HOMOLOGICAL CONJECTURES AND RADICAL-FULL EXTENSIONS

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ABSTRACT. This survey paper is based on my lectures giving at the '42nd Symposium on Ring Theory and Representation Theory' held at Osaka Kyoiku University, Japan, 10-12 October 2009. In this paper, we consider the finitistic dimension and the strong no loop conjectures (and related other homological conjectures). We approach these conjectures by the so-called radical-full extensions, and reduce the verification of these conjectures to the following question: Suppose that $B \subseteq A$ is a radical-full extension such that the radical of B is a left ideal in A, and that one of these conjectures is true for A, is it possible to prove that the same conjecture is true for B ? We shall provide basic definitions and examples, and report current results on the two conjectures in this direction.

Key Words: Algebra, homological conjecture, module, radical-full extension, syzygy.

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1. INTRODUCTION TO TWO HOMOLOGICAL CONJECTURES

In the modern representation theory of algebras, homological methods are used quite often to describe algebraic invariants and properties of modules and algebras. These homological aspects nowadays become interesting topics, and stimulate many deep investigations in different directions. It has turned out that many homological conjectures on algebras and modules arise (see [1]). Among them are the finitistic dimension and the strong no loop conjectures, on which we will concentrate in the present paper. In this section, we shall give the precise statements of the conjectures, and mention other related conjectures; In Section 2, we propose a new idea to understand these two conjectures, namely we want to approach the conjectures by algebra extensions, in this way, one may use external information of an algebra with simple representation theory to investigate homological conjectures for another algebra with usually complicated representation theory, and show that this new method may be useful for attacking the conjectures. In Section

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The detailed version of this paper has been submitted for publication elsewhere.

3, we introduce a special extension of algebras, namely the radical-full extension, and reduce the consideration of our homological conjectures to questions related to radical-full extensions of algebras. We shall give two kinds of examples for obtaining radical-full extensions. In Section 4 and Section 5, we summarize current results on the finitistic dimension and the strong no loop conjectures under our setting, respectively.

Let us fix some notations. Let A be a finite-dimensional k-algebra over a field k. By a module we mean a finitely generated left module, and by A-mod we denote the category of all A-modules. For a module $M \in A$-mod, we denote by $\mathrm{pd}({}_AM)$ (respectively, $\mathrm{id}({}_AM)$) the projective (respectively, injective) dimension of M, and by $\mathrm{gl.dim}(A)$ the global dimension of A. The finitistic dimension of A is defined as

$$\mathrm{fin.dim}(A) = \sup\{\mathrm{pd}({}_AM) \mid M \in A\text{-mod}, \mathrm{pd}({}_AM) < \infty\}$$

The following question on finitistic dimension was mentioned in a paper [2] of H.Bass in 1960, which now becomes a conjecture (see [1]), and will be called the finitistic dimension conjecture in this paper.

Finitistic dimension conjecture: For a finite-dimensional k-algebra A, $\mathrm{fin.dim}(A)$ is finite.

As is known, this conjecture is related to many other homological conjectures in homological algebra and in the representation theory of Artin algebras. Among them are the following:

- **Wakamatsu tilting conjecture**: Suppose that T is a Wakamatsu tilting A-module over a finite-dimensional algebra A, If $\mathrm{pd}({}_AT) < \infty$, then T is a tilting A-module.

  Recall that an A-module T is called a *Wakamatsu tilting module* if $\mathrm{Ext}^n_A(T,T) = 0$ for all $n > 0$, and there is an exact sequence

  $$0 \to {}_AA \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} \cdots \longrightarrow T_n \xrightarrow{f_n} T_{n+1} \to \cdots$$

  in A-mod with $T_i \in \mathrm{add}(T)$ such that $\mathrm{Ext}^1_A(T, \mathrm{Im}(f_i)) = 0$ for all $i \geq 0$, where $\mathrm{add}(T)$ stands for the additive subcategory of A-mod generated by T, and $\mathrm{Im}(f_i)$ denotes the image of f_i.

- **Tilting complement conjecture**: An almost tilting A-module has only finitely many non-isomorphic indecomposable tiling complements.

  Recall that an A-module T is called an *almost tilting module* if $\mathrm{pd}({}_AT) < \infty$, $\mathrm{Ext}^i_A(T,T) = 0$ for all $i > 0$, and the number of non-isomorphic indecomposable summands of T is equal to the number of non-isomorphic simple A-modules minus 1. Given an almost tilting module T, an indecomposable A-module M is called a *tilting complement* to T if $T \oplus M$ is a tilting module.

- **Nakayama Conjecture**: If all injective A-modules I_j in a minimal injective resolution $0 \to {}_AA \to I_0 \to I_1 \to \cdots$ of A are projective, then A is self-injective, that is, ${}_AA$ is an injective A-module.

- **General Nakayama conjecture**: Every indecomposable injective A-module is isomorphic to a direct summand of some I_j in a minimal injective resolution of A: $0 \to {}_AA \to I_0 \to I_1 \to \cdots$.

- **Strong Nakayama conjecture**: If M is a non-zero A-module, then there is an $n \geq 0$ such that $\mathrm{Ext}_A^n(M, A) \neq 0$.
- **Gorenstein symmetry conjecture**: For an algebra A, if $\mathrm{id}({}_AA) < \infty$, then $\mathrm{id}(A_A) < \infty$.

The relationship between these conjectures is that if the finitistic dimension conjecture is true for all Artin algebras, then each of these other conjectures is true for all Artin algebras.

For further discussion on the links between these conjectures, we refer the reader to [17].

Now, we turn to introducing the strong no loop conjecture. In the papers [10] and [6], it was shown that if the global dimension of a finite-dimensional algebra A is finite, then $\mathrm{Ext}_A^1(S, S) = 0$ for all simple A-modules S. Thus, in this case, the quiver of the algebra A has no loops. In [6], a strong version of this result was proposed:

Strong no loop conjecture: If a simple A-module S satisfies $\mathrm{Ext}_A^1(S, S) \neq 0$, then $\mathrm{pd}({}_AS) = \infty$.

We notice that all conjectures listed here are still open.

2. Main ideas and questions

To understand the finitistic dimension and the strong no loop conjectures, we will use certain extensions of algebras. Our idea is to employ external information of bigger algebras A with relatively simple representation theory to investigate the conjectures for subalgebras B with, usually, a relatively complicated representation theory. In this way, we may work out a method for understanding these conjectures, which is applicable to general finite-dimension algebras instead of a special class of algebras.

If A and B are algebras such that B is a subalgebra of A with the same identity, then we say that A is an extension of B. In this case, we also say that $B \subseteq A$ is an extension of algebras.

We consider the following question:

Let $B \subseteq A$ be an extension of algebras. Suppose that a conjecture is true for A, is it possible to show that the same conjecture is true for B ?

Clearly, for an arbitrary extension, we could not say much about this question. So we confront immediately with the following questions that we have to think about:

(a) What kind of extensions should we choose ?
(b) What kind of A should be considered ?
(c) Does such an idea make sense ?

To question (c): On the one hand, every finite-dimensional algebra can be embedded into a full matrix algebra, this experience tells us that a bigger algebra may have a relatively simple representation theory and homological property. On the other hand, for any algebra A given by quiver and relations, if the quiver contains at least two arrows, then A contains a subalgebra of infinite global dimension and of infinite representation type. This means that in general subalgebras of an algebra may be more complicated than the algebra itself. Also, the content of the finitistic dimension conjecture itself does not tell us any information or indication about algebras and modules that we are concerning, so some external information for looking at this "black box" may be needed. From these points of view, our idea may make sense.

To question (b): Transparently, we should choose algebras for which the conjectures hold true. Moreover, we would like to replace the bigger algebras A by some algebras that are "equivalent" to A. For equivalences we here choose stable equivalences of Morita type and derived equivalences since the finiteness of finitistic dimension is preserved under these two kinds of equivalences (see [11]). In fact, it is easy to see that stable equivalences of Morita type even preserve finitistic dimension. This leads us to considering the invariants and constructions of these equivalences, a topic which we shall not touch in this paper.

To question (a): Of course, we cannot choose arbitrary extensions since they do not provide us desired information. So we would like to choose certain idealized extensions and the so-called radical-full extensions, both of which involve the Jacobson radicals of algebras. This topic will be discussed in the next section.

3. Radical-full extensions

In literature, there are many types of extensions, for example, separable extension, semisimple extension, H-separable extension, Frobenius extension, and so on. For our purpose, we shall introduce an extension related to the Jacobson radicals of algebras (see [12] and [13], for example).

An extension $B \subseteq A$ of Artin algebras is called *radical-idealized* if $\mathrm{rad}(B)$ is a left ideal in A, and *radical-full* if $\mathrm{rad}({}_BA) = \mathrm{rad}({}_AA)$, that is, $\mathrm{rad}(A) = \mathrm{rad}(B)A$. A special case of a radical-full extension is the radical-equal extension, that is, an extension $B \subseteq A$ with $\mathrm{rad}(B) = \mathrm{rad}(A)$. Similarly, one can define a right version of these notions by using right modules.

The following propositions show that our approach to the finitistic dimension and the strong no loop conjectures by radical-full extensions may be useful.

Proposition 1. *Let k be a perfect field. Then the following are equivalent:*

(1) *For all k-algebras A,* $\mathrm{fin.dim}(A) < \infty$.

(2) *For any radical-idealized, radical-full extension $C \subseteq B$ of k-algebras, if* $\mathrm{fin.dim}(B) < \infty$*, then* $\mathrm{fin.dim}(C) < \infty$.

(3) *For any radical-idealized extension $C \subseteq B$ of k-algebras, if* $\mathrm{fin.dim}(B) < \infty$*, then* $\mathrm{fin.dim}(C) < \infty$.

(4) *For any extension $C \subseteq B$ of k-algebras such that* $\mathrm{rad}(C)$ *is an ideal in B, if* $\mathrm{fin.dim}(B) < \infty$*, then* $\mathrm{fin.dim}(C) < \infty$.

Similarly, for the strong no loop conjecture, we have the following equivalent conditions. Note that when we say that the strong no loop conjecture is true for an algebra A, we mean that for every simple A-module S with $\mathrm{Ext}^1_A(S,S) \neq 0$, we have $\mathrm{pd}({}_AS) = \infty$.

Proposition 2. *Let k be a perfect field. Then the following are equivalent:*

(1) *The strong no loop conjecture is true for all k-algebras A.*

(2) *For any radical-idealized, radical-full extension $C \subseteq B$ of k-algebras, if the strong no loop conjecture is true for B, then so is it for C.*

(3) *For any radical-idealized extension* $C \subseteq B$ *of* k*-algebras, if the strong no loop conjecture is true for* B*, then so is it for* C*.*

(4) *For any radical-idealized extension* $C \subseteq B$ *of* k*-algebras such that* $\mathrm{rad}(C)$ *is an ideal in* B*, if the strong no loop conjecture is true for* B*, then so is it for* C*.*

Thus, from the above two propositions, it is sufficient to investigate the question in Section 2 for radical-idealized and radical-full extensions. An immediate question is how to get such extensions.

Now let us give three constructions of radical-full extensions.

Suppose that $A = kQ/I$ is a finite-dimensional algebra (over a field k) presented by a quiver $Q = (Q_0, Q_1)$ with relations, where I is an admissible ideal in the path algebra kQ of Q. Note that the composition of two arrows $\alpha, \beta \in Q_1$ is written as $\alpha\beta$, where α comes first and then β follows. As usual, for $i \in Q_0$, we denote by e_i the primitive idempotent element in A corresponding to the vertex i.

(1) **Gluing vertices**

Suppose we are given a partition of the vertex set Q_0, say $Q_0 = \cup_{j=1}^{m} I_j$. Let $f_j = \sum_{i \in I_j} e_i$ for $j = 1, 2, \cdots, m$. Let B be the subalgebra of A generated by $f_1, f_2, \cdots, f_m$ and $\mathrm{rad}(A)$. Then we see that $B \subseteq A$ is a radical-equal extension. The quiver of B is obtained from that of A by gluing all vertices in I_j together for every I_j.

(2) **Unifying arrows**

Let $\{1, 2, \cdots, n\}$ be a subset of Q_0, and let α_i be n distinct arrows in Q_1 such that α_i has the terminus i and that all α_j have a common starting vertex. We define $\overline{Q}_0 = Q_0 \setminus \{1, 2, \cdots, n\}$, $\overline{Q}_1 = Q_1 \setminus \{\alpha_i \mid i = 1, 2, \cdots, n\}$, $e = \sum_{i=1}^{n} e_i$, and $\alpha = \sum_{i=1}^{n} \alpha_i$. Let B be the subalgebra of A generated by the idempotent elements e, e_j, with $j \in \overline{Q}_0$ and the arrows α, β, with $\beta \in \overline{Q}_1$. Note that if α_n is a loop in A, then we have $\alpha_n \alpha = \alpha^2$. It is not hard to see that $\mathrm{rad}(B)$ is a left ideal in A and $\mathrm{rad}(A) = \mathrm{rad}(B)S = \mathrm{rad}(B)A$, where S is the maximal semisimple subalgebra of A generated by all e_i with $i \in Q_0$. Thus the extension $B \subseteq A$ is radical-idealized and radical-full. The quiver of B is obtained from that of A by gluing all vertices in $\{1, 2, \cdots, n\}$ together into one vertex, and unifying all arrows $\alpha_1, \alpha_2, \cdots, \alpha_n$ into one arrow.

(3) **Triangulation**

Suppose that we are given an algebra B with a decomposition $B = S \oplus \mathrm{rad}(B)$, where S is a maximal semisimple subalgebra of B. Let n be the nilpotency of $\mathrm{rad}(B)$. We define $\bar{B} = B/\mathrm{rad}^{n-1}(B)$, and

$$A = \begin{pmatrix} S & 0 \\ \mathrm{rad}(B) & \bar{B} \end{pmatrix}.$$

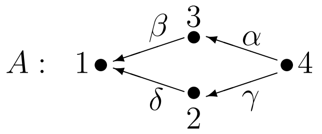
Then there is an embedding of B into A such that this extension is radical-idealized and radical-full, namely

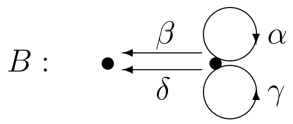
$$B \subseteq A, \qquad b = s + r \mapsto \begin{pmatrix} s & 0 \\ r & \bar{b} \end{pmatrix},$$

where $\bar{b}$ is the image of $b \in B$ under the canonical surjection from B to $\bar{B}$.

Now we display two concrete examples to illustrate the first two constructions.

Example 1. Let A and B be the following two algebras presented by quivers with relations, respectively:

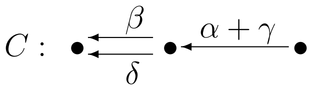




$\alpha\beta = \gamma\delta.$ $\qquad$ $\alpha\beta = \gamma\delta, \alpha\delta = \gamma\beta = \alpha^2 = \gamma^2 = \gamma\alpha = \alpha\gamma = 0.$

We can see that B is obtained by gluing the vertices 2, 3 and 4 in the quiver of A. Thus the extension $B \subseteq A$ is radical-equal. Note that A is representation-finite and has finite global dimension, while the subalgebra B of A is representation-infinite and of infinite global dimension.

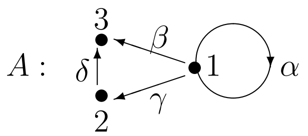
If we unify the arrows α and γ in the quiver of A, then we get the following subalgebra C of A:

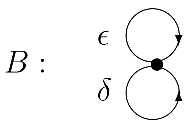


$(\alpha + \gamma)\beta = (\alpha + \gamma)\delta.$

Thus $C \subseteq A$ is a radical-full extension. Again, the subalgebra C of A is representation-infinite. Clearly, the radical of C is properly contained in the radical of A.

Example 2. Let A and B be algebras presented by the following quivers with relations, respectively:





$\alpha\beta - \alpha\gamma\delta = \alpha^2 = 0.$ $\qquad$ $\delta^2 = \epsilon^3 = \delta\epsilon = 0.$

Clearly, we see that B can be obtained from A by unifying the arrows α, β and γ into one arrow $\epsilon = (\alpha + \beta + \gamma)$. In this procedure, the arrow δ in the quiver of A becomes a loop in the quiver of B.

Finally, we mention some facts on radical-full extensions from [12] and [15].

Assume that $B \subseteq A$ is a radical-idealized extension of Artin algebras. Then

(1) for any B-module ${}_BX$, the B-module $\Omega^j_B(X)$ is an A-module for $j \geq 2$, where Ω^i_B is the i-th syzygy operator of B.

(2) For each A-module Y, we have $\Omega_A(A \otimes_B Y) \simeq \Omega_B(Y)$ as A-modules.

(3) If the extension is radical-full, then $\mathrm{add}\big({}_B(A/\mathrm{rad}(A))\big) = \mathrm{add}(B/\mathrm{rad}(B))$. Thus every simple B-module is a direct summand of the restriction of a simple A-module to B.

A direct consequence of the facts (1) and (2) is the following proposition.

Proposition 3. *Suppose that $B \subseteq A$ is a radical-idealized extension of Artin algebras. If* $\mathrm{pd}(A_B) < \infty$, *then* $\mathrm{fin.dim}(B) \leq \mathrm{fin.dim}(A) + \mathrm{pd}(A_B) + 2$.

Proof. Let $n = \mathrm{pd}(A_B)$. Pick a B-module X, define $Y := \Omega^{n+2}_B(X)$, which is an A-module by (1), and consider a minimal projective resolution of ${}_BY$:

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to Y \to 0.$$

By tensoring this sequence, we get a sequence:

$$(*) \quad 0 \to A \otimes_B P_m \to \cdots \to A \otimes_B P_1 \to A \otimes_B P_0 \to A \otimes_B Y \to 0.$$

Since $\mathrm{Tor}_j^B(A_B, Y) = \mathrm{Tor}_j^B(A_B, \Omega_B^{n+2}(X)) = \mathrm{Tor}_{n+2+j}^B(A_B, X) = 0$ for $j \geq 1$, we see that this sequence is exact. Furthermore, we can show by the fact (2) that the sequence is also a minimal projective resolution of the A-module $A \otimes_B Y$. Thus $\mathrm{pd}({}_BY) = \mathrm{pd}({}_AA \otimes_B Y) \leq$ fin.dim(A), and therefore we have the estimation in the proposition.

4. Recent results on the finitistic dimension conjecture

In this section we present some results along the idea of algebra extensions.

In [12], we showed the following result.

Theorem 4. *Let $C \subseteq B \subseteq A$ be three Artin algebras with the same identity such that both $C \subseteq B$ and $B \subseteq A$ are radical-idealized. If A is representation-finite, then C has finite finitistic dimension.*

An open question is to extend this result to a chain containing four or more than four algebras. A positive answer to this question for finite chain of algebras would solve the finitistic dimension conjecture [12]. The next result involves global dimension [13].

Theorem 5. *Let $B \subseteq A$ be a radical-idealized, radical-full extension of Artin algebras. If* gl.dim$(A) \leq 4$, *then* fin. dim$(B) < \infty$.

The case of gl.dim$(A) \geq 5$ is open. It would be interesting to generalize this result.

When considering an extension, we may automatically think of the notion of relatively projective modules, and the one of relative global dimension.

Recall that, given an extension $B \subseteq A$ of algebras, an A-module X is called *relatively projective* if the multiplication map $\mu : {}_AA \otimes_B X \longrightarrow {}_AX$ of A-modules is a split-epimorphism, that is, there is a homomorphism $\varphi : X \longrightarrow A \otimes_B X$ of A-modules such that $\varphi\mu$ is the identity map on X. In this case we also say that X is (A, B)-projective. A short exact sequence of A-modules is called (A, B)-exact if it splits as an exact sequence of B-modules. The relative projective dimension of an A-module can be defined by (A, B)-projective modules and exact (A, B)-sequences. We leave the precise formulation of this notion to the reader. We denote by gl.dm(A, B) the relative global dimension of the extension $B \subseteq A$. For more details on relative homological algebra one may look at the paper [4].

It is known that gl.dim$(A, B) = 0$ if and only if the extension $B \subseteq A$ is semisimple, that is, every A-module is (A, B)-projective. Examples of semisimple extension are radical-equal extensions.

Related to (A, B)-projective modules, we have the following results in [15].

Theorem 6. *Let $B \subseteq A$ be a radical-idealized extension of Artin algebra. Suppose the category of all finitely generated (A, B)-projective A-modules is closed under taking A-syzygies (for example, the extension is semisimple, or A_B is projective). If* fin.dim$(A) < \infty$, *then* fin.dim$(B) < \infty$.

In [14] there is another approach to finitistic dimension conjecture, namely we use the pair $eAe \subseteq A$ with $e^2 = e \in A$, and try to understand the finitistic dimension of eAe by that of A. For details we refer to the paper [14]. Recently, Huard, Lanzilotta and Mendoza use socle or top layers of a module to approach the finitistic dimension conjecture. Again, I refer the details to the paper [5].

5. Recent results on the strong no loop conjecture

Concerning the strong no loop conjecture, not much is known. There are only a few papers dealing with this conjecture in literature. It was verified for monomial algebras [6], quasi-monomial algebras [3], special biserial algebras and quasi-stratified algebras [8, 9], and algebras (over an algebraically closed field) of radical-cube-zero with two simple modules [7].

Along the approach by extensions, we have the following result in [16].

Theorem 7. *Let $B \subseteq A$ be a radical-idealized, radical-full extension of Artin algebras. If* $\mathrm{gl.dim}(A) \leq 2$*, then the strong no loop conjecture is true for B.*

If we stress the condition on extension, we have the following result.

Theorem 8. *Let $B \subseteq A$ be a radical-idealized extension of Artin algebras with gl.dim$(A, B) = 0$. If the strong no loop conjecture is true for A, then it is true for B.*

Thus, if we glue vertices from an algebra A given by quiver and relations, then we get a new algebra B for which the strong no loop conjecture is true. Moreover, if we start with algebra of global dimension at most 2 (for example, with an Auslander algebra), and unify arrows, then the strong no loop conjecture is true for the new algebra.

Finally, we remark that $\mathrm{gl.dim}(A, B) \leq 1$ for any radical-idealized, radical-full extension $B \subseteq A$ of Artin algebras. Thus, if we could extend Theorem 6 and Theorem 8 to the case of $\mathrm{gl.dim}(A, B) \leq 1$, we would prove both the finitistic dimension conjecture and the strong no loop conjecture.

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