ with $\tau_v \in \mathbb{Z}$ and $\tau_v = 0$ for almost all $v \in Q$, the exponent τ_v is called the **order** $\tau_v = \mathrm{ord}_v \delta$.

Consider a subfield k_o of an algebraically closed field k. In [137] we consider an affine curve over k_o as a k_o-quasivariety $\Omega(R)$ for some prime affine P.I. algebra R over k_o having Krull dimension 1. By a result of L. Small such an algebra is a finite module over its centre. If $n = p.i.\mathrm{deg}R$ then $M \in \Omega(R)_n$ correspond to $m \in \Omega(C)$, $L = Z(R)$, such that $R \otimes_k k_C(m) = M_n(k)$. For $k_o \neq k$ it is still true that $k_R(M) = R \otimes_{k_o} k_o(m)$ is a central simple algebra and $P \in \Omega_n(R)$ if and only if $k_R(P)$ has degree n (dimension : n^2) if and only if $P \cap C$ is non split.

An algebraic function field K in one variable over k_o is an extension K of k_o such that k_o is algebraically closed in K and K is separable of t.d. over k_o.

A **function algebra** in one variable over k_o is a central simple K-algebra A. For an affine prime P.I. algebra over k_o there is equivalence between $\Omega(R)$ (the space of maximal ideals) being an affine k_o-curve and $Q(R)$ being a function algebra in one variable. The complement of unramified points in $Z(R)$ is the ramification divisor of $\Omega(C)$ for Q, these correspond to the maximal ideals of C that are split in R.

Let $C_{k_o}(R)$ be the set of all k_o-valuation rings O_v of K (those are discrete). For every $O_v \in C_{k_o}(K)$ we choose and fix a maximal order Λ_v over O_v and write $C_{k_o}(Q)$ for this set. This choice can'be made such that almost all Λ_v contain a suitable Azumaya algebra (obtained as $\cap_{P\in\Omega_n(R)} R_P$ for some R ascending the curve). Write D_Q for the group of divisors generated by $C_{k_o}(Q)$. The degree of a divisor $\delta \in D_Q, \delta = \Sigma f_v \mathrm{ord}_v \delta$, where f_v is the absolute residue class degree u.e. $f_v = \dim_{k0} k_v$, k_v the residue field of O_v. We say that $\delta_1|\delta_2$ if for all $v \in Q$, $\mathrm{ord}_v\delta_1 \leq \mathrm{ord}_v\delta_2$.

1.1 Lemma. *If $\delta_1|\delta_2$ then :*

$$\dim_{k_o}(\Gamma(\delta_1(S))/\Gamma(\delta_2(S)) = \deg\ \delta_2 - \deg\ \delta_1$$

where for any finite subset S of the algebra of valuation vectors $V_Q, \Gamma(\delta|S) = \{a \in R, v(a) \geq \mathrm{ord}_v\delta, \text{all } v \in S\}$ (cf. [130], [137]).

In particular if $S = V_A$ then we define $L(S)$ as $\Gamma(\delta|V_A)$ and $l(\delta) = \dim_{k_o} L(\delta)$. Valuation forms can now also be defined in the noncommutative case and by using the reduced trace map for Q every valuation form is of the form $w(Tr(a-))$ for some $a \in Q$ and fixed valuation form w.

1.2 Theorem. Riemann-Roch for n.c. curves *Let $\beta \in D_Q$ be arbitrary and δ "canonical" (see Proposition XI.3.9. p. 376 of [137]), then :*

$$\deg\beta + l(\beta) = l(\beta^{-1}\delta^{-1}) + 1 - g_Q$$

where g_Q is a constant, called the **genus** *of Q.*

The ring $l = \cap\{\Lambda_v, \Lambda_v \in C_{k_o}(Q)\}$ is the ring of k_o-constants it is algebraic over k_o and a central simple algebra finite dimensional over K_0 (XI.2.14 of [137]).

1.3 Corollary.

i) $\ell(\delta^{-1}) = n - 1 + g_Q, n = \dim_{k_o}\ell$.

ii) $\deg(\delta^{-1}) = 2 - 2y_Q$

iii) $g_Q = N_{g_K} - N + 1 + \frac{1}{2}\Sigma f_v(r_v - 1)$, *where f_v is the residual degree r_v the ramification index of v,* $\mathbb{N} = \dim[Q : K]$.

1.4 Theorem. *Let $k = k_o$ and $X = \Omega(R)$ an affine k-curve with central curve $Y = \Omega(Z(R))$ then : $g_X = Ng_Y - N + 1$ (since k is algebraically closed $Q = Q(R) = M_n(K)$ by Tsen's theorem.*

1.5 Remark. If $Q = M_r(\Delta)$ then $g_Q = r^2 g_\Delta - r^2 + 1$. The Brauer group of K (not trivial if k_o is not algebraically closed) yields invariants g_Q for every $[Q] \in BrK$. What is the relation between the commutative geometry of the central curve and these invariants ?

1.A. Project : Noncommutative Invariants of Varieties

After [130], [137], Van den Bergh, Van Geel obtained a cohomological Riemann-Roch result for higher dimensional noncommutative varieties. The foregoing question may be generalized to this higher dimensional situation using the ingredients (invariants) stemming from the Riemann Roch theorem.

In dimension more than two there are noncommutative invariants stemming from the Brauer group of the function field that is now not trivial even in the case where k_o is algebraically closed. There is some work of M. Artin about maximal orders over surfaces (see [9]) but a complete noncommutative version of the work of O. Zariski on surfaces remains to be developed. In general the set of discrete primes of a central simple algebra provides us with something like s noncommutative Riemann surface. The theory of a.p.v's works well if some arithmetical ring is given but it should be extended to more genertal situations using rings in which ideals do not commute and noncommutative (totally ordered) value groups.

1.B. Project : Valuations of Weyl Algebras, Enveloping Algebras etc..

The theory of valuations also extends to the non P.I. case; O. Schilling (cf. [117]) already introduced noncommutative valuations on skewfields not necessarily finite dimensional over the centre. However, the valuation theory for most quantized algebras nowadays popular remains unexplored. In a paper with L. Willaert, I investigated valuations of the Weyl skewfield and this led to the discovery of a subring of the Weyl skewfield having it as a ring of fractions (therefore in some sense birational to the Weyl algebra $K[X][\frac{\partial}{\partial X}] \cong K < X, Y > /(YX - XY - 1)$) and being a kind of antipode for the Weyl algebra. This ring appearing as the intersection of noncommutative valuation rings is a "duo ring" i.e. each one sided ideal is two sided and localizations at prime ideals correspond to valuation over rings. A divisor theory for the Weyl field remained to be worked out. Up to a particiular application related to Sklyanin algebras, the noncommutative valuation theory remains to be applied. For example, it is an unpublished consequence of some results in the Ph. D. thesis of L. Hellström (Lund T. U., Sweden) that one may construct large families of noncommutative valuations of the skewfield appearing as the ring of fractions of the enveloping algebra of a finite dimensional Lie algebra. Further characterization of these n.c. valuations and calculations similar to a divisor calculus should be undertaken and these results should have meaning in the structure theory of Lie algebras or at least in the noncommutative geometry of their enveloping algebras. In particular some rings appearing as intersections of families of n.c. valuation rings could shed new light on the algebraic structure ?

2. Schematic Algebras and Noncommutative Schemes

Algebraic geometry is built upon the correspondence between quotients of polynomial rings and varieties embedded in affine (projective) spaces. In noncommutative algebra the generic algebra i.e. the free algebra, is not too well behaved and the formation of products (tensor products) is also somewhat problematic. Is it possible to fix a class of algebras such that most operations from scheme theory may be performed whilst keeping a good duality with noncommuative algebra constructions ? For a given noncommutative algebra one may of course try to extend the algebraic techniques appearing in commutative algebraic geometry to it without trying to associate a "geometric" space to it. This works to some extent in several cases but it is perhaps not guaranteed that one is really studying a noncommutative geometry, it is noncommutative algebra in disguise. I always wanted some kind of topological space and (coherent) sheaves on it to correspond to the module of some ring of functions via some noncommutative version of J.P. Serre's global section theorem. This led to the introduction of noncommutative topology and schematic algebras.

2.1 Noncommutative Spaces and Localization

Perhaps a few historical remarks concerning the development of this subject. During my stay at Cambridge University in 1972-73 I worked with D. Murdoch (Vancouver University B. C.) on localization theory and we constructed the first structure sheaf for a noncommutative ring yielding the ring as global sections, cf. [93]. For me this was connected to the primes or pseudoprimes I introduced in my thesis and I combined the ideas into a theory of prime spectra for noncommutative rings in [134] where I also started the projective theory by constructing Proj for a noncommutative positively graded ring. This was also related to my search for an answer to a question J. Murre (University of Leiden) asked me concering a purely algebraic description of the Brauer group of a projective variety during our stay at Cambridge. Since maximal orders were at the centre of all these problems I started a seminar on this topic at the University of Antwerp (UIA) which attracted many students and visitors. With J. Van Geel, E. Nauwelaerts and visitor H. Marubayashi and later L. Willaert we continued in the direction of primes of noncommutative algebras; with A. Verschoren and L. Le Bruyn in the direction of localization and noncommutative schemes, with L. Le Bruyn, E. Jespers, P. Wauters in the direction of graded orders and later with M. Van den Bergh in projective noncommutative geometry. The work with A. Verschoren (resulting in the first book with the title "Noncommutative Geometry", cf. [137]) was noticed by M. Artin and after a stay of A. Verschoren at the M.I.T. there was a growing group of people involved in the development, including M. Artin, W. Schelter etc... Starting from regularity conditions from homological algebra, M. Van den Bergh then cooperated with M. Artin, J. Tate (cf. [6],[7]) and later with T. Stafford, P. Smith and many more, specially on low dimensional noncommutative varieties. On the other hand the graded constructions in the constructions of proj created a cooperation with C. Năstăsescu on graded ring theory, cfr. [97], [95], [94]. Meanwhile it turned out that the answer to the question of J. Murre fitted completely in the framework of graded localization. After I introduced the graded Brauer group of a $\mathbb{Z}$-graded ring, A. Verschoren and I described the Brauer group of a projective variety algebraically as the Brauer group of the category (in modern language) appearing as a quotient category of the finitely generated graded modules modulo those of finite length i.e. the graded quotient category associated to the graded localization at the positive cone of the positively graded coordinate ring. This continued in work on the cohomology of graded rings with S. Caenepeel [32] and extended to Brauer groups of other actions and conditions leading to the Brauer group of a quantun group, cf. [33], [34], and later with Y. H. Zhang to a theory of Brauer groups of braided categories, cf. [142], [143]. After the beginning of the interest in graded rings, filtrations also became interesting, particularly because of the use of the Rees (blow up) ring; this makes for a transfer between graded and filtered ring theory allowing several nice applications to for example rings of differential operators and (generalized) Weyl algebras cfr. work with Li Huishi [79], [80], and later with V. Bavula [17] [18]. The connection with representation theory was also explored at several places and developed mainly by L. Le Bruyn and M. Van den Bergh e.g. in the geometry of path algebras for quivers cf. [69],[68]. This shows how the original ideas concerning a kind of noncommutative geometry has branched into many directions that have achieved nowadays a good level of popularity.

So originally we considered a noncommutative variety or scheme as a structure sheaf on the prime spectrum, that prime spectrum was either determined in terms of prime ideals (Murdoch, Van Oystaeyen) or prime torsion theories (J. Golan, J. Raynand, F. Van Oystaeyen cf. [47]). However in the noncommutative case, the construction was not functorial (I remember to have proved, unpublished, that functionality forces commutativity) but it was possible to view Spec as a (localization) functor on the category of modules and to relate a ring morphism to a natural transformation of the Spec functor. This has convinced me that the construction of a (noncommutative) topology was more essential than the choice of points, in fact one could work with a pointless topology and sheaf theory over that. This gave rise to the construction of virtual topology and functor geometry, a very abstract framework for categorical algebraic geometry, cf. [135].

A noncommutative ring R is said to be (affine) **schematic** if there exists a finite set of nontrivial Ore sets $S_1, \ldots, S_n$ such that for every choice of $s_i \in S_i, i = 1, \ldots, n$ we have that $\sum_{i=1}^{n} Rs_i = R$ or equivalently $\cap_i \mathcal{L}(S_i) = \{R\}$ where $\mathcal{L}(S_i)$ is the Gabriel filter of S_i. Recall that a left **Ore set** S of R is a multiplicatively closed subset of R, $0 \notin S, 1 \in S$, such that for given $r \in R, s \in S$ there exists $r' \in R, s' \in S$ such that : $s'r = r's$ and moreover if $rs = 0$ then there is an $s'' \in S$ such that $s''r = 0$. The right version is defined symmetrically. For an Ore set S the ring of fractions $S^{-1}R$ exists and in this case the left localization at S and the right localization coincide. For an R-module M the S-torsion part of M is $M_s = \{m \in M, sm = 0 \text{ for some } s \in S\}$ and $S^{-1}M = S^{-1}R \otimes_R M$ is the (left) localization of M at S. Clearly $M/t_S M$ is S-torsion free i.e. $t_S(M/t_S M) = 0$ and we have the standard localization morphism in $j_S : M \to S^{-1}M$ with $\ker j_S = t_S M$ and $\mathrm{Im} j_S \cong M/t_S M$. In case R is the free algebra it only has trivial Ore sets i.e. contained in the ground field and hence already invested in the ring. So free algebras, the generic algebras in the associative situation, are not schematic. On the other hand all rings frequently encountered in noncommutative algebra seem to be schematic. For example the ring of generic matrices, the Weyl algebras, the coordinate ring of quantum 2×2-matrices $Q_q(M_2(\mathbb{C}))$ is schematic (1.2.11 of [131]), quantum Weyl algebras $A_n(q)$ (1.2.14 of [131]), rings that are finite modules over their centre, the Sklyanin algebra $S_K(a, b, c)$ (1.2.17 of [131]), E. Witten's gauge algebras $W(\mathbb{C})$, (1.2.21 of [131]), quantum sl_2 (1.2.23 of [131]). Let A be a K-algebra and positvely graded such that $A = K \oplus A_2 \oplus \ldots$ we write A_+ for $A_1 \oplus A_2 \oplus \ldots$ and K_+ for the torsion theory with Gabriel filter $\mathcal{L}(K_+) = \{L \text{ left ideal of } A, L \supset A_+^n \text{ for some } n \in \mathbb{N}\}$. We say A is **schematic** (projective) if there exists a finite set of homogeneous Ore sets, say $\mathcal{I}$, such that for every $S \in \mathcal{I}$, $S \cap A_+ \neq \emptyset$ and such that for any $s_i \in S_i, i \in \mathcal{I}$, there exists an $m \in \mathbb{N}$ such that $(A_+)^m \subset \sum_{i\in\mathcal{I}} As_i$ (or equivalently : $\cap\, \mathcal{L}(S_i) = \mathcal{L}(K_+)$ where $\mathcal{L}(S_i) = \{L \subset A, L \supset As_i$ for some $s_i \in S_i\}$, or equivalently $\kappa_+ = \Lambda_{i\in\mathcal{I}}\mathcal{L}(S_i)$ holds in the lattice of torsion theories on A-gr, the category of graded A-modules). A schematic positively graded K-algebra need not be affine schematic, we have a weaker notion **weakly affine schematic** defined as (projective) schematic plus the fact that the $S_i \in \mathcal{I}$ are such that $A = \cap_{i\in\mathcal{I}} S_i^{-1}A$.

If we have a positively filtered K-algebra A with filtration ... $\subset F_{n-1}A \subset F_nA \subset \ldots \subset A$ then the associated graded algebra is $G(A) = \oplus_{n\in\mathbb{N}} F_nA|F_{n-1}$
$A = K \oplus F_1(A)|K \oplus \ldots$ and the Rees algebra (or the blow-up algebra of FA) is $\widetilde{A} \cong \sum_n F_n(A)T^n \subset A[T]$. It is easy to see that $G(A) = \widetilde{A}|\widetilde{A}T, A = \widetilde{A}|(T-1)\widetilde{A}$ and T is a central regular element homogeneous of degree 1 in $\widetilde{A}$. In the positively filtered situation (this is a discrete filtration) the filtration will be Zariskian in the sense of [79] exactly when $\widetilde{A}$ is Noetherian which in this case is equivalent to $G(A)$ being Noetherian.

2.1.1 Theorem. *If FA is a positive Zariskian filtration on A such that $F_0A = K$, then if $G(A)$ is schematic it follows that $\widetilde{A}$ is schematic too.*

2.1.2 Corollary. *If in the situation of the theorem $G(A)$ is commutative then $\widetilde{A}$ is schematic. It follows from this that rings of differential operators on varieties (non-singular) and enveloping algebras of Lie algebras have schematic Rees rings.*

When trying to introduce a scheme theory on Proj$A = Y$ for some positively graded noncommutative K-algebra $A = K \oplus A_1 \oplus A_2 \oplus \ldots\ldots$, a good idea could be to replace an affine open, something like $Y(f)$ in the commutative case, by a homogeneous Ore set S of A and the ring of sections (in the commutative case $(A_f)_o$) by $(S^{-1}A)_o$. If A is schematic then we have covered Y by "opens" corresponding to the $S_i, i \in \mathcal{I}$. For commutative A if $Y(f_i)$ cover Y then $Y(ff_i)$ cover $Y(f)$ and for modules of sections we have $M_f = \varprojlim_i \{M_{ff_i}, i\}$ where M_f stand for the localization at the multiplicative set $\{1, f, f^2, \ldots\}$. The straightforward generalization of this property would require that the canonical map :

$$(*) \qquad Q\kappa_{S_1} \wedge \ldots \wedge \kappa_{S_d}(M) \longrightarrow \varprojlim \left(\begin{tikzcd} Q_{S_i}(M) \arrow[dr] & \\ \cdots & Q_{S_i \vee S_j}(M) \\ Q_{S_j}(M) \arrow[ur] & \end{tikzcd} \right)$$

has to be an isomorphism for all $M \in A$-gr. Looking at just two Ore sets S and T, (*) will be an isomorphism if and only if Q_S and Q_T commute, i.e. if and only if : $\kappa_S Q_T = Q_T \kappa_S$ and $\kappa_T Q_S = Q_S \kappa_T$. This compatibility does not always hold and the solution is to introduce more "open sets" i.e. to define a suitable noncommutative Grothendieck topology defined in terms of localization functors on a suitable category. For Proj$A$ the category on which the scheme structure is defined is A-gr localized at κ_T, i.e. (A, κ_+)-gr or the finitely generated objects in this. Let us write $\mathcal{O}(A)$ for the set of homogeneous left Ore sets S of A such that $1 \in S, 0 \notin S$ and $S \cap A_+ \neq \emptyset$. The free monoid on $\mathcal{O}(A)$ is denoted by $\mathcal{D}(A)$. If $W = S_1, \ldots, S_n \in \mathcal{W}(A)$ then we write $w \in W$ meaning that w is of the form $s_1 \ldots s_n$ with $s_i \in S_i, i = 1, \ldots, n$. The category $\underline{\mathcal{W}}$ is defined by taking the elements of $W(A)$ for the objects while for words $W = S_1 \ldots S_n, W' = T_1 \ldots T_m$ we define : $\mathrm{Hom}(W', W) = \{W' \to W\}$ or $\emptyset$ depending on whether there exists a strictly increasing map $\alpha : \{1, \ldots, n\} \to \{1, \ldots, m\}$ for which $S_i = T_{\alpha(i)}$ or not. So $\mathrm{Hom}(W'W)$ is a singleton if it is not empty. Put $Q_W(M) = (Q_{S_n} \circ \ldots \circ Q_{S_1})(M) = Q_{S_n}(A) \otimes_A \ldots \otimes Q_{S_1}(A) \oplus_A M$. To W we associate a filter of left ideals of A, $\mathcal{L}(W) = \{L, w \in L \text{ for some } w \in W\}$. For $w, w' \in W$ there are $a, b \in A$ such that : $aw = bw' = w'' \in W$, also for $w \in W, a \in A$ there are $w' \in W, b \in A$ such that $w'a = bw$. For $M \in A$-mod, $\kappa_W(M) = \{x \in M, wx = 0 \text{ for some } w \in W\}$; this κ_W is an exact preradical on A-mod and it is not necessarily idempotent. $\mathcal{L}(W)$ has a cofinal system of graded left ideals so it induces on exact preradical of A-gr. If $W' \to W$ in $\underline{W}$ then $\mathcal{L}(W) \subset \mathcal{L}(W')$ and for every $V \in \underline{W}, W'W \to WV$, as well as $VW' \to VW$, are morphisms in $\underline{W}$. A global cover of $Y = \mathrm{Proj}A$ is just a finite subset $\{W_i, i \in \mathcal{I}\}$ of objects of $\underline{W}$ such that $\bigcap_{i \in \mathcal{I}} \mathcal{L}(W_i) = \mathcal{L}(\kappa_+)$; the existence of at least one global cover given by words consisting of one letter, is ensured by the schematic constitution for A. For $W \in \underline{W}$ we let $\mathrm{cov}(W)$ be $\{W_i W \to W, i \in \mathcal{W}\}$. The category $\underline{W}$ together with the sets $\mathrm{cov}(W)$ form a noncommutative Grothendieck topology. Global covers induce covers because of :

2.1.3 Lemma. *If $\{W_i, i \in \mathcal{I}\}$ is a global cover then for all $V \in \underline{W}$ we have that $\mathcal{L}(V) = \bigcap_{i \in \mathcal{I}} \mathcal{L}(W_i V)$.*

A presheaf Q on $\underline{W}$ is now a contravariant functor from $\underline{W}$ to A-gr such that for all $w \in \underline{W}$ the sections $Q(W)$ of Q over W form a graded $Q_S(R)$-module where S is the last letter of W. For $W = 1$ we demand $Q(1)$ to be a $Q_{\kappa_+}(A)$-module, we write $\Gamma_*(Q) = Q(1)$. It is straightforward to define sheaves by introducing separatedness and glueing conditions.

For any graded A-module M we obtain a structure presheaf $\underline{O}^g_M$ associating to W the $Q_W(M)$.

2.1.4 Theorem. *For any graded A-module M, A being a schematic K-algebra, the structure presheaves $\underline{O}^g_M$ and $\underline{O}_M = (\underline{O}^g_M)_o$ are in fact sheaves !*

The affine-like properties follows from :

2.1.5 Proposition. *Let A be a schematic K-algebra and suppose that $A = K[A_1]$. For every homogeneous Ore set $S \in \mathcal{O}(A)$ (thus $S \cap A_+ \neq \emptyset$) the ring $S^{-1}A = Q_S(A)$ is strongly graded.*

Recall that a R**-graded ring** is said to be strongly graded if $R_n R_{-n} = R_0 = R_{-n} R_n$ for all n, or equivalently $R_1 R_{-1} = R_0 = R_{-1} R_1$. For a strongly graded ring $R - \mathrm{gr} \cong R_0$-mod. On the basic opens $Q_S(A)$ is a (strongly) graded ring and $Q_S(M)$ is a graded $Q_S(A)$-module ! This need not hold with respect to Q_W for general W ! To $S \in \mathcal{O}(A)$ we associate a basic open $Y(S)$ given by $Q_S(A)$-gr equivalent to $Q_S(A)_o$-fgmod, (fg stands for finitely generated) the latter may be viewed as "Spec$Q_S(A)_o$".

2.1.6 Definition. A noncommutative projective scheme Proj$A$ is defined by $(A, \kappa_+)_{fg}$-gr with a non-commutative Grothendieck topology $\underline{W}$ with affines $Y(S_i)$ generating the topology by intersections. We may view $Y(S)$ as Spec$A_{(S)}$, where $A_{(S)} = (S^{-1}A)_o$, defined in a categorical way.

2.2 Noncommutative Topology and Categorical Theory

The correspondence between coherent sheaves and module categories over the ring of global sections is in the commutative case given by J. P. Serre's fundamental global sections theorem. In the noncommutative case a sheaf $\mathcal{F}$ on $\underline{W}$ is **quasi-coherent** if there is an affine cover $\{T_i, i \in J\}$ for $Y = \text{Proj}A$ together with graded $Q_{T_i}(A)$-modules M_i such that for any morphism $V \to W$ in $\underline{W}$ we obtain a commutative diagram, the vertical arrows representing isomorphisms in A-gr.

$$\begin{tikzcd} \mathcal{F}(T_iW) \arrow[r] \arrow[d] & \mathcal{F}(T_iV) \arrow[d] \\ Q_W(M_i) \arrow[r] & Q_V(M_i) \end{tikzcd}$$

A quasi-coherent $\mathcal{F}$ is said to be coherent if all M_i, $i \in J$, one finitely generated $Q_{T_i}(A)$-modules.

2.2.1 Theorem. *If $\mathcal{F}$ is a quasi-coherent sheaf on $\underline{W}$ and $\Gamma_*(\mathcal{F})(= \mathcal{F}(1))$ denotes its global section A-module then $\mathcal{F}$ is sheaf isomorphic to the structure sheaf of $\Gamma_*(\mathcal{F})$.*

2.2.2 Theorem. (Noncommutative version of J. P. Serre's global section theorem) *For a schematic K-algebra A, the category of quasi-coherent sheaves on $\underline{W}$ is equivalent to (A, κ_+)-gr. The category of coherent scheaves on $\underline{W}$ is equivalent to* Proj(A), *i.e.* $(A, \kappa_+)_{fg}$*-gr.*

These results are due to L. Willaert, F. Van Oystaeyen, see [141] or also Theorem 2.1.5. in [131].

The Rees ring $\widetilde{A}$ of a Noetherian positively filtered K-algebra A is isomorphic to $\sum F_nAT^n \subset A[T]$ and inverting the central homogeneous element of degree $1, T$, we obtain $\widetilde{A}_T = A[T, T^{-1}]$. We may view $Y(T)$ in $Y = \text{Proj}\widetilde{A}$ with sections $A[T, T^{-1}]_{fg}$-gr $= A\text{-mod}_{fg} \simeq \text{Spec}A$.

A filtered K-algebra A as above such that $G(A)$ is a schematic domain has an $\widetilde{A}$ which is again a schematic domain; let $\pi : \widetilde{A} \to \widetilde{A}/T\widetilde{A} \cong G(A)$ be the canonical epimorphism. The Ore sets $S_1, \ldots, S_n$ defining the schematic property for $G(A)$ yield $T_i = \mathcal{L}\pi^{-1}(S_i)$ plus the special Ore set $< T >= S_T$ central in $\widetilde{A}$ and thus compatible to all the $T_i, i = 1, \ldots, n$. The images $\overline{T}_i$ in A via $\widetilde{A} \to A = \widetilde{A}/(T-1), \widetilde{A}$ are saturated Ore sets such that $\sigma(\overline{T}_i) = S_i$ where $\sigma : A \to G(A)$ is the principal symbol map sending $a \in F_nA - F_{n-1}A$ to $\overline{a} = a \text{ mod} F_{n-1}A$ in $G(A)_n = F_nA/F_{n-1}A$. The Rees ring is the homogenization of A with respect to FA, geometrically this means :

2.2.3 Proposition. *For A as above,* Proj$G(A)$ *is a closed subscheme of* $Y = \text{Proj}\widetilde{A}$ *and* $\text{Proj}\widetilde{A} = \text{Proj}G(A) \cup \text{Spec}A$ *(cf. Proposition 2.1.10 of [131]).*

So we may think of Proj$G(A)$ as the part at ∞ for the projective closure of Spec$A$. The part Spec$A$ corresponds to the T-torsionfree class of objects from $(A, \kappa_+)_{fg}$-gr, the part Proj$G(R)$ corresponds to the T-torsion objects. Using microlocalizations of filtered rings one may define quantum section, in [131] Section 2.3. many examples of quantum sections are calculated and given by generators and relations, e.g. for the Weyl algebra $\mathbb{A}_1(\mathbb{C})$, enveloping algebras, colour Lie superalgebras, quantized Weyl algebras. We may look at almost commutative geometry by studying filtered rings A as before, but with $G(A)$ an affine commutative algebra generated by homogeneous elements of degree one. For such rings microlocalization functors do commute and sheaf theory becomes more easy, Section 2.4. in [131].

The latter results provide us with more hints that a completely categorical version of noncommutative geometrical may be possible in terms of arbitrary localizations (torsion theories or quotient categories) and a formally defined P noncommutative topology. This was the aim of [135]. We consider a poset Λ with $0, 1$ and take operations $\wedge, \vee$ on Λ satisfying :

A.1. For $x, y \in \Lambda$, $x \wedge y \leq y$

A.2. For $x \in \Lambda, x \wedge 1 = 1 \wedge x, 0 \wedge x = x \wedge 0 = 0$, moreover $x \wedge \ldots \wedge x = 0$ if and only if $x = 0$

A.3. For $x, y, z \in \Lambda, x \wedge y \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z)$

A.4. For $a \leq b$ in Λ and $x, y \in \Lambda$ we obtain : $x \wedge a \leq x \wedge b, a \wedge y \leq b \wedge y$ (it follows that $x \wedge y \leq x$ too !)

A.5. For $x, y \in \Lambda, y \leq x \vee y$

A.6. For $x \in \Lambda$, $1 \vee x = x \vee 1 = 1$, $x \vee 0 = x = 0 \vee x$, moreover $x \vee \ldots \vee x = 1$ if and only if $x = 1$

A.7. For $x, y, z \in \Lambda, x \vee (y \vee z) = x \vee y \vee z = (x \vee y) \vee z$.

A.8. For $a \leq b$ in Λ and $x, y \in \Lambda$ we obtain : $x \vee a \leq x \vee b$, $a \vee y \leq b \vee y$ (it follows that $x, y \leq x \vee y$).

A.9. (weak modularity). Let $i_\wedge(\Lambda)$ be the $\wedge$-idempotent elements i.e. the $x \in \Lambda$ such that $x \wedge x = x$, then for $x \in i_\wedge(\Lambda)$ and $x \leq z$ in Λ we have :

$$x \vee (x \wedge z) \leq (x \vee x) \wedge z$$
$$x \vee (z \wedge x) \leq (z \vee z) \wedge z$$

(if Λ satisfies A.1.... A.9., then $i_1(\Lambda) \subset i_\vee(\Lambda)$ where $i_\vee(\Lambda)$ consists of $z \in \Lambda$ such that $z \vee z = z$).

A.10. For any global cover $1 = \lambda_1 \vee \ldots \vee \lambda_n$ and any $z \in \Lambda$ we have : $(x \wedge \lambda_1) \vee \ldots \vee (x \wedge \lambda_n) = x$. The presheaves on Λ with values in a Grothendieck (abelian) category is again a Grothentieck (abelian) category but this fails for the category of sheaves (defined suitably), this category is not a topos. If $x \wedge x = x$ for all x in Λ then Λ is an abelian operation in Λ so the noncommutativity of the topology is exactly characterized by the existence of nontrivial selfintersection.

The definition of a noncommutative Grothendieck topology may be given by "symmetrizaion" of the classical definition. A category $\underline{\mathcal{C}}$ such that for each object U of $\underline{\mathcal{C}}$ a set Cov(U) is given, consisting of subsets of morphisms with common target U, is a noncommutative Grothendieck topology if it satisfies the following properties :

G.1. $\{U \to U\} \in \mathrm{Cov}(U)$
G.2. If $\{U_i \to U, i \in I) \in \mathrm{Cov}(U)$ and $\{U_{ij} \to U_i, j \in J\} \in \mathrm{Cov}(U)$ for all $i \in I$, then $\{U_{ij} \to U, i \in I, \in J\} \in \mathrm{Cov}(U)$.
G.3. For given $U' \to U$ and $\{U_i \to U, i \in I\} \in \mathrm{Cov}(U)$ there is a cover $\{U' \times_U U_i \to U', i \in I\}$ satisfying : for $V \to U_i$, $V \to U'$ and $T \to U_i, T \to U'$ there exist $V \wedge T \to U' X_U U_i$ and $T \wedge V \to U' \times_U U_i$ fitting in the commutative diagram :

$$\begin{tikzcd}
& & U & & \\
& U_i \arrow[ur] & & U' \arrow[ul] & \\
T \arrow[ur] \arrow[urrr] & & U' \times_U U_i \arrow[ul] \arrow[ur] & & V \arrow[ul] \arrow[ulll] \\
& T \wedge V \arrow[ul] \arrow[ur] & & V \wedge T \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Taking $T = V$ in the foregoing, one obtains the obvious non-idempotent versions of the pullback property reducing to G.3. in case $T = T \wedge T$.

2.2.4 Example. Any modular lattice satisfies A.1. …A.9. A distributive lattice satisfies A.1.…A.10. The lattice of all torsion theories on R-mod for a associative ring R, say R-tors, is a complete modular lattice; we shall look at the torsion theories by their kernel functors. If the idempotent kernel functors σ, τ are given by their Gabriel filters $\mathcal{L}(\sigma), \mathcal{L}(\tau)$ resp. then $\sigma \wedge \tau$ and $\sigma \vee \tau$ are defined by $\mathcal{L}(\sigma \wedge \tau), \mathcal{L}(\sigma \vee \tau)$ resp. Define $W(R)$ is the set of filters obtained by evaluating expressions involving products and intersections of filters corresponding to elements of R-tors. For $w, w' \in W(R)$ put $w \leq w'$ if and only if $\mathcal{L}(w') \subset \mathcal{L}(w)$. We define $w \vee w'$ by $\mathcal{L}(w) \cap \mathcal{L}(w')$, hence $\vee$ is a commutative operation here. Put $\mathcal{L}(ww')$ equal to $\{L \in R, L \supset J'J, J' \in \mathcal{L}(w'), J \in \mathcal{L}(w)\}$, this defines $w \wedge w'$ and the corresponding function $Q_w Q_{w'}$.

2.2.5 Proposition. $W(R)$ *consists of exact preradicals and it is a noncommutative topology with respect to the structures defined above.*

A categorical version of noncommutative algebraic geometry can now be developed, cf. [135]; there are many open questions in this theory, I refer to loc. cit. for many exercises and research projects. The example obtained from Ore sets has some interesting applications using Çech-cohomology on the noncommutative topology one may calculate a moduli space for the left ideals of the Weyl algebra (work of L. Willaert, F. Van Oystaeyen, recovering a result of L. Le Bruyn). This technique may very probably be applied to several other quantized algebras where we can calculate enough Ore localizations.

3. Regularity and Filter-Graded Transfer

3.1 Graded Homological Algebra and Regularity

A ling is **left regular** if every finitely generated R-moduli has finite projective dimension. For a graded ring R **left**gr**-regularity** is defined in terms of objects of R-gr. For a left

Noetherien R we have $\text{gldim}R[X_1,\ldots,X_n] = n + \text{gldim}R$ and Auslander's theorem learns that for a Noetherian R, $\text{rgldim} = \text{lgldim}R$. For graded rings the graded versions of several dimensions can be defined (and used) in the obvious way. For example if R is a graded Noetherian ring, then the left and right (graded) global dimensions coincide.

3.1.1 Theorem. *Let R be a Zariski filtered ring (in the positive case $G(R), R, \widetilde{R}$ are Noetherian rings) then :*

(1) *If $G(R)$ is left gr-regular then $\widetilde{R}$ is left regular*
(2) *We have :*

$$\text{grgldim}\widetilde{R} \leq 1 + \text{grgldim}G(R)$$
$$\text{gldim}\widetilde{R} \leq 1 + \text{gldim}G(R)$$

and equalities hold in case $G(R)$ has finite (gr-)global dimension.

It is now possible to obtain left regularity of a.o. the following rings : $A[X,\sigma.\delta]$ where δ is a σ-derivation of the left regular A and σ automorphism of A, the crossed product $A*G$ where A is left regular and G is poly-infinite cyclic, the crossed product $A*U(g)$ where A is a left regular K-algebra and $U(g)$ the K-enveloping algebra of a finite dimensional Lie algebra $g, \ldots$. For a survey on $GK\text{dim}$ and a new dimension, the schematic dimension Sdim we refer to [131] Section 3.1.

For a left Noetherian R and a finitely generated R-module M we have $\text{pdim}_R M = n < \infty$ if and only if $\text{Ext}_R^{n+1}(M,N) = 0$ for all finitely generated R-modules N, consequently $\text{Ext}_R^n(M,R) \neq 0$. For any R-module M the **grade number** $j_R(M)$ is the unique smallest integer such that $\text{Ext}_R^{j_R(M)}(M,R) \neq 0$; if such integer does not exist then we put $j_R(M) = \infty$. We say that M satisfies the Auslander condition if for $k \geq 0$ and any R-submodule N of $\text{Ext}_R^k(M,R)$ it follows that $j_R(N) \geq k$. If we have an exact sequence of R-modules :

$$0 \to M' \to M \to M'' \to 0$$

then if M', M'' satisfy the Auslander condition so does M. In case M satisfies the Auslander condition then $j_R(M) = \inf\{j_R(M'), j_R(M'')\}$. A left and right Noetherian ring R of finite global dimension is **Auslander regular** if every finitely generated left or right R-module satisfies the Auslander condition.

3.1.2 Theorem. (Li Huishi, F. Van Oystaeyen, cf. [131] Theorem 5) *If R is a (left and right) Zariski filtered ring such that $G(R)$ is Auslander regular then R is Auslander regular. The theorem yields Auslander regularity of the following rings : $U(g)$ for a finite dimensional Lie algebra g, the n-th Weyl algebra $\mathbb{A}_n(K)$, the ring $\mathcal{D}(V)$ of $\mathbb{C}$-linear differential operators on irreducible smooth subvarieties V of affine n-space, the ring $\mathcal{D}_1$ of O-linear differential operators on the reular local ring O_n of convergent power series in n-variables over $\mathbb{C}$, the stalks $\mathcal{E}_P$ of the sheaf of microlocal differential operators,...*

3.1.3 Theorem. *Let R be a Zariski filtered ring with $G(R)$ Auslander regular then for every filtered R-module M with good filtration we have $j_R(M) = j_{G(R)}(G(M)) = j_{\widetilde{R}}(\widetilde{M})$ where $\widetilde{M}$ is the Rees module of M with respect to FM.*

Concerning Auslander regularity of the Rees ring we obtain the following result :

3.1.4 Theorem. *If R is a filtered ring such that $\widetilde{R}$ is Noetherian then Auslander regularity of R and $G(R)$ implies Auslander regularity of $\widetilde{R}$. If moroever R is Zariski filtered and $G(R)$ is Auslander regular then $\widetilde{R}$ is Auslander regular.*

Applying the foregoing and some corollaries of it one arrives at the following examples of Auslander regular rings constructed over an Auslander regular A : $A[X,\sigma]$ σ an automorphism of A, $A[[X,\sigma]]$ the X-completion of $A[X,\sigma]$, $A[X,\sigma,\delta]$ where δ is a σ-derivation of A, the crossed product $A * G$ where G is the poly-infinite cyclic group, $A * U(g)$ where A is a K-algebra and g a Lie algebra of finite K-dimension. In particular one finds that the following rings are Auslander regular too : coordinate ring of quantum 2×2 matrices, quantum Weyl algebras $\mathbb{A}_n(q)$, Witten gauge algebras $W(\mathbb{C})$ and "quantum sl_2" $W_q(\mathrm{sl}_2)$.

Using injective resolutions and injective dimension instead of projective resolutions and projective dimension one obtains a similar theory with respect to so-called Auslander-Gorenstein regularity.

A lot of work has gone into the classification of low dimensional algebras e.g. M. Artin, W. Schelter [5]. All 3-dimensional regular algebras have been classified by P. R. Stephenson. Using Cohen-Macauley modules point and line modules over a 3-dimensional quadratic algebra are classified by their homological properties. If R is graded and $R = R_0[R_1]$ then a (left) point module is a cyclic graded R-module $M = \oplus_{n\geq 0} M_n$ such that $M = RM_0$ and the Hilbert series $H_M(t)$ is $(1-t)^{-1}$. A (left) line module is as before but with $H_M(t) = (1-t)^{-2}$.

The point modules, force the Hibert series to look as in the commutative case, maybe too commutative in spirit to yield a good tool in noncommutative geometry. In fact there exist higher dimensional regular algebras with finitely many, say 20, point modules, even there are some without points. Some of these nice algebras, having very few point modules are graded (generic) Clifford algebras.

Put $C = \mathbb{C}[Y_1,\ldots,Y_n], \alpha \in M_n(C)$ a symmetric matrix (α_{ij}) where each α_{ij} is a homogeneous linear polynomial. The **Clifford algebra** $A(\alpha)$ associated to α is defined as the K-algebra with generators $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n\}$ and defining relations :

$$\begin{cases} X_iX_j + X_jX_i = \alpha_{ij}, \text{ for } i,j = 1,\ldots,n \\ [Y_i,X_j] = 0 = [Y_i,Y_j] \text{ for } i,j, = 1\ldots n \end{cases}$$

The gradation of $A(\alpha)$ is defined by putting $X_i \in A(\alpha)_1$ for $i = 1,\ldots,n$ and $Y_j \in A(\alpha)_2$ for $j = 1,\ldots,n$. Expanding $\alpha = \alpha_1 Y_1 + \ldots + \alpha_n Y_n$ where $\alpha_1,\ldots,\alpha_n$ are symmetric matrices in $M_n(K)$ we associate to α an n-dimensional linear system of quadrics $Q_1,\ldots,Q_n \subset \mathbb{P}^{n-1}(K), \underline{Q} = KQ_1 + \ldots + KQ_n$ where $Q_i = \{z \in \mathbb{P}^{n-1}, z^t\alpha z = 0\}$. A base point of $\underline{Q}$ is any point in the intersection of $Q_1,\ldots,Q_n$.

3.1.5 Proposition. (L. Le Bruyn, J. of Algebra 177, 1995)*A Clifford algebra $A(\alpha)$ is a quadratic Auslander regular algebra of dimension n if and only if $\underline{Q}$ has no base points.*

For $n = 4$, M. Van den bergh proved that $A(\alpha)$ has exactly 20 point modules for generic α, an explicit construction of such algebra was given by M. Van Cliff, K. Van Rompay, L. Willaert.

Several references provide a good starting point for reading about low dimensional regular algebras. Results now exist up to dimension 4 as fas as I know. There are interesting research problems here e.g. the relation between graded deformations and so-called rolled up Rees rings (cf. [68]).

3.2 Examples

Several examples of well-known rings popular nowadays have been referred to with reference to the literature. As an appendix I include some of them with full definition.

4. Applications and other Directions

4.1 Cayley Smooth Orders

As a consequence of project 1.1. there always was a tendency to try to relate noncommutative information to the commutative theory (via the centre of the algebras used). This is the case too with the theory related to canonical resolutions of quotient singularities. For a finite group G acting on the vector space $\mathbb{C}^d$ (freely away from the origin), the quotient spqce $\mathbb{C}^d/G$ is an isolated singularity and resolutions $Y \twoheadrightarrow \mathbb{C}^d|G$ were constructed using the skew groupring $\mathbb{C}[X_1, \ldots, X_d] * G$ which is an order having the fix-ring $\mathbb{C}[X_1, \ldots, X_d]^G$ for its centre. In case $d = 2$ we are in the situation of Kleinian singularities this yields minimal resolutions. In case $d = 3$ the skew groupring appears via the superpotential and commuting matrices (in Physics) or via the McKay quiver. For abelian G the study leads to "crepant" resolutions, for general G one obtains partial resolutions with remaining manifold singularities. In [69] L. Le Bruyn obtains lists of types of singularities contained in partial resolutions of the quotient variety $\mathbb{C}^d/G$.

Smoothness of R-orders , R a commutative ring e.g. the coordinate ring of some (quotient) variety, is defined in two ways :

(1) J. P. Serre smoothness i.e. the R-order A has finite global dimension plus Auslander regularity and the Cohen-Macauley property
(2) Cayley-smooth of the corresponding G-variety is smooth. The Zariski and étale covers are used.

Cayley-Hamilton algebras are introduced as algebras with a nice trace map (Definition 1.4. in [69]) and every R-order in a central simple algebra is a CH-algebra. C. Procesi proved the reconstruction of orders and their centers from the G-equivariant geometry of the quotient variety in case $G = PGL_n$. The category of CH-algebras of degree n with trace preserving morphisms constitutes a version of noncommutative geometry. A Cayley smooth algebra A is an object of the foregoing category with a lifting property i.e. if R is in the category and I is a nilpotent ideal of B such that $B|I$ is in the category and the natural $B \to B|I$ preserves the trace than any trace preserving $\emptyset : A \to BI$ lifts to $A \to B$. These Cayley smooth algebras correspond to smooth PGL_n-varieties. The noncommutative structure sheaf of an R-order is then used as the noncommutative geometry, as explained in the first part of this survey. Via the representation theory (marked) quiver settings are associated to the orders which connect the Zariski and étale structure to quivers (Theorem 7.9. of [69]. This leads to quiver-recognition of isolated singularities (Theorem 1.12. loc. cit) and noncommutative desingularizations. We refer to loc. cit. for complete detail.

4.2 Hopf Algebras and twisted Algebras

The effect of a Hopf algebra structure on the noncommutative geometry remains largely to be studied. Of course one may consider braided categories and localizations thereof but this does not connect nicely to something like a noncommutative algebraic group. For P.I. rings we know that the assumption of a group variety structure on its prime spectrum makes it into a commutative variety, perhaps one should look for a theory of noncommutative algebraic semigroups ? In the direction of valuation theory there has been some work by Aly Farahat, F. Van Oystaeyen on Hopf valuations and related Hopf orders. An interesting consequence of this theory (cf. [44]) is the appearance of new maximal orders over specific number rings leading to very concrete examples.

On the other hand, a replacement of the geometric product may be found by using the twisted product of algebras. A general theory of twisting algebras appeared in the paper [83] by X. Lopez, F. Panaite, F. Van Oystaeyen. An example is given by A. Connes quantum space that turns out to be a twisted product of quantum planes. The twisted product can be iterated under some pentagonal diagramme condition, cf. [89]. The algebraic properties of general twisted products of low dimensional algebras (e.g. with the quaternions $\mathbb{H}$ over the reals) should be further investigated. Since connections behave well with respect to twisted products some further relations with A. Conne's noncommutative geometry remain to be investigated.

4.3 Simple Modules

The classification of simple (left) modules of a noncommutative algebra is a basic problem relating to representation theory on one side and to some kind of noncommutative geometry on the other side. For algebras of quantized type (deformations) not many cases have been completely solved. For example the case of the second Weyl algebras remained open for a while till V. Bavula, F. Van Oystaeyen obtained a classification by pairs of elements in twisted Laurent polynomials in [17]. They continued this for rings of differential operators on surfaces that are products of curves in [18]. The techniques make use of a gradation and graded module theory as well as G/K-dimension.

5. Appendix : Some Examples

5.1 Quantum 2×2-matrices

The $\mathbb{C}$-algebra generated by a, b, c, d with defining relations :

$$\begin{aligned} &ba = q^{-2}ab, ca = q^{-2}ac, bc = cb, \\ &db = q^{-2}bd, dc = q^{-2}cd, ad - da = (q^2 - g^{-2})bc \end{aligned}$$

is called the algebra of quantum 2×2-matrices $M_q(2)$. Then $M_q(2)$ is a schematic algebra and a Noetherian domain as it is an iterated Ore extension of a nice kind :

$R_1 = \mathbb{C}[a]$
$R_2 = \mathbb{C}[a, b]/(ba - q^{-2}ab)$

$R_3 = \mathbb{C}[a,b,c]/(ba - q^{-2}ab, ca - q^{-2}ac, bc - cb)$
$R_2 = R_1[b, \rho_1]$ where the automorphism ρ_1 of R_1 is determined by $\rho_1(a) = q^{-2}a$.
$R_3 = R_2[c, \rho_2]$ where $\rho_2(a) = q^{-2}a,\ \rho_2(b) = b$
Finally $M_q(2) = R_3[d, \rho_3, \delta]$ where the automorphism ρ_3 is determined by $\rho_3(a) = a, \rho_3(b) = q^{-2}b, \rho_3(c) = q^{-2}c$ and the ρ_3-derivation δ is given by $\delta(a) = (q^2 - q^{-2})bc$ and $\delta(b) - \delta(c) = 0$.

5.2 Quantum Weyl Algebras

Look at $(\lambda_{ij}) \in M_n(k)$ with $\lambda_{ij} \in k^*$, together with a row $(q_1, \ldots, q_n), q_i \in k^*$. The **quantum Weyl algebra** $A_n(\overline{q}, \Lambda)$ in the R-algebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$ with defining relations : (putting $\mu_{ij} = \lambda_{ij}q_i$), for $i \subset j$:

$$\begin{aligned}
x_i x_j &= \mu_{ij} x_j x_i \\
x_i y_j &= \lambda_{ji} y_j x_i \\
y_j y_i &= \lambda_{ji} y_i y_j \\
x_j y_i &= \mu_{ij} y_i x_j \\
x_j y_j &= q_j y_i x_j + 1 + \textstyle\sum_{i<j} (q_i - 1) y_i x_i
\end{aligned}$$

We may again establish that $A_n(\overline{q}, \Lambda)$ is an iterated Ore extension by adding the variables in the order : $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$. The associated graded rings with respect to the standard filtrations may be calculated and one obtains the fact that $A_n(\overline{q}, \Lambda)$ is affine schematic (and its Rees rings too) and also schematic.

5.3 The Sklyanin Algebra

The 3-dimensional algebra generated over k by three homogeneous elements of degree $1, X, Y, Z$ say, with defining relations :

$$\begin{aligned}
aXY + bYX + cZ^2 &= 0 \\
aYZ + bZY + cX^2 &= 0 \\
aZX + bXZ + cY^2 &= 0
\end{aligned}$$

$(a, b, x \in k)$ is said to be the Sklyanin algebra $S_k(a, b, c)$. This algebra is schematic.

5.4 Color Lie Superalgebras

Consider an abelian group Γ and $\epsilon : \Gamma \times \Gamma \to \mathbb{C}^*$ satisfying : $\epsilon(\alpha, \beta)\ \epsilon(\beta, \alpha) = 1,\ \epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\ \epsilon(\alpha, \gamma),\ \epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\ \epsilon(\beta, \gamma)$.

Let $L = \oplus_{\gamma \in \Gamma} L_\gamma$ be a Γ-graded vector space together with a graded bilinear mapping $< \ldots, \ldots >$ satisfying for $a \in L_\alpha, b \in L_\beta, c \in L_\gamma, \alpha, \beta, \gamma \in \Gamma, < a, b >= -\epsilon(\alpha, \beta) < b, a >$
$0 = \epsilon(\gamma, \alpha) < a, < b, c >> + \epsilon(\alpha, \beta) < b, < c, a >> + \epsilon(\beta, \gamma) < c, < a, b >>$

Consider the tensor algebra $T(L)$ and let $J(L)$ be the ideal generated by all

$$a \otimes b - \epsilon(\alpha, \beta) b \otimes a - < a, b >$$

for $a \in L_\alpha, b \in L_\beta$. The algebra $T(L)/J(L)$ is the universal enveloping algebra of L, it is a Γ-graded ring and it has also a positive filtration by taking $F_n U_K(L)$ to be the image of $T(K)_n$. ¿From the generalized Poincaré-Birkhoff-Witt theorem it follows that the associated $\mathbb{Z}$-graded algebra $G(U(L))$ is a $\mathbb{Z} \times \Gamma$-graded algebra isomorphic to $T(L)$ modulo the ideal generated by all $a \otimes b - \epsilon(\alpha, \beta) b \otimes a$, for $a \in L_\alpha, b \in L_\beta$. We have established (Theorem 1.2.19 in [131]) that $G(U(L))$ is schematic, $U(L)$ is weakly affine schematic and the Rees ring $U(L)^\sim$ is schematic.

5.5 Witten's Gauge Algebras

Consider the $\mathbb{C}$-algebra W generated by X, Y, Z subjected to the relations :

$$\begin{aligned}
&XY + \alpha YX + \beta Y = 0\\
&YZ + \gamma ZY + \delta X^2 + \epsilon X = 0\\
&ZX + \xi XZ + \eta Z = 0
\end{aligned}$$

Total degree on X, Y, Z defines the standard filtration on W. It is not hard to verify that $G(W)$ is defined by the relations :

$$\begin{aligned}
&XY + \alpha YX = 0\\
&YZ + \gamma ZY + \delta X^2 = 0\\
&ZX + \xi XZ = 0
\end{aligned}$$

The algebra $G(W)$ is quadratic and represents a quantum space in the sense of Y. Manin. The algebra $W(\mathbb{C})$ is weakly affine schematic, $G(W)$ and the Rees ring $\widetilde{W}$ are schematic.

5.6 Quantum sl_2 (Woronowicz)

Let $W_q(\mathrm{sl}_2)$ be the $\mathbb{C}$-algebra generated by X, Y, Z subjected to the following defining relations :

$$\begin{aligned}
&\sqrt{q}XZ - \sqrt{q}^{-1}ZX = \sqrt{q+q^{-1}}Z\\
&\sqrt{q}^{-1}XY = \sqrt{q}YX = -\sqrt{q+q^{-1}}Y\\
&YZ - ZY = (\sqrt{q} - \sqrt{q}^{-1})X^2 - \sqrt{q+q^{-1}}X
\end{aligned}$$

(classically $q = \exp\left(\frac{2\pi i}{k+2}\right)$ and k is the Chern coupling constant.

In $W_q(\mathrm{sl}_2)$ there is a central quadratic element, the deformed Casimir operator $C = \sqrt{q}^{-1}ZY + \sqrt{q}YZ + X^2$. Put $A = 1 - C(\sqrt{q} - \sqrt{q}^{-1})(\sqrt{q+q^{-1}})$ and write :

$$\begin{aligned}
&x = (X - (\sqrt{q} - \sqrt{q}^{-1})\sqrt{q+q^{-1}}^{-1}c)\sqrt{q+q^{-1}}A^{-1}\\
&y = Y(\sqrt{q+q^{-1}}^{-1})\sqrt{A}^{-1}\\
&z = Z(\sqrt{q+q^{-1}}^{-1})\sqrt{A}^{-1}
\end{aligned}$$

which is posible **up to inverting the central element** A ! The relations rewrite in the new arguments x, y, z as

$$\begin{aligned}
&\sqrt{q}\; xz - \sqrt{q}^{-1}zx = z\\
&\sqrt{q}^{-1}xy - \sqrt{q}\; yx = y\\
&q^{-1}zy - qyz = x
\end{aligned}$$

One calculates from this the relations for the associated graded rings in the standard filtration :

$$\begin{aligned}
&\sqrt{q}\; xz - \sqrt{q}^{-1}zx = 0\\
&\sqrt{q}^{-1}xy - \sqrt{q}\; yx = 0\\
&q^{-1}zy - qyz = 0
\end{aligned}$$

The Rees ring $W_q(\mathrm{sl}_2)^\sim$ can be written by homogenizing the relations between the x, y, z.

Looking at the Witten algebra W defined by putting $\delta = 0$ and making obvious choices for the $\alpha, \beta, \gamma, \ldots$ it is clear that the special Witten algebra then obtained contains A^{-1} as a normalizing element. This means that $W_q(\mathrm{sl}_2)$ and the special Witten algebra are birational in the noncommutative sense (up to inverting a central element in the first and a normalizing element in the second they yield the same localization but up to the quadratic extension obtained by adding $\sqrt{A}$. Again $G(W_q(\mathrm{sl}_2))$ and the Rees ring $W_q(\mathrm{sl}_2)^{\sim}$ are schematic.

All foregoing examples are Auslander regular (3.2.17. of [131]).

References

[1] E. Abe, *Hopf Algebras*, Cambridge Univ. Press, Cambridge, 1977.

[2] M. Artin, *Geometry of Quantum Planes*, in : Azumaya Algebras, Actions and Groups, (eds. D. Haile and J. Osterberg, Contemp. Math. Vol 124, Amer. Math. Soc., Providence, 1992.

[3] M. Artin, *Grothendiek Topologies*, Harvard Univ. Lect. Notes, Cambridge Mass. 1962.

[4] M. Artin, A. Grotendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lect. Notes in Math. 269, 270, 305, Springer Verlag, Berlin, 1972-73.

[5] M. Artin, W. Schelter, *Graded algebras of global dimension 3*, Adv. Math. **66** (1987), 171-216.

[6] M. Artin, J. Tate, M. Van den Bergh, *Modules over Regular Algebras of Dimension 3*, Invent. Math. **106** (1991), 335-388.

[7] M. Artin, J. Tate, M. Van den Bergh, *Some algebras associated to automorphisms of curves*, in The Grothendieck Festschrift (Eds. P. Cartier et al), Birkhauser, Basel, 1990.

[8] M. Artin, J. Zhang, *Noncommutative Projective Schemes*, Adv. in Math. **109** (1994), 228-287

[9] M. Artin, *Local Structure of Maximal Orders on Surfaces* in : LNM 917, Springer Verlag, Berlin 1982, pp. 146-181.

[10] M.J. Asensio, M. Van den Bergh, F. Van Oystaeyen, *A new algebraic approach to microlocalization of filtered rings*, Trans. Amer. Math. Soc.**316**(2) (1989), 537-555.

[11] B. Auslander, *The Brauer Group of a Ringed Space*, J. Alg. **4** (1966), 220-273.

[12] M. Auslander, O. Goldman, *The Brauer Group of a Commutative Ring*, Trans. Amer. Math. Soc. **97** (1960), 367-409

[13] M. Auslander, *On the dimension of modules and algebras III*, Nagoya Math. J. **9** (1955), 67-77.

[14] M. Awami, F. Van Oystaeyen, *On filtered rings with Noetherian associated graded rings, in Proceedings of Ring Theory Meeting*, Granada 1986, Springer Verlag, LNM.

[15] G. Azumaya, *On Maximally Central Algebras*, Nagoya Math. J. **2** (1951), 119-150.

[16] H. Bass, *Algebraic K-Theory*, Benjamin, New York, 1968.

[17] V. Bavula, F. Van Oystaeyen, *Simple Holonomic Modules over the Second Weyl Algebra*, Adv. Math. **150**(1) (2000), 80-116.

[18] V. Bavula, F. Van Oystaeyen, *Simple Holonomic Modules over Rings of Differential Operators with Regular Coefficients of Krull Dimension 2*, Trans. Amer. Math. Soc. **353**(6) (2001), 2193-2214.

[19] G. Bergman, *Everybody Knows What a Hopf Algebra is.* Contemp. Math. **43** (1985), 25-48.

[20] I.N. Bernstein, *Algebraic theory of $\mathcal{D}$-modules*, Preprint, 1982.

[21] I.N. Bernstein, *On the dimension of modules and algebras III*, Direct limits, Nagoya Math. J. **13** (1958), 83-84.

[22] J.-E. Björk, *The Auslander condition on Noetherian rings*, Proc. Sem. Malliavin, LNM, 1404, Springer Verlag, 1988-173.

[23] J.-E. Björk, *Filtered Noetherian rings*, Math. Surveys and Monographs, No. 24, 1987, 59-97.

[24] J.-E. Björk, *Rings of differential operators*, Math. Library 21, North Holland, Amsterdam, 1979.

[25] R. Bocklandt, L. Le Bruyn, G. Van de Weyer, *Smooth Order Singularities*, J. of Algebra, Appl. **2**(4) (2003), 365-395.

[26] R. Bocklandt, *Graded Calabi-Yau Algebras of Dimension 3*, J. Pure Appl. Algebra **212**(1) (2008), 14-32.

[27] M. Boratinski, *A change of rings theorem and the Artin Rees property*, Proc. Amer. Math Soc. **53** (1975), 307-320.
[28] A. Borel, *Algebraic $\mathcal{D}$-modules*, Academic press, London-New York, 1987.
[29] W. Borho, J. L. Brylinski, *Differential Operators in Homogeneous Spaces Irreducibility of the associated Variety*, Invent. Math. **69** (1982), 437-476.
[30] D. Bulacu, F. Panaite, *A Generalization of the Quasi-Hopf Algebra $D^\omega(G)$*, Comm. Alg. , to appear.
[31] S. Caenepeel, *Brauer Groups, Hopf Algebras and Galois Theory*, Kluwer Academic Publ., Dordrecht, 1998.
[32] S. Caenepeel, F. Van Oystaeyen, *Brauer Groups and the Cohomology of Graded Rings*, Monographs. vol. 121, M. Dekker Publ., New York, 1989
[33] S. Caenepeel, F. Van Oystaeyen, Y. H. Zhang, *The Brauer Group of Yetter-Drinfel'd Module Algebras*, Trans. Amer. Math. Soc. **349** (1997), 3737-3771.
[34] S. Caenepeel, F. Van Oystaeyen, Y. H. Zhang, *Quantum Yang-Baxter Module Algebras*, K-Theory **8** (1994), 231-255.
[35] H. Cartan, S. Eilenberg, *Homological Algebra*, Princeton Univ. Press., Princeton 1956.
[36] A. Connes, *C^*-Algébres et Géometrie Differentielle*, C.R. Acad. Sc. Paris **290** (1980), 599-604.
[37] P. Deligne, J. S. Milne, *Tannakian Categories*, Lect. Notes in Math. 900, Springer-Verlag, Berlin 1982.
[38] F. De Meyer, E. Ingraham, *Separable Algebras over Commutative Rings*, Lect. Notes in Math. 181, Springer Verlag, Berlin 1971.
[39] R. Dijkgraaf, V. Pasquier, P. Roche, *Quasi-Hopf Algebras*, Group Cohomology and Orbifold Models, Nuclear Phys. B. Proc. Suppl. 18 B, 1990, 60-72.
[40] J. Dixmier, *Enveloping algebras*, Math. Library, North Holland, Vol. 14, Amsterdam, 1977.
[41] V.G. Drinfel'd, *Quantum Groups*, in Proc. ICM, Amer. Math. Soc., Providence, 1987, 798-820
[42] V.G. Drinfel'd, *Quasi-Hopf Algebras*, Leningrad Math. J. **1** (1990), 1419-1457.
[43] V.G. Drinfel'd, *Quasi-Hopf Algebras*, Algebra i Amaliz **1** (1989), 114-148.
[44] A. Farahat, F. Van Oystaeyen, *Hopf Filtrations and Larson-type Orders in Hopf Algebras*, J. of Algebra **267**(2) (2003), 756-772.
[45] R. M. Fossum, A. Griffith, I. Reiten, *Trivial extensions of abelian categories, homological algebra of trivial extensions of abelian categories with applications to ring theory*, LNM. 456, Springer-Verlag, 1975.
[46] O. Gabber, *The integrability of characteristic varieties*, Amer. J. Math. **103** (1981), 445-468.
[47] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323-448.
[48] P. Gabriel, R. Rentschler, *Sur la dimension des anneaux et ensembles ordonnées*, C. R. Acad. Sci. Paris **265** (1967), 712-715.
[49] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Invent. Math. **84** (1986), 327-402.
[50] J. Golan, *Localization of noncommutative rings*, M. Dekker, New York, 1975.
[51] O. Goldman, *Rings and Modules of Quotients*, J. of Algebra, **13** (1969), 10-47.
[52] R. Gordon, J. C. Robson, *Krull Dimension, Memoirs of the AMS*, No. 133, 1973.
[53] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Math. 52, Springer Verlag, Berlin, 1977.
[54] M. Hussein, F. Van Oystaeyen, *Discrete Valuations Extend to Certain Algebras of Quantum Type*, Comm. in Algebra **24**(8) (1996), 2551-2566.
[55] M. Jimbo, *A q-difference Analog of $U(g)$ and the Yang-Baxter Equation*, Lett. Math. Phys. **10** (1985), 63-69.
[56] D.A. Jordan, *Iterated skew Polynomial Rings and Quantum Groups*, J. Alg. **156** (1993), 194-218.
[57] A. Joyal, R. Street, *Braided Monoidal Categories*, Adv. Math. **102** (1993), 20-78.
[58] A. Joyal, R. Street, *Tortile Yang-Baxter Operators in Tensor Categories*, J. Pure Appl. Algebra **71** (1991), 43-51.
[59] M. Kashiwara, *B-function and holonomic systems*, Invent. Math. **38** (1976), 33-53.
[60] C. Kassel, *Quantum Groups*, Springer-Verlag, Berlin, 1995.
[61] C. Kassel, V. Turaev, *Double Construction for Monoidal Categories*, Acta Math. **175** (1995), 1-48.
[62] M. Kontsevich, *Triangulated Categories and Geometry*, Course a É cole Norm. Sup., Paris, 1998.

[63] G. R. Krause, T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Research Notes in Mathematics, 116, Pitman Advanced Publishing Program, 1985.

[64] R. G. Larson, J. Towber, *Two Dual Classes of Bialgebras Related to the Concepts of Quantum Groups and Quantum Lie Algebras*, Comm. Algebra **19** (1991), 3295-3345.

[65] L. Le Bruyn, *Conformal sl_2 Enveloping Algebras*, Comm. Alg. **23**(4), (1995), 1325-1362.

[66] L. Le Bruyn, *Homological Properties of Braided Matrices*, J. of Alg. **170**(2) (1994), 596-607.

[67] L. Le Bruyn, F. Van Oystaeyen, *Quantum Sections and Gauge Algebras*, Publ. Math. **36**(2A) (1992), 693-714.

[68] L. Le Bruyn, M. Van den Bergh, F. Van Oystaeyen, *Graded Orders*, Birkhauser Verlag, 1988.

[69] L. Le Bruyn, *Nomcommutative Geometry and Cayley-smooth Orders*, in Pure and Applied Mathematics, Chapman & Hall, Lect. Notes Vol. 290, Boca Raton, 2006.

[70] Huishi Li, *Note on microlocalization of filtered rings and the embedding of rings in skew fields*, Bull. Math. Soc. Belgique, (serie A) **43** (1991), 49-57.

[71] Huishi Li, *Note on pure module theory over Zariskian filtered ring and the generalized Roos theorem*, Comm. Alg. **19** (1991), 843-862.

[72] Huishi Li, *Structure Sheaves on Almost Commutative Algebras*, Comm. in Algebra, to appear.

[73] Huishi Li, M. Van den Bergh, F. Van Oystaeyen, *The global Dimension and Regularity of Rees Rings for Non-Zariskian Filtration*, Comm. Alg. **18** (1990), 3195-3208.

[74] Huishi Li, F. Van Oystaeyen, *Dehomogenization of Gradings to Zariskian Filtrations and Applications to Invertible Ideals*, Proc. Amer. Math. Soc. **115**(1) (1992), 1-11.

[75] Huishi Li, F. Van Oystaeyen, *Filtrations on Simple Artinian Rings*, J. Alg. **132** (1990), 361-376.

[76] Huishi Li, F. Van Oystaeyen, *Global dimension and Auslander regularity of graded rings*, Bull. Math. Soc. Belgique, (serie A) **43** (1991), 59-87.

[77] Huishi Li, F. Van Oystaeyen, *Strongly filtered rings, applied to Gabber's integrability theorem and modules with regular singularities*, Proc. Sem. Malliavin, LNM. 1404, Springer-Verlag, 1988.

[78] Huishi Li, F. Van Oystaeyen, E. Wexler-Kreindler, *Zariski Rings and Flatness of Completion*, J. Alg. **138** (1991), 327-339.

[79] Huishi Li, F. Van Oystaeyen, *Zariskian Filtrations*, Kluwer Monographs, Vol. 2, Kluwer Acad. Publ., 1995.

[80] Huishi Li, F. Van Oystaeyen, *Zariskian Filtrations*, Comm. Alg. **17** (1989), 2945-2970.

[81] F. W. Long, *The Brauer Group of Dimodule Algebras*, J. Alg. **31** (1974), 559-601.

[82] F.W. Long, *Generalization of the Brauer Group of Graded Algebras*, Proc. London Math. Soc. **29** (1974), 237-256.

[83] X. Lopez, F. Panaite, F. Van Oystaeyen, *General Twisting of Algebras*, Adv. Math. **212**(1) (2007), 315-337.

[84] S. Maclane, *Categories for the Working Mathematician*, Springer-Verlag, New York, 1974.

[85] S. Majid, *Algebras and Hopf Algebras in Braided Categories*, Lect. Notes in Pure Applied Math. 158, 1994, 55-105.

[86] S. Majid, *Doubles of Quasitriangular Hopf Algebras*, Comm. Alg. **19** (1991), 3061-3073.

[87] S. Majid, *Tannaka-Krein Theorem for Quasi-Hopf Algebras and Other Results*, Contemp. Math. **134** (1992), 219-232.

[88] Yu I Manin, *Quantum Groups and Noncommutative Geometry*, Publ. Centre Math. Montreal, 1988.

[89] P. J. Martinez, J. Lopez, F. Panaite, F. Van Oystaeyen, *On Iterated Twisted Tensor Products of Algebra*, Internat. J. Math. **19**(9) (2008), 1053-1101.

[90] J. C. McConnell, *The K-theory of filtered rings and skew Laurent extensions*, LNM. 1146, Springer-Verlag, 1987.

[91] J.C. McConnell, J.C. Robson, *Noncommutative Noetherian rings*, J. Wiley, London, 1987.

[92] S. Montgomery, *Hopf Algebras and Their Actions on Rings*, Amer. Math. Soc., Providence, 1993.

[93] D. Murdoch, F. Van Oystaeyen, *Noncommutative Localization and Sheaves*, J. of Algebra **35** (1975), 500-515.

[94] C. Năstăsescu, E. Nauwelaerts, F. Van Oystaeyen, *Arithmetically Graded Rings Revisited*, Comm. Alg. **14**(10) (1986), 1991-2017.

[95] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable Functors Applied to Graded Rings*, J. Alg. **123** (1989), 397-413.
[96] C. Năstăsescu, F. Van Oystaeyen, *Dimensions of Ring Theory*, D. Reidel Publ. Co., 1987.
[97] C. Năstăsescu, F. Van Oystaeyen, *Graded Ring Theory*, Library of Math. 28, North-Holland, Amsterdam 1982.
[98] D.G. Northcott, *An Introduction to Homological Algebra*, Cambridge Univ. Press, London and New York, 1960.
[99] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge Univ. Press, London and New York, 1968.
[100] A. V. Odeskii, B. L. Feigin, *Elliptic Sklyanin Algebras*, (in Russian), Funkt. Anal. i Priloz **23** (1989), 45-54
[101] F. Panaite, F. Van Oystaeyen, *Quasi-Hopf Algebras and the Centre of a Tensor Category*, Preprint 1998.
[102] B. Pareigis, *The Brauer Group of a Symmetric Monoidal Category*, in Lect. Notes in Math. 549, Springer Verlag, Berlin, 1976.
[103] B. Pareigis, *Non-additive Ring and Module Theory IV, The Brauer Group of a Symmetric Monoidal Category*, Lect. Notes in Math. 549, 1976, 112-133.
[104] B. Pareigis, *Non-additive Ring and Module Theory I, III.* Publ. Math. Debrecen **24** (1977), 189-204, Publ. Math. Debrecen **25** (1978), 177-186.
[105] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, Academic Press, London, 1973.
[106] C. Procesi, *Rings with Polynomial Identities*, Marcel Dekker INC. , New York, 1977.
[107] D. Radford, *The Structure of Hopf Algebras with a Projection*, J. Alg. **92** (1987), 322-347.
[108] A. Radwan, F. Van Oystaeyen, *Coherent Sheaves over Microstructure Sheaves.*, Bull. Soc. Math. Belg, 1993.
[109] A. Radwan, F. Van Oystaeyen, *Microstructure Sheaves and Quantum Sections over Projective Schemes* in : Hermann, Paris 1995.
[110] R. Rentschler, P. Gabriel, *Sur la Dimension des Anneaux et Ensembles Ordonnés*, C.R. Acad. Sci. Paris **265** (1967), 712-715.
[111] J.-E. Roos, *Bidualité et structure des foncteurs dérivés d'un anneau régulier*, C. R. Acad. Sci. Paris **254** 1962.
[112] J.-E. Roos, *Compléments à l' étude des quotients primitifs des algèbres enveloppantes des algébres de Lie sémi-simple*, C. R. Acad. Sci. Paris. Sér. A-B.
[113] A. Rosenberg, *Noncommutative Geometry*, Reidel Publ. G., Amsterdam, 1997.
[114] J. J. Rotman, *An introduction to homological algebra*, New York, 1979.
[115] R. Sallam, F. Van Oystaeyen, *A Microstructure Sheaf and Quantum Section over a Projective Scheme*, J. Alg. **158**(1) (1993), 201-225.
[116] P. Schapira, *Microdifferential Systems in the Complex Domain*, Springer Verlag, Berlin, 1985.
[117] O. Schilling, *The Theory of Valuations*, AMS. Math. Suur. IV, 1950.
[118] J.P. Serre, *Algébre Locale, Multiplicités*, LNM. 11, Springer-Verlag, 1975.
[119] J.P. Serre, *Faisceaux Algébriques Cohérents*, Ann. Math. **61** (1955), 197-278.
[120] G. Sjödin, *On filtered modules and their associated graded modules*, Math. Scand. **33** (1973), 229-240.
[121] E. K. Sklyanin, *Some Algebraic Structures connected to the Yang-Baxter Equation*, Func. Anal. Appl. **16** (1982), 27-34
[122] T. Springer, *Microlocalization algébrique*, Sem. Algèbre Dubreil- Malliavin, LNM, Springer Verlag, Berlin, 1984.
[123] J. T. Stafford, *Non-holonomic modules over Weyl algebras and enveloping algebras*, Invent. Math. **79** (1984), 619-638.
[124] M. E. Sweedler, *Hopf Algebras*, Benjamin, N.Y., 1969.
[125] B. R. Tennison, *Sheaf Theory*, LMS Lect. Notes Series, vol. 20, Cambridge Univ. Press, 1975.
[126] K.H. Ulbrich, *On Hopf Algebras and Rigid Monoidal Categories*, Isp. J. of Math. **72** (1990), 252-256.

[127] M. Van den Bergh, F. Van Oystaeyen, *Lifting maximal orders*, Comm. Alg. **2**(2) (1989), 341-349.

[128] M. Van den Bergh, *Noncommutative Crepant Resolutions*, Springer verlag, Berlin, pp. 749-770, 2004.

[129] A. Van den Essen, *Modules with regular singularities over filtered rings*, P.R.I. for Mathematical Science, Kyoto Univ., Vol. 22, No. 5, 1986.

[130] J. Van Geel, *Places and valuations in noncommutative ring theory*, Lecture notes, Vol. 71, Marcel Dekker, New York, 1981.

[131] F. Van Oystaeyen, *Algebraic Geometry for Associative Algebras*, Monographs, Pure and Applied Math., Vol. 232, M. Dekker, New York, 2000.

[132] F. Van Oystaeyen, *Graded P.I. Rings and Generalized Crossed Product Azumaya Algebras*, Chin. Ann. Math. B **8**(1) (1987), 13-21.

[133] F. Van Oystaeyen, *A Note on Graded P. I. Rings*, Bull. Soc. Math. Belgique **32** (1980), 22-28.

[134] F. Van Oystaeyen, *Prime Spectra in Noncommutative Algebra*, Lect. Notes in Math. 444, Springer-Verlag, Berlin, 1978.

[135] F. Van Oystaeyen, *Virtual Topology and Functor Geometry*, Lect. Notes in Pure and Applied Math. Vol. 256, Chapman & Hall, Boca Raton, 2008.

[136] F. Van Oystaeyen, R. Sallam, *A Microstructure Sheaf and Quantum Sections over a Projective Scheme*, J. of Algebra **158** (1993), 201-225.

[137] F. Van Oystaeyen, A. Verschoren, *Non-commutative Algebraic Geometry*, Lect. Notes in Math. vol. 887, Springer Verlag, Berlin, 1981.

[138] F. Van Oystaeyen, A. Verschoren, *Relative Invariants of Rings*, I and II. Monographs in Pure Appl. Math. vol 79 and vol. 86, M. Dekker, New York, 1983 and 1984.

[139] F. Van Oystaeyen, L. Willaert, *Cohomology of Schematic Algebras*, J. of Algebra **183**(2) (1996), 359-364.

[140] F. Van Oystaeyen, L. Willaert, *Examples and Quantum Sections of Schematic Algebras*, J. Pure Applied Algebra **120** (1997), 195-211.

[141] F. Van Oystaeyen, L. Willaert, *Grothendieck Topology, Coherent Sheaves and Serre's Theorem for Schematic Algebras*, J. Pure Applied Algebra **104** (1995), 109-122.

[142] F. Van Oystaeyen, J. H. Zhang, *The Brauer Group of a Braided Monoidal Category*, J. Alg. **202** (1998), 96-128.

[143] F. Van Oystaeyen, Y. H. Zhang, *Brauer Groups of Actions, in "Rings, Hopf Algebras and Brauer Groups*, Vol. 197 of Lecture Notes in Pure and Applied Math., S. Caenepeel and A. Verschoren (eds.), Marcel Dekker, 1998, 299-311.

[144] F. Van Oystaeyen, Y. H. Zhang,*Embedding the Hopf Automorphism Group Into the Brauer Group of a Hopf Algebra*, Bull. Canad. Math. Soc. to appear.

[145] C.T.C. Wall, *Graded Brauer Groups*, J. Reine, Angew Math. **213** (1964), 187-199.

[146] W.C. Waterhouse, *Introduction to Affine Group Schemes*, Graduate Texts in Math. 66, Springer Verlag, Berlin 1979.

[147] L. Willaert, *Schematic Algebras and the Auslander Gorenstein Property*, UIA preprint Soc. Math. France, 149-156.

[148] A. Yekuteli, J. Zhang, *Serre Duality for Noncommutative Projective Schemes*, preprint.

[149] D. N. Yetter, *Quantum Groups and Representations of Monoidal Categories*, Math. Proc. Cambridge Philos Soc. **108** (1990), 261-290.

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