THE CLASSIFICATION OF TILTING MODULES OVER HARADA ALGEBRAS

KOTA YAMAURA

Abstract. In the 1980s, Harada introduced a class of algebras now called Harada algebras, which give a common generalization of quasi-Frobenius algebras and Nakayama algebras. In this paper, we classify tilting modules over Harada algebras by giving a bijection between tilting modules over Harada algebras and tilting modules over direct products of upper triangular matrix algebras over K. A combinatorial description of tilting modules over upper triangular matrix algebras over K is known. These facts allow us to classify tilting modules over a given Harada algebra.

1. Introduction

Two classes of algebras have been studied for a long time. The first is Nakayama algebras and the second is quasi-Frobenius algebras. In the 1980s, Harada introduced a class of algebras now called Harada algebras, which give a common generalization of quasi-Frobenius algebras and Nakayama algebras. Many authors have studied the structure of Harada algebras (e.g. [7, 8, 17, 18, 19, 20, 21, 22]). Now let us recall that left Harada algebras as defined from a structural point of view as follows.

Definition 1. Let R be a basic algebra and $\mathrm{Pi}(R)$ be a complete set of orthogonal primitive idempotents of R. We call R a *left Harada algebra* if $\mathrm{Pi}(R)$ can be arranged such that $\mathrm{Pi}(R) = \{e_{ij}\}_{i=1,j=1}^{m,\ n_i}$ where

(a) $e_{i1}R$ is an injective R-module for any $i = 1, \cdots, m$,
(b) $e_{ij}R \simeq e_{i,j-1}J$ for any $i = 1, \cdots, m$, $j = 2, \cdots, n_i$.

Here J is the Jacobson radical of R.

Then we put

$$P_{ij} := e_{i1}J^{j-1} \simeq e_{ij}R \qquad (1 \leq i \leq m,\ 1 \leq j \leq n_i) \tag{1.1}$$

for simplicity. By the above conditions (1) and (2), we have a chain

$$P_{i1} \supset P_{i2} \supset \cdots \supset P_{in_i}$$

of indecomposable projective R-modules.

It follows from definition that left Harada algebras satisfy the property QF-3 which is the condition that the injective hull of the algebra is projective. This property is called 1-Gorenstein by Auslander (and dominant dimension at least one by Tachikawa) [5, 12, 14, 15, 24], and often plays an important role in the representation theory. Left Harada algebras form a class of 1-Gorenstein algebras, and their indecomposable projective modules have "nice" structure.

The detailed version of this paper will be submitted for publication elsewhere.

In this paper, we classify tilting modules over left Harada algebras. Tilting modules provide a powerful tool in the representation theory of algebras and are due to [4, 9, 10].

Definition 2. Let R be an algebra. An R-module T is called a *partial tilting module* if it satisfies the following conditions.

(1) $\mathrm{proj.dim} T \leq 1$.
(2) $\mathrm{Ext}^1_R(T, T) = 0$.

A partial tilting R-module T is called a *tilting module* if it satisfies the following condition.

(3) There exists an exact sequence

$$0 \longrightarrow R_R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

where T_0, $T_1 \in \mathrm{add} T$.

We can see from the above definition that tilting modules are a generalization of progenerators. Morita theory shows that any progenerator P over an algebra R induces a categorical equivalence between $\mathrm{mod} R$ and $\mathrm{mod}(\mathrm{End}_R(P))$. This result is generalized by Brenner-Butler. It says that any tilting module T over an algebra R induces two categorical equivalences between certain full subcategories of $\mathrm{mod} R$ and of $\mathrm{mod}(\mathrm{End}_R(T))$. As a consequence, R and $\mathrm{End}_R(T)$ share a lot of homological properties (e.g. finiteness of global dimension). By this reason, tilting modules are important for the study of algebras and finding a classification of tilting modules over a given algebra is an important problem in representation theory.

Now we give notion which gives an essential class of tilting modules.

Definition 3. Let T be a module over an algebra R and $T \simeq \oplus_{i=1}^n T_i$ an indecomposable decomposition of T. Then we call T *basic* if T_i and T_j are not isomorphic to each other for any $i \neq j$.

Thanks to Morita theory, it is enough to consider basic tilting modules. We denote by $\mathrm{tilt}(R)$ the set of isomorphism classes of basic tilting modules over an algebra R.

The aim of this paper is to give a classification of tilting modules over a left Harada algebra. We present our main theorem which return the classification of tilting modueles over left Harada algebras to that of tilting modules over upper triangular matrix algebras over K. We dente by $\mathrm{T}_n(K)$ an $n \times n$ upper triangular matrix algebra over K.

Theorem 4. *Let R be a left Harada algebra as in Definition 1. Then there is a bijection*

$$\mathrm{tilt}(R) \longrightarrow \mathrm{tilt}(\mathrm{T}_{n_1}(K)) \times \mathrm{tilt}(\mathrm{T}_{n_2}(K)) \times \cdots \times \mathrm{tilt}(\mathrm{T}_{n_m}(K)).$$

We will construct the above bijection in Section 2, and give outline of the proof in Section 3.

In Section 4, we give a description of tilting $\mathrm{T}_n(K)$-modules by using non-crossing partitions of regular polygons. Then we can completely classify tilting modules over a given left Harada algebra.

In Section 5, we show an example of the classification of tilting modules over left Harada algebras.

Throughout this paper, an algebra means a finite dimensional associative algebra over an algebraically closed field K. We always deal with finitely generated right modules over algebras. We denote by J the Jacobson radical of an algebra R.

2. Main results

In this section, let R be a left Harada algebra as in Definition 1. We use the notation (1.1). We consider a factor algebra $\overline{R} = R/I$ of R which is isomorphic to direct product of upper triangular matrix algebras over K. $\overline{R}$ contains important information of R which is seen in Lemma 10 and Proposition 11. After introducing $\overline{R}$, we define a functor $F : \text{mod}R \longrightarrow \text{mod}\overline{R}$ which induces the bijection of Theorem 4, and give the precise statement of Thorem 4.

We start by giving the ideal I of R. We put

$$e_{ij}R \supset I_{ij} := e_{ij}J^{n_i-j+1} \ (1 \leq i \leq m, 1 \leq j \leq n_i),$$

$$R \supset I := \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_i} I_{ij}.$$

Obviously I is a right ideal of R. But it can be seen that I is also a left ideal of R. Thus we have the following lemma.

Lemma 5. *I is an ideal of R.*

By Lemma 5, we can consider a factor algebra

$$\overline{R} := R/I.$$

We show that $\overline{R}$ is isomorphic to direct product of upper triangular matrix algebras over K. To show this, we describe all indecomposable projective $\overline{R}$-modules as factor modules of indecomposable projective R-modules. Since I is contained in J,

$$\{\overline{e}_{ij} := e_{ij} + I \mid 1 \leq i \leq m, \ 1 \leq j \leq n_i\}$$

is a complete set of orthogonal primitive idempotents of $\overline{R}$. Thus

$$\{\overline{e}_{ij}\overline{R} \mid 1 \leq i \leq m, \ 1 \leq j \leq n_i\}$$

is a complete set of indecomposable projective $\overline{R}$-modules. Obviously we have

$$\overline{e}_{ij}\overline{R} \simeq e_{ij}R/e_{in_i}J \simeq P_{ij}/(P_{in_i}J).$$

By the structure of R in Definition 1, indecomposable projective $\overline{R}$-modules have the following unique composition series.

$$\begin{array}{ccccccc}
P_{11}/(P_{1n_1}J) & \supset & P_{12}/(P_{1n_1}J) & \supset \cdots\cdots \supset & P_{1n_1}/(P_{in_i}J) & \supset & 0 \\
P_{21}/(P_{2n_2}J) & \supset & P_{22}/(P_{2n_2}J) & \supset \cdots\cdots \supset & P_{2n_2}/(P_{2n_2}J) & \supset & 0 \\
\vdots & & & & & & \\
P_{m1}/(P_{mn_m}J) & \supset & P_{m2}/(P_{mn_m}J) & \supset \cdots\cdots \supset & P_{mn_m}/(P_{mn_m}J) & \supset & 0
\end{array}$$

We note that composition factors of the above composition series are not isomorphic to each other.

We put

$$\overline{e}_i := \overline{e}_{i1} + \overline{e}_{i2} + \cdots + \overline{e}_{in_i}$$

for any $1 \leq i \leq m$. Then by the above argument, we have the following result.

Proposition 6. *We have the following algebra isomorphisms.*

(1) $\overline{e}_i\overline{R}\,\overline{e}_j \simeq \mathrm{Hom}_{\overline{R}}(\overline{e}_j\overline{R}, \overline{e}_i\overline{R}) \simeq \begin{cases} \mathrm{T}_{n_i}(K) & (i=j), \\ 0 & (i \neq j). \end{cases}$

(2) $\overline{R} \simeq \mathrm{T}_{n_1}(K) \times \mathrm{T}_{n_2}(K) \times \cdots \times \mathrm{T}_{n_m}(K)$.

Next we consider a functor

$$F := - \otimes_R \overline{R} : \mathrm{mod}R \longrightarrow \mathrm{mod}\overline{R}.$$

This functor plays a key role for our main theorem.

Now we state a theorem which gives a bijection between $\mathrm{tilt}(R)$ and $\mathrm{tilt}(\overline{R})$ by using the functor F.

Theorem 7. *We have a bijection*

$$F : \mathrm{tilt}(R) \ni T \longmapsto F(T) \in \mathrm{tilt}(\overline{R}).$$

As a consequence of Theorem 7, we have the following result immediately.

Corollary 8. *We have a bijection*

$$\mathrm{tilt}(R) \ni T \longmapsto (F(T)e_1, \cdots, F(T)e_m) \in \mathrm{tilt}(\overline{R}e_1) \times \cdots \times \mathrm{tilt}(\overline{R}e_m).$$

Hence by Proposition 6, we have Theorem 4.

3. Proof of Theorem 7

In this section, we keep the notations from the previous section. We show outline of the proof of Theorem 7.

First we give a more stronger result than our main theorem. Namely we classify indecomposable R-modules whose projective dimension is equal to one. Obviously projective dimension of P_{ik}/P_{il} is equal to one for any $1 \leq i \leq m$, $1 \leq k < l \leq n_i$. The following theorem shows that the converse holds.

Theorem 9. *A complete set of isomorphism classes of indecomposable R-modules whose projective dimension is equal to one is given as follows.*

$$\{P_{ik}/P_{il} \mid 1 \leq i \leq m,\ 1 \leq k < l \leq n_i\}.$$

Next we consider the restriction on F to full subcategories $\mathcal{P}$ or $\mathcal{P}_i$ of $\mathrm{mod}R$ which are defined by

$$\mathcal{P} := \{M \in \mathrm{mod}R \mid \mathrm{proj.dim}M \leq 1\}$$

and

$$\mathcal{P}_i := \mathrm{add}\{P_{ij},\ P_{ik}/P_{il} \mid 1 \leq j \leq n_i,\ 1 \leq k < l \leq n_i\}$$

for any $1 \leq i \leq m$. By Theorem 9, we have

$$\mathcal{P} = \mathrm{add}(\mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m).$$

The restriction on F to $\mathcal{P}$ has two important properties. First property is the following lemma which is proved by easy calculations.

Lemma 10. *The following hold.*

(1) *The restriction on F to $\mathcal{P}$ induces a bijection from isomorphism classes of $\mathcal{P}$ to that of* $\mathrm{mod}\overline{R}$.

(2) *The restriction on F to $\mathcal{P}_i$ induces a bijection from isomorphism classes of $\mathcal{P}_i$ to that of* $\mathrm{mod}(\overline{R}\overline{e}_i)$.

We remark that the restriction on F to $\mathcal{P}$ is not faithful in general, in particular, it is not an equivalence.

Second property is that F preserves vanishing property of first extension group on $\mathcal{P}$.

Proposition 11. *For any M, $N \in \mathcal{P}$, $\mathrm{Ext}^1_R(M, N) = 0$ if and only if $\mathrm{Ext}^1_{\overline{R}}(F(M), F(N)) = 0$.*

Finally by using the following well-known charactarization of tilting module, we can prove Theorem 7.

Proposition 12. [3] *Let R be a general algebra. Let T be a partial tilting module. Then the following are equivalent.*

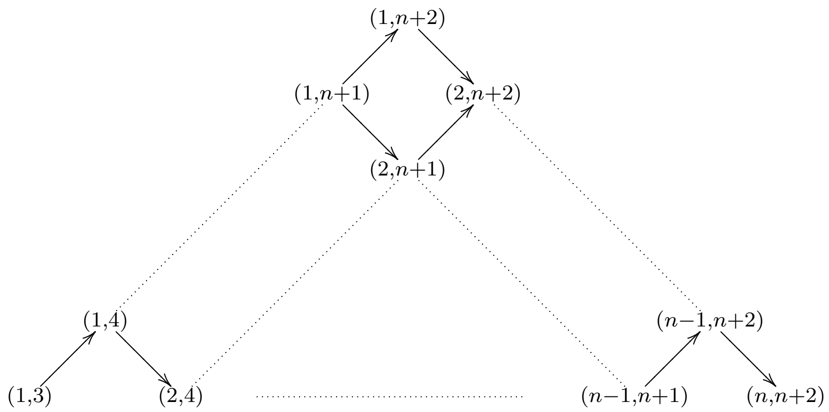
(1) *T is a tilting module.*
(2) *The number of pairwise nonisomorphic indecomposable direct summands of T is equal to that of pairwise nonisomorphic simple R-modules.*

Now we prove Theorem 7. Let T be a basic tilting R-module. It is enough to show that $F(T)$ is a basic tilting $\overline{R}$-module. First by proj.dim$T \leq 1$, we have $T \in \mathcal{P}$. Next by $\mathrm{Ext}^1_R(T, T) = 0$ and Proposition 11, we have $\mathrm{Ext}^1_{\overline{R}}(F(T), F(T)) = 0$. Therefore $F(T)$ is a basic partial tilting $\overline{R}$-module. Finally by Lemma 10 and Proposition 12, we can see that the number of pairwise nonisomorphic indecomposable direct summands of $F(T)$ is equal to that of pairwise nonisomorphic simple $\overline{R}$-modules. Consequently by Proposition 12, $F(T)$ is a basic tilting $\overline{R}$-module. □

4. Combinatorial description of tilting $\mathrm{T}_n(K)$-modules

In this section, we show a classification of basic tilting $\mathrm{T}_n(K)$-modules by constructing a bijection between $\mathrm{tilt}(\mathrm{T}_n(K))$ and the set of non-crossing partitions of the regular $(n+2)$-polygon. We remark that our classification should be well-known for experts [2, 11, 16, 23].

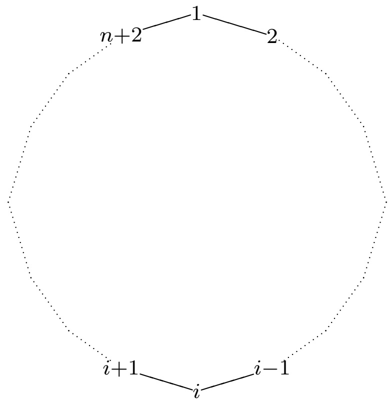
First we introduce coordinates in the AR-quiver of $\mathrm{T}_n(K)$ as follows.



We remark that the vertex (i, j) corresponds the $\mathrm{T}_n(K)$-module

$$M_{ij} = \left(0 \cdots 0 \overset{j-2}{K} \cdots \overset{1}{K}\right) / \left(0 \cdots \overset{i}{0} K \cdots \overset{1}{K}\right) = \left(0 \cdots 0 \overset{j-2}{K} \cdots \overset{i}{K} 0 \cdots \overset{1}{0}\right).$$

Next we consider a regular $(n+2)$-polygon R_{n+2} whose vertices are numbered as follows.



We denote by $D(R_{n+2})$ the set of all diagonals of R_{n+2} except edges of R_{n+2}. We call a subset S of $D(R_{n+2})$ a *non-crossing partition* of R_{n+2} if S satisfies the following conditions.

(1) Any two distinct diagonals in S do not cross except at their endpoints.
(2) R_{n+2} is divided into triangles by diagonals in S.

We denote by $\mathcal{P}_{n+2}$ the set of an non-crossing partitions of R_{n+2}.

Now we construct the correspondence Φ from $\mathcal{P}_{n+2}$ to $\mathrm{tilt}(\mathrm{T}_n(K))$. We take $S \in \mathcal{P}_{n+2}$. We remark that non-crossing partition of R_{n+2} consists of $n-1$ diagonals. We denote by (i, j) the diagonal between i and j for $i < j$ and put

$$S = \{(i_1, j_1),\ (i_2, j_2),\ \cdots,\ (i_{n-1}, j_{n-1})\}.$$

Then we define

$$\Phi(S) := M_{1,n+2} \oplus \left(\bigoplus_{k=1}^{n-1} M_{i_k, j_k} \right).$$

It is shown that this is a basic tilting $\mathrm{T}_n(K)$-module.

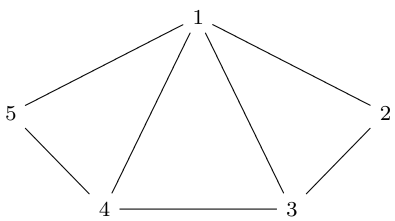
Then the following hold.

Theorem 13. *The above correspondence Φ is a bijection.*

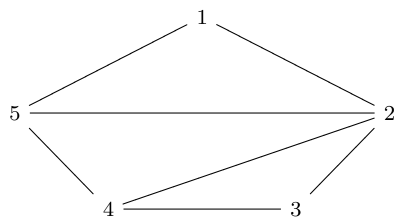
Theorem 13 gives a constructive bijection.

Example 14. We consider $n = 3$ case. We classify basic tilting $\mathrm{T}_3(K)$-modules by using Theorem 13. The partitions of the regular pentagon into triangles are given as follows.

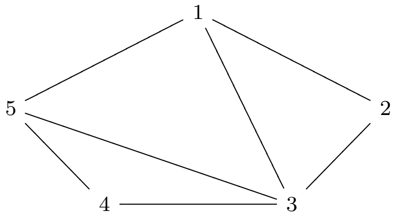
(1)



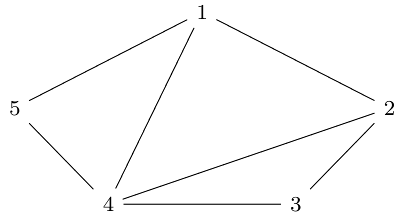
(2)



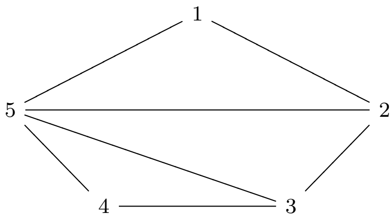
(3)



(4)



(5)



Therefore the number of basic tilting $\mathrm{T}_3(K)$-modules is equal to 5 and all of basic tilting $\mathrm{T}_3(K)$-modules are given as follows.

(1) $(\,K\ K\ K\,) \oplus (\,0\ K\ K\,) \oplus (\,0\ 0\ K\,)$,
(2) $(\,K\ K\ K\,) \oplus (\,K\ K\ 0\,) \oplus (\,0\ K\ 0\,)$,
(3) $(\,K\ K\ K\,) \oplus (\,K\ 0\ 0\,) \oplus (\,0\ 0\ K\,)$,
(4) $(\,K\ K\ K\,) \oplus (\,0\ K\ K\,) \oplus (\,0\ K\ 0\,)$,
(5) $(\,K\ K\ K\,) \oplus (\,K\ K\ 0\,) \oplus (\,K\ 0\ 0\,)$.

5. Example

In this section, we show an example of the classifications of tilting modules over Harada algebras.

Example 15. Let R be a basic QF-algebra whose complete set of orthogonal primitive idempotents is given by $\{e, f\}$. Then we can represent R as the following matrix form.

$$R \simeq \begin{pmatrix} eRe & eRf \\ fRe & fRf \end{pmatrix} =: \begin{pmatrix} Q & A \\ B & W \end{pmatrix}.$$

Now we consider the *block extension* (c.f. [8, 22])

$$R(n_1, n_2) := \left(\begin{array}{ccc|ccc} Q & \cdots & Q & A & \cdots & A \\ & \ddots & \vdots & \vdots & & \vdots \\ J(Q) & & Q & A & \cdots & A \\ \hline B & \cdots & B & W & \cdots & W \\ \vdots & & \vdots & & \ddots & \vdots \\ B & \cdots & B & J(W) & & W \end{array}\right)$$

for $n_1, n_2 \in \mathbb{N}$ of R which is a subalgebra of $\mathrm{End}_R((eR)^{n_1} \oplus (fR)^{n_2})$. We can show that

(a) the first and (n_1+1)-th rows are injective modules,
(b) the i-th row is the Jacobson radical of the $(i-1)$-th row for $2 \leq i \leq n$ and $n+2 \leq i \leq n+m$.

In particular $R(n_1, n_2)$ is a left Harada algebra with $m = 2$ in Definition 1.

We classify basic tilting $R(n_1, n_2)$-modules. By easy calculation, we can see that the ideal I which is defined in Section 2 of $R(n_1, n_2)$ is given by

$$I = \left(\begin{array}{ccc|ccc} J(Q) & \cdots & J(Q) & A & \cdots & A \\ \vdots & & \vdots & \vdots & & \vdots \\ J(Q) & \cdots & J(Q) & A & \cdots & A \\ \hline B & \cdots & B & J(W) & \cdots & J(W) \\ \vdots & & \vdots & \vdots & & \vdots \\ B & \cdots & B & J(W) & \cdots & J(W) \end{array}\right).$$

Hence we have

$$\overline{R} = R/I = \left(\begin{array}{ccc|ccc} Q/J(Q) & \cdots & Q/J(Q) & 0 & \cdots & 0 \\ & \ddots & \vdots & \vdots & & \vdots \\ 0 & & Q/J(Q) & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & W/J(W) & \cdots & W/J(W) \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & & W/J(W) \end{array}\right) \simeq \mathrm{T}_{n_1}(K) \times \mathrm{T}_{n_2}(K).$$

By Theorem 7, The functor

$$F = - \otimes \overline{R} : \mathrm{mod}R \longrightarrow \mathrm{mod}\overline{R}$$

induces a bijection

$$\mathrm{tilt}(R(n_1, n_2)) \longrightarrow \mathrm{tilt}(\mathrm{T}_{n_1}(K)) \times \mathrm{tilt}(\mathrm{T}_{n_2}(K)).$$

We can obtain all basic tilting $R(n_1, n_2)$-modules from the above bijection and Theorem 13.

References

[1] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules (second edition)*, Graduate Texts in Math. 13, Springer-Verlag, Heidelberg/New York/Berlin (1991).

[2] T. Araya, *Exceptional Sequences over path algebras of type A_n and Non-crossing Spanning Trees*, arXiv: 0904. 2831.

[3] I. Assem, D. Simson, A. Skowronski, *Elements of the Representation Theory of Associative Algebras*, London Mathematical Society Student Texts 65, Cambridge university press (2006).

[4] M. Auslander, M. I. Platzeck, I. Reiten, *Coxeter functors without diagrams.* Trans. Amer. Math. Soc. 250 (1979), 1–46

[5] M. Auslander, I. Reiten, *k-Gorenstein algebras and syzygy modules*, J. Pure Appl. Algebra 92 (1994), 1–27.

[6] M. Auslander, I. Reiten, S. Smalo, *Representation Theory of Artin Algebras*, Cambridge Studies in Advanced Mathematics 36, Cambridge university press (1995).

[7] Y. Baba and K. Iwase, *On quasi-Harada rings*, J. Algebra 155 (1996), 415–434.

[8] Y. Baba and K. Oshiro, *Classical Artinian Rings and Related Topics*, World Scientific (2009).

[9] S. Brenner, M. C. R. Butler, *Generalizations of the Bernstein Gelfand Ponomarev reflection functors*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., 832, Springer, Berlin-New York (1980), 103–169.

[10] I. N. Bernstein, I. M. Gelfand, V. A. Ponomarev, *Coxeter functors and Gabriel's theorem*, (Russian) Uspehi Mat. Nauk 28 no. 2(170) (1973), 19–33.

[11] P. Caldero, F. Chapoton, R. Schiffler, *Quivers with relations arising from clusters (A_n case)*, Trans. Amer. Math. Soc. 358 no. 3 (2006), 1347–1364.

[12] R. Fossum, P. Griffith, I. Reiten, *Trivial extensions of abelian categories*, Lecture Notes in Mathematics Vol. 456. Springer-Verlag, Berlin-New York, (1975).

[13] M. Harada, *Non-small modules and non-cosmall modules*, Ring Theory. Proceedings of 1978 Antwerp Conference, New York (1979), 669–690.

[14] Z. Huang, O. Iyama, *Auslander-type conditions and cotorsion pairs*, J. Algebra 318 no. 1 (2007), 93–100.

[15] K. Igusa, S. O. Smalo, G. Todorov, *Finite projectivity and contravariant finiteness.* Proc. Amer. Math. Soc. 109 no. 4 (1990), 937–941.

[16] O. Iyama, *Higher-dimensional Auslander-Reiten theory on maximal orthogonal subcategories*, Adv. Math. 210 no. 1 (2007), 22–50.

[17] K. Koike, *Almost self-duality and Harada rings*, J. Algebra 254 (2002), 336–361.

[18] K. Oshiro, *Lifting modules, extending modules and their applications to QF-rings*, Hokkaido Math. J. 13 (1984), 310–338.

[19] ———, *Lifting modules, extending modules and their applications to generalized uniserial rings*, Hokkaido Math. J. 13 (1984), 339–346.

[20] ———, *On Harada rings I*, Math. J. Okayama Univ. 31 (1989), 161–178.

[21] ———, *On Harada rings II*, Math. J. Okayama Univ. 31 (1989), 169–188.

[22] ———, *On Harada rings III*, Math. J. Okayama Univ. 32 (1990), 111–118.

[23] C. Riedtmann, *Representation-finite self-injective algebras of class A_n*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), Lecture Notes in Math., 832, Springer, Berlin (1980), 449–520.

[24] H. Tachikawa, *Quasi-Frobenius rings and generalizations*, Lecture Notes in Mathematics Vol. 351. Springer-Verlag, Berlin-New York, (1973).

[25] Thrall, R. M., *Some generalization of quasi-Frobenius algebras*, Trans. Amer. Math. Soc. 64 (1948), 173–183.

Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 Japan
E-mail address: m07052d@math.nagoya-u.ac.jp