HOMOLOGICAL APPROACH TO THE FACE RING OF A SIMPLICIAL POSET

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ABSTRACT. A finite poset P is called *simplicial*, if it has the smallest element $\hat{0}$, and every interval $[\hat{0}, x]$ is a boolean algebra. The face poset of a simplicial complex is a typical example. Generalizing the Stanley-Reisner ring of a simplicial complex, Stanley assigned the graded ring A_P to P. This ring has been studied from both combinatorial and topological perspective. In this paper, we will give a concise description of a dualizing complex of A_P and some related results.

1. INTRODUCTION

All posets (partially ordered sets) in this paper will be assumed to be finite. By the order given by inclusion, the power set of a finite set can be seen as a poset, and it is called a *boolean algebra*. We say a poset P is *simplicial*, if it admits the smallest element $\hat{0}$, and the interval $[\hat{0}, x] := \{ y \in P \mid y \leq x \}$ is isomorphic to a boolean algebra for all $x \in P$. For the simplicity, we denote $\operatorname{rank}(x)$ of $x \in P$ just by $\rho(x)$. If P is simplicial and $\rho(x) = m$, then $[\hat{0}, x]$ is isomorphic to the boolean algebra $2^{\{1,\dots,m\}}$.

Let Δ be a finite simplicial complex (with $\emptyset \in \Delta$). The face poset (i.e., the set of the faces of Δ with order given by inclusion) is a simplicial poset. Any simplicial poset P is obtained as the face poset of a regular cell complex, which we denote by $\Gamma(P)$. For $\hat{0} \neq x \in P$, $c(x) \in \Gamma(P)$ denotes the open cell corresponds to x. Clearly, $\dim c(x) = \rho(x) - 1$. While the closure $\overline{c(x)}$ of $c(x)$ is always a simplex, the intersection $\overline{c(x)} \cap \overline{c(y)}$ for $x, y \in P$ is not necessarily a simplex. For example, if two d-simplices are glued along their boundaries, then it is not a simplicial complex, but gives a simplicial poset.

For $x, y \in P$, set

$$[x \vee y] := \text{the set of the minimal elements of } \{ z \in P \mid z \geq x, y \}.$$

More generally, for $x_1, \dots, x_m \in P$, $[x_1 \vee \cdots \vee x_m]$ denotes the set of the minimal elements of the common upper bounds of $x_1, \dots, x_m$.

Set $\{ y \in P \mid \rho(y) = 1 \} = \{y_1, \dots, y_n\}$. For $U \subset [n] := \{1, \dots, n\}$, we simply denote $[\bigvee_{i \in U} y_i]$ by $[U]$. If $x \in [U]$, then $\rho(x) = \#U$. For each $x \in P$, there exists a unique U such that $x \in [U]$. Let $x, x' \in P$ with $x \geq x'$ and $\rho(x) = \rho(x') + 1$, and take $U, U' \subset [n]$ such that $x \in [U]$ and $x' \in [U']$. Since $U = U' \coprod \{i\}$ for some i in this case, we can set

$$\alpha(i, U) := \#\{ j \in U \mid j < i \} \qquad \text{and} \qquad \epsilon(x, x') := (-1)^{\alpha(i,U)}.$$

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Then ϵ gives an incidence function of the cell complex $\Gamma(P)$, that is, for all $x, y \in P$ with $x > y$ and $\rho(x) = \rho(y) + 2$, we have $\epsilon(x, z) \cdot \epsilon(z, y) + \epsilon(x, z') \cdot \epsilon(z', y) = 0$, where $\{z, z'\} = \{ w \in P \mid x > w > y \}$.

As is well-known, the Stanley-Reisner ring of a finite simplicial complex is a powerful tool for combinatorics. Generalizing this idea, Stanley [6] assigned the commutative ring A_P to a simplicial poset P. For the definition of A_P, we remark that if $[x \vee y] \neq \emptyset$ then $\{ z \in P \mid z \leq x, y \}$ has the largest element $x \wedge y$. Let $\Bbbk$ be a field, and $S := \Bbbk[\, t_x \mid x \in P \,]$ the polynomial ring in the variables t_x. Consider the ideal

$$I_P := (\, t_x t_y - t_{x \wedge y} \sum_{z \in [x \vee y]} t_z \mid x, y \in P \,) + (\, t_{\hat{0}} - 1 \,)$$

of S (if $[x \vee y] = \emptyset$, we interpret that $\sum_{z \in [x \vee y]} t_z = 0$), and set

$$A_P := S/I_P.$$

We denote A_P just by A, if there is no danger of confusion. Clearly, $\dim A_P = \operatorname{rank} P = \dim \Gamma(P) + 1$. For a rank 1 element $y_i \in P$, set $t_i := t_{y_i}$. If $\{x\} = [U]$ for some $U \subset [n]$ with $\# U \geq 2$, then $t_x = \prod_{i \in U} t_i$ in A, and t_x is a "dummy variable". Clearly, A is a graded ring with $\deg(t_x) = \rho(x)$. If $\Gamma(P)$ is a simplicial complex, then A_P is generated by degree 1 elements, and coincides with the Stanley-Reisner ring of $\Gamma(P)$.

Note that A also has a $\mathbb{Z}^n$-grading such that $\deg t_i \in \mathbb{N}^n$ is the ith unit vector. For each $x \in P$, the ideal

$$\mathfrak{p}_x := (t_z \mid z \not\leq x)$$

of A is a prime ideal with $\dim A/\mathfrak{p}_x = \rho(x)$, since $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x \,]$.

Recently, M. Masuda and his coworkers studied A_P with a view from *toric topology*, since the *equivariant cohomology* ring of a torus manifold is of the form A_P (cf. [4, 5]). In this paper, we will introduce another approach.

Let R be a noetherian commutative ring, $\operatorname{Mod} R$ the category of R-modules, and $\operatorname{mod} R$ its full subcategory consisting of finitely generated modules. The *dualizing complex* $D_R^\bullet$ of R gives the important duality $\mathbf{R}\operatorname{Hom}_R(-, D_R^\bullet)$ on the bounded derived category $\mathsf{D}^b(\operatorname{mod} R)$. If R is a (graded) local ring with the maximal ideal $\mathfrak{m}$, then the (graded) Matlis dual of $H^{-i}(D_R^\bullet)$ is the local cohomology $H^i_{\mathfrak{m}}(R)$.

We have a concise description of the dualizing complex A_P as follows.

Theorem 1. *Let P be a simplicial poset with $d = \operatorname{rank} P$, and set $A := A_P$. The complex*

$$I_A^\bullet : 0 \to I_A^{-d} \to I_A^{-d+1} \to \cdots \to I_A^0 \to 0,$$

given by

$$I_A^{-i} := \bigoplus_{\substack{x \in P, \\ \rho(x) = i}} A/\mathfrak{p}_x,$$

and

$$\partial_{I_A^\bullet}^{-i} : I_A^{-i} \supset A/\mathfrak{p}_x \ni 1_{A/\mathfrak{p}_x} \longmapsto \sum_{\substack{\rho(y) = i-1, \\ y \leq x}} \epsilon(x, y) \cdot 1_{A/\mathfrak{p}_y} \in \bigoplus_{\substack{\rho(y) = i-1, \\ y \leq x}} A/\mathfrak{p}_y \subset I_A^{-i+1}$$

is isomorphic to the dualizing complex $D_A^\bullet$ of A in $\mathsf{D}^b(\operatorname{Mod} A)$.

To prove this, it might be possible to use the description of $H^i_{\mathfrak{m}}(A)$ by Duval ([1]). However, we will take more conceptual approach. In [8], the author defined a *squarefree module* over a polynomial ring, and many applications have been found. (For example, regarding A as a squarefree module over the polynomial ring $\operatorname{Sym} A_1$, Duval's formula of $H^i_{\mathfrak{m}}(A)$ mentioned above can be proved quickly. See Remark 15.) We will extend this notion to modules over A, and use it in the proof of Theorem 1.

The category $\operatorname{Sq} A$ of square free A-modules is an abelian category with enough injectives, and $A/\mathfrak{p}_x$ is an injective object. Hence $I_A^\bullet$ is a complex in $\operatorname{Sq} A$, and $\mathbb{D}(-) := \underline{\operatorname{Hom}}_A^\bullet(-, I_A^\bullet)$ gives a duality on $\mathsf{K}^b(\text{Inj-Sq})\,(\cong \mathsf{D}^b(\operatorname{Sq} A))$. Moreover, via the forgetful functor $\operatorname{Sq} A \to \operatorname{Mod} A$, $\mathbb{D}$ coincides with the duality $\mathbf{R}\operatorname{Hom}_A(-, D_A^\bullet)$ on $\mathsf{D}^b(\operatorname{mod} A)$.

As [9, 10], we can assign a squarefree A-module M the constructible sheaf M^+ on (the underlying space of) $\Gamma(P)$. In this context, the duality $\mathbb{D}$ corresponds to the Poincaré-Verdier duality for the constructible sheaves on X up to translation. In particular, the sheafification of the complex $I_A^\bullet[-1]$ coincides with the Verdier dualizing complex of X with the coefficients in $\Bbbk$, where $[-1]$ represents a translation by -1.

Using this argument, we can show the following. At least for the Cohen-Macaulay case, it has been shown in Duval [1]. However our proof gives new perspective.

Corollary 2 (see, Theorem 16)**.** *The Cohen-Macaulay (resp. Gorenstein*, Buchsbaum properties) and Serre's condition (S_i) of A_P are topological properties of the underlying space of $\Gamma(P)$. Here we say A_P is Gorenstein*, if A_P is Gorenstein and the $\mathbb{Z}$-graded canonical module ω_{A_P} is generated by its degree 0 part.*

2. Preparation

In the rest of the paper, P is a simplicial poset with $\operatorname{rank} P = d$. As in the preceding section, we use the convention that $A = A_P$, $\{\, y \in P \mid \rho(y) = 1 \,\} = \{y_1, \ldots, y_n\}$, and $t_i := t_{y_i} \in A$.

For a subset $U \subset [n] = \{\, 1, \ldots, n \,\}$, A_U denotes the localization of A by the multiplicatively closed set $\{\, \prod_{i \in U} t_i^{a_i} \mid a_i \geq 0 \,\}$. If $[U] = \emptyset$, then $A_U = 0$. For $x \in [U]$,

$$u_x := \frac{t_x}{\prod_{i \in U} t_i} \in A_U$$

is an idempotent. Moreover, $u_x \cdot u_{x'} = 0$ for $x, x' \in [U]$ with $x \neq x'$, and $1_{A_U} = \sum_{x \in [U]} u_x$. Hence we have a $\mathbb{Z}^n$-graded direct sum decomposition

$$A_U = \bigoplus_{x \in [U]} A_U \cdot u_x.$$

Let $\operatorname{Gr} A$ be the category of $\mathbb{Z}^n$-graded A-modules, and $\operatorname{gr} A$ its full subcategory consisting of finitely generated modules. Here a morphism $f : M \to N$ in $\operatorname{Gr} A$ is an A-homomorphism with $f(M_{\mathbf{a}}) \subset N_{\mathbf{a}}$ for all $\mathbf{a} \in \mathbb{Z}^n$. As usual, for M and $\mathbf{a} \in \mathbb{Z}^n$, $M(\mathbf{a})$ denotes the shifted module of M with $M(\mathbf{a})_{\mathbf{b}} = M_{\mathbf{a}+\mathbf{b}}$. For $M, N \in \operatorname{Gr} A$,

$$\underline{\operatorname{Hom}}_A(M, N) := \bigoplus_{\mathbf{a} \in \mathbb{Z}^n} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(\mathbf{a}))$$

has a $\mathbb{Z}^n$-graded A-module structure. Similarly, $\underline{\mathrm{Ext}}^i_A(M,N) \in \mathrm{Gr}\,A$ can be defined. If $M \in \mathrm{gr}\,A$, the underlying module of $\underline{\mathrm{Hom}}_A(M,N)$ is isomorphic to $\mathrm{Hom}_A(M,N)$, and the same is true for $\underline{\mathrm{Ext}}^i_A(M,N)$.

If $M \in \mathrm{Gr}\,A$, then $M^\vee := \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} \mathrm{Hom}_\Bbbk(M_{-\mathbf{a}}, \Bbbk)$ can be regarded as a $\mathbb{Z}^n$-graded A-module, and $(-)^\vee$ gives an exact contravariant functor from $\mathrm{Gr}\,A$ to itself, which is called the *graded Matlis duality functor*.

Lemma 3. *(1) $E_A(x) := (A_U \cdot u_x)^\vee$ is injective in $\mathrm{Gr}\,A$. Conversely, any indecomposable injective in $\mathrm{Gr}\,A$ is isomorphic to $E_A(x)(\mathbf{a})$ for some $x \in P$ and $\mathbf{a} \in \mathbb{Z}^n$.*

(2) For $M \in \mathrm{Gr}\,A$, set $M_{\geq \mathbf{0}} := \bigoplus_{\mathbf{a}\in\mathbb{N}^n} M_{\mathbf{a}}$. Then we have a canonical isomorphism

$$\phi_x : A/\mathfrak{p}_x \xrightarrow{\cong} E_A(x)_{\geq \mathbf{0}}.$$

The Cěch complex $C^\bullet$ of A with respect to $t_1, \ldots, t_n$ is of the form

$$0 \to C^0 \to C^1 \to \cdots \to C^d \to 0 \quad \text{with} \quad C^i = \bigoplus_{\substack{U\subset[n] \\ \#U=i}} A_U$$

(note that if $\#U > d$ then $A_U = 0$). The differential map is given by

$$C^i \supset A_U \ni a \longmapsto \sum_{\substack{U'\supset U \\ \#U'=i+1}} (-1)^{\alpha(U'\setminus U, U)} f_{U',U}(a) \in \bigoplus_{\substack{U'\supset U \\ \#U'=i+1}} A_{U'} \subset C^{i+1},$$

where $f_{U',U} : A_U \to A_{U'}$ is the natural map.

Since the radical of the ideal $(t_1, \ldots, t_n)$ is the maximal ideal $\mathfrak{m} := (t_x \mid \hat{0} \neq x \in P)$, the cohomology $H^i(C^\bullet)$ of $C^\bullet$ is isomorphic to the local cohomology $H^i_\mathfrak{m}(A)$. Moreover, $C^\bullet$ is isomorphic to $\mathbf{R}\Gamma_\mathfrak{m}(A)$ in $\mathsf{D}^b(\mathrm{Mod}\,A)$. Here $\mathbf{R}\Gamma_\mathfrak{m}$ is the right derived functor of $\Gamma_\mathfrak{m} : \mathrm{Mod}\,A \to \mathrm{Mod}\,A$ given by $\Gamma_\mathfrak{m}(M) = \{s \in M \mid \mathfrak{m}^i s = 0 \text{ for } i \gg 0\}$. The same is true in the $\mathbb{Z}^n$-graded context. We may regard $\Gamma_\mathfrak{m}$ as a functor from $\mathrm{Gr}\,A$ to itself, and let ${}^*\mathbf{R}\Gamma_\mathfrak{m}$ be its right derived functor. Then $C^\bullet \cong {}^*\mathbf{R}\Gamma_\mathfrak{m}(A)$ in $\mathsf{D}^b(\mathrm{Gr}\,A)$.

Let ${}^*D^\bullet_A$ be a $\mathbb{Z}^n$-graded normalized dualizing complex of A. By the $\mathbb{Z}^n$-graded version of the local duality theorem [2, Theorem V.6.2], we have a quasi-isomorphism $({}^*D^\bullet_A)^\vee \longrightarrow {}^*\mathbf{R}\Gamma_\mathfrak{m}(A)$. Taking the Matlis dual, we get a quasi-isomorphism ${}^*\mathbf{R}\Gamma_\mathfrak{m}(A)^\vee \longrightarrow {}^*D^\bullet_A$. Hence

$${}^*D^\bullet_A \cong {}^*\mathbf{R}\Gamma_\mathfrak{m}(A)^\vee \cong (C^\bullet)^\vee$$

in $\mathsf{D}^b(\mathrm{Gr}\,A)$. Since

$$(C^i)^\vee \cong \bigoplus_{\substack{x\in P \\ \rho(x)=i}} E_A(x)$$

and each $E_A(x)$ is injective in $\mathrm{Gr}\,A$, $(C^\bullet)^\vee$ actually coincides with ${}^*D^\bullet_A$. Hence ${}^*D^\bullet_A$ is of the form

$$0 \to \bigoplus_{\substack{x\in P \\ \rho(x)=d}} E_A(x) \to \bigoplus_{\substack{x\in P \\ \rho(x)=d-1}} E_A(x) \to \cdots \to E_A(\hat{0}) \to 0,$$

where the cohomological degree is given by the same way to $I^\bullet_A$. We will show that this ϕ is a quasi-isomorphism.

For each $i \in \mathbb{Z}$, we have an injection $\phi^i : I^i_A \to {}^*D^i_A$ given by the injection $\phi_x : A/\mathfrak{p}_x \to E_A(x)$ of Lemma 3. Then $\phi := (\phi_i)_{i\in\mathbb{Z}}$ is a chain map $I^\bullet_A \hookrightarrow {}^*D^\bullet$.

Since $\mathfrak{p}_x$ is a $\mathbb{Z}^n$-graded ideal, ${}^*D^\bullet_{A/\mathfrak{p}_x} := \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, {}^*D^\bullet_A)$ is a $\mathbb{Z}^n$-graded (or $\mathbb{Z}^{\rho(x)}$-graded) dualizing complex of $A/\mathfrak{p}_x$, and quasi-isomorphic to its non-negative part $I^\bullet_{A/\mathfrak{p}_x} := ({}^*D^\bullet_{A/\mathfrak{p}_x})_{\geq \mathbf{0}}$ (the latter statement is the polynomial ring case of Theorem 1, and it is a well-known result). We have the following.

Lemma 4. *For all $x \in P$, $\phi : I^\bullet_A \to {}^*D^\bullet_A$ induces a quasi-isomorphism*

$$I^\bullet_{A/\mathfrak{p}_x} = \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, I^\bullet_A) \longrightarrow \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, {}^*D^\bullet_A) = {}^*D^\bullet_{A/\mathfrak{p}_x},$$

3. Squarefree Modules over A_P, and The Proof of Theorem 1

Let $R = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring, and regard it as a $\mathbb{Z}^n$-graded ring. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$, set $\mathrm{supp}(\mathbf{a}) := \{\, i \mid a_i \neq 0 \,\} \subset [n]$, and let $x^{\mathbf{a}}$ denote the monomial $\prod x_i^{a_i} \in R$.

Definition 5 ([8])**.** With the above notation, a $\mathbb{Z}^n$-graded R-module M is called *squarefree*, if it is finitely generated, $\mathbb{N}^n$-graded (i.e., $M = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} M_{\mathbf{a}}$), and the multiplication map $M_{\mathbf{a}} \ni s \longmapsto x^{\mathbf{b}} s \in M_{\mathbf{a}+\mathbf{b}}$ is bijective for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ with $\mathrm{supp}(\mathbf{a}) \supset \mathrm{supp}(\mathbf{b})$.

To define a squarefree module over the face ring $A = A_P$ of a simplicial poset P, we equip A with a finer "grading", where the index set is no longer a monoid.

Recall the convention that $\{\, y \in P \mid \rho(y) = 1 \,\} = \{y_1, \dots, y_n\}$ and $t_i = t_{y_i} \in A$. For each $x \in P$, set

$$\mathbb{M}(x) := \bigoplus_{y_i \leq x} \mathbb{N}\, \mathbf{e}_i^x,$$

where $\mathbf{e}_i^x$ is a basis element. So $\mathbb{M}(x) \cong \mathbb{N}^{\rho(x)}$ as additive monoids. For x, z with $x \leq z$, we have an injection $\iota_{z,x} : \mathbb{M}(x) \ni \mathbf{e}_i^x \longmapsto \mathbf{e}_i^z \in \mathbb{M}(z)$ of monoids. Set

$$\mathbb{M} := \varinjlim_{x \in P} \mathbb{M}(x),$$

where the direct limit is taken with respect to $\iota_{z,x} : \mathbb{M}(x) \to \mathbb{M}(z)$ for $x, z \in P$ with $x \leq z$. Note that $\mathbb{M}$ is no longer a monoid, just a set. Since all $\iota_{z,x}$ is an injection, we can regard $\mathbb{M}(x)$ as a subset of $\mathbb{M}$. For each $\underline{\mathbf{a}} \in \mathbb{M}$, $\{x \in P \mid \underline{\mathbf{a}} \in \mathbb{M}(x)\}$ has the smallest element, which is denoted by $\sigma(\underline{\mathbf{a}})$.

We say a monomial $\mathsf{m} = \prod_{x \in P} t_x^{n_x} \in A$, $n_x \in \mathbb{N}$, is *standard*, if $\{\, x \in P \mid n_x \neq 0 \,\}$ is a totally ordered set. The set of the standard monomials forms a $\Bbbk$-basis of A. Let $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathbb{M}$. If $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})] \neq \emptyset$, then we can take the sum $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}(x)$ for each $x \in [\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]$. Unless $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]$ consists of a single element, we cannot define $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}$. Hence we denote each $\underline{\mathbf{a}} + \underline{\mathbf{b}} \in \mathbb{M}(x)$ by $(\underline{\mathbf{a}} + \underline{\mathbf{b}})|x$.

Definition 6. $M \in \mathrm{Mod}\, A$ is said to be $\mathbb{M}$-*graded* if the following are satisfied;

1. $M = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{M}} M_{\underline{\mathbf{a}}}$ as $\Bbbk$-vector spaces;
2. For $\underline{\mathbf{a}}, \underline{\mathbf{b}} \in \mathbb{M}$, we have

$$t^{\underline{\mathbf{a}}} M_{\underline{\mathbf{b}}} \subset \bigoplus_{x \in [\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})]} M_{(\underline{\mathbf{a}}+\underline{\mathbf{b}})|x}.$$

Hence, if $[\sigma(\underline{\mathbf{a}}) \vee \sigma(\underline{\mathbf{b}})] = \emptyset$, then $t^{\underline{\mathbf{a}}} M_{\underline{\mathbf{b}}} = 0$.

Clearly, A itself is an $\mathbb{M}$-graded module with $A_{\underline{\mathbf{a}}} = \Bbbk\, t^{\underline{\mathbf{a}}}$. Since there is a natural map $\mathbb{M} \to \mathbb{N}^n$, an $\mathbb{M}$-graded module can be see as a $\mathbb{Z}^n$-graded module.

If M is an $\mathbb{M}$-graded A-module, then

$$M_{\not\leq x} := \bigoplus_{\underline{\mathbf{a}} \notin \mathbb{M}(x)} M_{\underline{\mathbf{a}}}$$

is an $\mathbb{M}$-graded submodule for all $x \in P$, and

$$M_{\leq x} := M / M_{\not\leq x}$$

is a $\mathbb{Z}^{\rho(x)}$-graded module over $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x\,]$.

Definition 7. We say an $\mathbb{M}$-graded A-module M is *squarefree*, if $M_{\leq x}$ is a squarefree module over the polynomial ring $A/\mathfrak{p}_x \cong \Bbbk[\, t_i \mid y_i \leq x\,]$ for all $x \in P$.

Clearly, A itself, $\mathfrak{p}_x$ and $A/\mathfrak{p}_x$ for $x \in P$, are squarefree. Let $\operatorname{Sq} A$ be the category of squarefree A-modules and their A-homomorphisms $f : M \to M'$ with $f(M_{\underline{\mathbf{a}}}) \subset M'_{\underline{\mathbf{a}}}$ for all $\underline{\mathbf{a}} \in \mathbb{M}$. For example, $I_A^\bullet$ is a complex in $\operatorname{Sq} A$.

The *incidence algebra* Λ of P over $\Bbbk$ is a finite dimensional associative $\Bbbk$-algebra with basis $\{\, e_{x,y} \mid x, y \in P,\ x \geq y \,\}$ whose multiplication is defined by

$$e_{x,y} \cdot e_{z,w} = \begin{cases} e_{x,w} & \text{if } y = z; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\operatorname{mod} \Lambda$ be the category of finitely generated left Λ-modules.

Proposition 8. *We have* $\operatorname{Sq} A \cong \operatorname{mod} \Lambda$. *Hence* $\operatorname{Sq} A$ *is an abelian category with enough injectives and the injective dimension of each object is at most d. An object $M \in \operatorname{Sq} A$ is an indecomposable injective if and only if $M \cong A/\mathfrak{p}_x$ for some $x \in P$.*

Let Inj-Sq be the full subcategory of $\operatorname{Sq} A$ consisting of all injective objects, that is, finite direct sums of $A/\mathfrak{p}_x$ for various $x \in P$. As is well-known, the bounded homotopy category $\mathsf{K}^b(\text{Inj-Sq})$ is equivalent to $\mathsf{D}^b(\operatorname{Sq} A)$. Since

$$\underline{\operatorname{Hom}}_A(A/\mathfrak{p}_x, A/\mathfrak{p}_y) = \begin{cases} A/\mathfrak{p}_y & \text{if } x \geq y, \\ 0 & \text{otherwise,} \end{cases}$$

we have $\underline{\operatorname{Hom}}^\bullet_A(J^\bullet, I^\bullet_A) \in \mathsf{K}^b(\text{Inj-Sq})$ for all $J^\bullet \in \mathsf{K}^b(M^\bullet)$. Moreover, $\underline{\operatorname{Hom}}^\bullet_A(-, I^\bullet_A)$ preserves homotopy equivalences, and gives a functor $\mathbb{D} : \mathsf{K}^b(\text{Inj-Sq}) \to \mathsf{K}^b(\text{Inj-Sq})^{\mathsf{op}}$.

On the other hand, $M^\bullet \longmapsto \underline{\operatorname{Hom}}^\bullet_A(M^\bullet, {}^*D^\bullet_A)$ gives the functor $\mathbf{R}\underline{\operatorname{Hom}}_A(-, {}^*D^\bullet_A) : \mathsf{D}^b(\operatorname{gr} A) \to \mathsf{D}^b(\operatorname{gr} A)^{\mathsf{op}}$ under the identification $\mathsf{D}^b_{\operatorname{gr} A}(\operatorname{Gr} A) \cong \mathsf{D}^b(\operatorname{gr} A)$. Combining $\mathbb{U} : \mathsf{K}^b(\text{Inj-Sq}) \xrightarrow{\cong} \mathsf{D}^b(\operatorname{Sq} A) \longrightarrow \mathsf{D}^b(\operatorname{gr} A)$ given by the forgetful functor $\operatorname{Sq} A \to \operatorname{gr} A$, we have the two functors $\mathbb{U} \circ \mathbb{D}$ and $\mathbf{R}\underline{\operatorname{Hom}}_A(-, {}^*D^\bullet_A) \circ \mathbb{U}$.

$$\begin{array}{ccc} (\mathsf{D}^b(\operatorname{Sq} A) \cong)\ \mathsf{K}^b(\text{Inj-Sq}) & \xrightarrow{\ \mathbb{U}\ } & \mathsf{D}^b(\operatorname{gr} A) \\ \Big\downarrow{\scriptstyle \mathbb{D}} & & \Big\downarrow{\scriptstyle \mathbf{R}\underline{\operatorname{Hom}}_A(-, {}^*D^\bullet_A)} \\ \mathsf{K}^b(\text{Inj-Sq})^{\mathsf{op}} & \xrightarrow[\ \mathbb{U}\ ]{} & \mathsf{D}^b(\operatorname{gr} A)^{\mathsf{op}} \end{array}$$

By the chain map $\phi : I^\bullet_A \to {}^*D^\bullet_A$ constructed in the end of the preceding section, we have a natural transformation $\Phi : \mathbb{U} \circ \mathbb{D} \to \mathbf{R}\underline{\mathrm{Hom}}_A(-, {}^*D^\bullet_A) \circ \mathbb{U}$.

Proposition 9. *Φ is a natural isomorphism. Hence $\mathbb{U} \circ \mathbb{D} \cong \mathbf{R}\underline{\mathrm{Hom}}_A(-, {}^*D^\bullet_A) \circ \mathbb{U}$.*

Proof. For $x \in P$, $\Phi(A/\mathfrak{p}_x)$ is the chain map $\underline{\mathrm{Hom}}_A(A/\mathfrak{p}_x, \phi) : \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, I^\bullet_A) \to \underline{\mathrm{Hom}}^\bullet_A(A/\mathfrak{p}_x, {}^*D^\bullet_A)$, which is a quasi-isomorphism as shown in Lemma 4. Since any indecomposable injectives in $\mathrm{Sq}\, A$ is isomorphic to $A/\mathfrak{p}_x$ for some $x \in P$, Φ is a natural isomorphism by [2, Proposition 7.1]. □

The proof of Theorem 1. Since $A \in \mathrm{Sq}\, A$, we have

$$I^\bullet_A = \mathbb{D}(A) \cong \mathbf{R}\underline{\mathrm{Hom}}_A(A, {}^*D^\bullet_A) = {}^*D^\bullet_A$$

by Proposition 9, where the isomorphism in the center is given by $\Phi(A)$. If we forget the $\mathbb{Z}^n$-grading, ${}^*D^\bullet_A$ is quasi-isomorphic to the usual (non-graded) dualizing complex $D^\bullet_A$. Hence $I^\bullet_A \cong D^\bullet_A$ in $\mathsf{D}^b(\mathrm{Mod}\, A)$. □

Remark 10. For $x \in P$ with $r = \rho(x)$, set $\underline{\mathbf{a}}(x) := (r, r, \dots, r) \in \mathbb{N}^r \cong \mathbb{M}(x) \subset \mathbb{M}$. If $x \geq y$, then there is a degree $\underline{\mathbf{a}}(x) - \underline{\mathbf{a}}(y) \in \mathbb{M}$ such that $t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot t^{\underline{\mathbf{a}}(y)} = t^{\underline{\mathbf{a}}(x)}$.

By $\mathsf{K}^b(\text{Inj-Sq}) \cong \mathsf{D}^b(\mathrm{Sq}\, A)$, $\mathbb{D}$ can be regarded as a duality on $\mathsf{D}^b(\mathrm{Sq}\, A)$. Then, through the equivalence $\mathrm{Sq}\, R \cong \mathrm{mod}\, \Lambda$, $\mathbb{D}$ coincides with the duality functor $\mathbf{D}$ on $\mathsf{D}^b(\mathrm{mod}\, \Lambda)$ defined in [10] up to translation. Hence, for $M^\bullet \in \mathsf{D}^b(\mathrm{Sq}\, A)$, the complex $\mathbb{D}(M^\bullet)$ has the following description: The term of cohomological degree p is

$$\mathbb{D}(M^\bullet)^p := \bigoplus_{i+\rho(x)=-p} (M^i_{\underline{\mathbf{a}}(x)})^* \otimes_{\Bbbk} A/\mathfrak{p}_x,$$

where $(-)^*$ denotes the $\Bbbk$-dual. The differential is given by

$$(M^i_{\underline{\mathbf{a}}(x)})^* \otimes_{\Bbbk} A/\mathfrak{p}_x \ni f \otimes 1_{A/\mathfrak{p}_x} \longmapsto \sum_{\substack{y \leq x, \\ \rho(y)=\rho(x)-1}} \epsilon(x,y) \cdot f_y \otimes 1_{A/\mathfrak{p}_y} + (-1)^p \cdot f \circ \partial^{i-1}_{M^\bullet} \otimes 1_{A/\mathfrak{p}_x},$$

where $f_y \in (M_{\underline{\mathbf{a}}(y)})^*$ denotes $M_{\underline{\mathbf{a}}(y)} \ni s \mapsto f(t^{\underline{\mathbf{a}}(x)-\underline{\mathbf{a}}(y)} \cdot s) \in \Bbbk$, and $\epsilon(x,y)$ is the incidence function.

Since $H^{-i}(\mathbb{D}(M)) \cong \underline{\mathrm{Ext}}^{-i}_A(M, {}^*D^\bullet_A) \cong H^i_{\mathfrak{m}}(M)^\vee$ in $\mathrm{Gr}\, A$, we have the following.

Corollary 11. *If $M \in \mathrm{Sq}\, A$, then the local cohomology $H^i_{\mathfrak{m}}(M)^\vee$ can be seen as a squarefree module.*

4. Sheaves and Poincaré-Verdier duality

The results in this section are parallel to those in [9, 10]. Recall that a simplicial poset P gives a regular cell complex $\Gamma(P)$. Let X be the underlying space of $\Gamma(P)$, and $c(x)$ the open cell corresponding to $\hat{0} \neq x \in P$. Hence, for each $x \in P$ with $\rho(x) \geq 2$, $c(x)$ is an open subset of X homeomorphic to $\mathbb{R}^{\rho(x)-1}$ (if $\rho(x) = 1$, then $c(x)$ is a single point), and X is the disjoint union of the cells $c(x)$. Moreover, $x \geq y$ if and only if $\overline{c(x)} \supset c(y)$.

As in the preceding section, let Λ be the incidence algebra of P. In [10], we assigned the constructible sheaf $N^\dagger$ on X to $N \in \mathrm{mod}\, \Lambda$. Through $\mathrm{Sq}\, A \cong \mathrm{mod}\, \Lambda$, we have the constructible sheaf M^+ on X corresponding to $M \in \operatorname{Sq} A$. Here we give a precise construction for the reader's convenience. For the sheaf theory, consult [3].

For $M \in \operatorname{Sq} A$, set

$$\operatorname{Sp\acute{e}}(M) := \bigcup_{\hat{0} \neq x \in P} c(x) \times M_{\underline{\mathbf{a}}(x)},$$

where $\underline{\mathbf{a}}(x) \in \mathbb{M}(x) \subset \mathbb{M}$ is the one defined in Remark 10. Let $\pi : \operatorname{Sp\acute{e}}(M) \to X$ be the projection map which sends $(p, m) \in c(x) \times M_{\underline{\mathbf{a}}(x)} \subset \operatorname{Sp\acute{e}}(M)$ to $p \in c(x) \subset X$. For an open subset $U \subset X$ and a map $s : U \to \operatorname{Sp\acute{e}}(M)$, we will consider the following conditions:

($*$) $\pi \circ s = \mathrm{id}_U$ and $s_p = t^{\underline{\mathbf{a}}(x) - \underline{\mathbf{a}}(y)} \cdot s_q$ for all $p \in c(x) \cap U$, $q \in c(y) \cap U$ with $x \geq y$. Here $s_p \in M_{\underline{\mathbf{a}}(x)}$ (resp. $s_q \in M_{\underline{\mathbf{a}}(y)}$) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).

($**$) There is an open covering $U = \bigcup_{i \in I} U_i$ such that the restriction of s to U_i satisfies ($*$) for all $i \in I$.

Now we define a sheaf M^+ on X as follows: For an open set $U \subset X$, set

$$M^+(U) := \{\, s \mid s : U \to \operatorname{Sp\acute{e}}(M) \text{ is a map satisfying } (**) \,\}$$

and the restriction map $M^+(U) \to M^+(V)$ for $U \supset V$ is the natural one. It is easy to see that M^+ is a constructible sheaf with respect to the cell decomposition $\Gamma(P)$. For example, A^+ is the $\Bbbk$-constant sheaf $\underline{\Bbbk}_X$ on X, and $(A/\mathfrak{p}_x)^+$ is (the extension to X of) the $\Bbbk$-constant sheaf on the closed cell $\overline{c(x)}$.

Let $\operatorname{Sh}(X)$ be the category of sheaves of $\Bbbk$-vector spaces on X. Since the stalk $(M^+)_p$ at $p \in c(x) \subset X$ is isomorphic to $M_{\underline{\mathbf{a}}(x)}$, the functor $(-)^+ : \operatorname{Sq} A \to \operatorname{Sh}(X)$ is exact.

As mentioned in the previous section, $\mathbb{D} : \mathsf{D}^b(\operatorname{Sq} A) \to \mathsf{D}^b(\operatorname{Sq} A)^{\mathsf{op}}$ corresponds to $\mathbf{T} \circ \mathbf{D} : \mathsf{D}^b(\operatorname{mod} \Lambda) \to \mathsf{D}^b(\operatorname{mod} \Lambda)^{\mathsf{op}}$, where $\mathbf{D}$ is the one defined in [10], and $\mathbf{T}$ is the translation functor (i.e., $\mathbf{T}(M^\bullet)^i = M^{i+1}$). Through $(-)^\dagger : \operatorname{mod} \Lambda \to \operatorname{Sh}(X)$, $\mathbf{D}$ gives the Poincaré-Verdier duality on $\mathsf{D}^b(\operatorname{Sh}(X))$, so we have the following.

Theorem 12. *For $M^\bullet \in \mathsf{D}^b(\operatorname{Sq} A)$, we have*

$$\mathbf{T}^{-1} \circ \mathbb{D}(M^\bullet)^+ \cong \mathbf{R}\mathcal{H}om((M^\bullet)^+, \mathcal{D}_X^\bullet)$$

in $\mathsf{D}^b(\operatorname{Sh}(X))$. In particular, $\mathbf{T}^{-1}((I_A^\bullet)^+) \cong \mathcal{D}_X^\bullet$, where $I_A^\bullet$ is the complex constructed in Theorem 1, and $\mathcal{D}_X^\bullet$ is the Verdier dualizing complex of X with the coefficients in $\Bbbk$.

The next result follows from results in [10].

Theorem 13. *For $M \in \operatorname{Sq} A$, we have the decomposition $H^i_{\mathfrak{m}}(M) = \bigoplus_{\underline{\mathbf{a}} \in \mathbb{M}} H^i_{\mathfrak{m}}(M)_{-\underline{\mathbf{a}}}$ by Corollary 11. The the following hold.*

(a) *There is an isomorphism*

$$H^i(X, M^+) \cong H^{i+1}_{\mathfrak{m}}(M)_{\mathbf{0}} \quad \text{for all } i \geq 1,$$

and an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M)_{\mathbf{0}} \to M_{\mathbf{0}} \to H^0(X, M^+) \to H^1_{\mathfrak{m}}(M)_{\mathbf{0}} \to 0.$$

(b) *If $0 \neq \underline{\mathbf{a}} \in \mathbb{M}$ with $x = \sigma(\underline{\mathbf{a}})$, then*

$$H^i_{\mathfrak{m}}(M)_{-\underline{\mathbf{a}}} \cong H^{i-1}_c(U_x, M^+|_{U_x})$$

for all $i \geq 0$. Here $U_x = \bigcup_{z \geq x} c(z)$ is an open set of X, and $H^\bullet_c(-)$ stands for the cohomology with compact support.

Let $\tilde{H}^i(X;\Bbbk)$ denote the ith *reduced cohomology* of X with coefficients in $\Bbbk$. That is, $\tilde{H}^i(X;\Bbbk) \cong H^i(X;\Bbbk)$ for all $i \geq 1$, and $\tilde{H}^0(X;\Bbbk) \oplus \Bbbk \cong H^0(X;\Bbbk)$, where $H^i(X;\Bbbk)$ is the usual cohomology of X. Recall that $H^i(X;\Bbbk)$ is isomorphic to the sheaf cohomology $H^i(X, \underline{\Bbbk}_X)$. In the Stanley-Reisner ring case, (the latter half of) the next result is nothing other than a famous formula of Hochster.

Corollary 14 (Duval [1, Theorem 5.9]). *We have*

$$[H^i_{\mathfrak{m}}(A)]_{\mathbf{0}} \cong \tilde{H}^{i-1}(X;\Bbbk) \quad \text{and} \quad [H^i_{\mathfrak{m}}(A)]_{-\underline{\mathbf{a}}} \cong H^{i-1}_c(U_x;\Bbbk)$$

for all $i \geq 0$ and all $\mathbf{0} \neq \underline{\mathbf{a}} \in \mathbb{M}$ with $x = \sigma(\underline{\mathbf{a}})$.

Here, $[H^i_{\mathfrak{m}}(A)]_{-\underline{\mathbf{a}}}$ is also isomorphic to the ith cohomology of the cochain complex

$$K^\bullet_x : 0 \to K^{\rho(x)}_x \to K^{\rho(x)+1}_x \to \cdots \to K^d_x \to 0 \quad \text{with} \quad K^i_x = \bigoplus_{\substack{z \geq x \\ \rho(z)=i}} \Bbbk\, b_z$$

(b_z is a basis element) whose differential map is given by

$$b_z \longmapsto \sum_{\substack{w \geq z \\ \rho(w)=\rho(z)+1}} \epsilon(w,z)\, b_w.$$

For this description, $\underline{\mathbf{a}}$ can be $\mathbf{0} \in \mathbb{M}$. In this case, $x = \hat{0}$.

Duval uses the latter description, and he denotes $H^i(K^\bullet_x)$ by $H^{i-\rho(x)-1}(\mathrm{lk}_P\, x)$.

Proof. The former half follows from Theorem 13. The latter part follows from that $H^i_{\mathfrak{m}}(A) \cong H^{-i}(\mathbb{D}(A))^\vee$ and that $(\mathbb{D}(A)^\vee)_{-\underline{\mathbf{a}}} = K^\bullet_x$ as complexes of $\Bbbk$-vector spaces by Remark 10. □

Remark 15. Consider the polynomial ring $T := \mathrm{Sym}\, A_1 \cong \Bbbk[t_1, \ldots, t_n]$ (note that T is *not* a subring of A). Since A is a squarefree module over T, the $\mathbb{Z}^n$-graded Hilbert function of $H^i_{\mathfrak{m}}(A)$ can be computed by [8, Theorem 2.10], and [1, Theorem 5.9] (essentially, the latter half of Corollary 14) follows rather quickly.

Similarly, we can easily describe $\mathbb{D}_T(A) \cong \mathbf{R}\underline{\mathrm{Hom}}_T(A, D^\bullet_T)$, and it coincides with $I^\bullet_A$ as a complex of T-modules. That is, the dualizing complex $D^\bullet_A$ becomes much easier if we regard it as a complex of T-modules.

Theorem 16 (c.f. Duval [1]). *Set $d := \mathrm{rank}\, P = \dim X + 1$. Then we have the following.*

(a) *A is Cohen-Macaulay if and only if $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ for all $i \neq -d+1$, and $\tilde{H}^i(X;\Bbbk) = 0$ for all $i \neq d-1$.*

(b) *Assume that A is Cohen-Macaulay and $d \geq 2$. Then A is Gorenstein*, if and only if $\mathcal{H}^{-d+1}(\mathcal{D}^\bullet_X) \cong \underline{\Bbbk}_X$. (When $d = 1$, A is Gorenstein* if and only if X consists of exactly two points.)*

(c) *A is Buchsbaum if and only if $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ for all $i \neq -d+1$.*

(d) *Set*

$$d_j := \begin{cases} \dim(\mathrm{supp}\, \mathcal{H}^{-j}(\mathcal{D}^\bullet_X)) & \text{if } \mathcal{H}^{-j}(\mathcal{D}^\bullet_X) \neq 0, \\ -1 & \text{if } \mathcal{H}^{-j}(\mathcal{D}^\bullet_X) = 0 \text{ and } \tilde{H}^j(X;\Bbbk) \neq 0, \\ -\infty & \text{if } \mathcal{H}^{-j}(\mathcal{D}^\bullet_X) = 0 \text{ and } \tilde{H}^j(X;\Bbbk) = 0. \end{cases}$$

Here $\operatorname{supp}\mathcal{F} = \{\, p \in X \mid \mathcal{F}_p \neq 0 \,\}$ *for a sheaf* $\mathcal{F}$ *on* X*. Then, for* $2 \leq i < d$*,* A *satisfies Serre's condition* (S_i) *if and only if* $d_j \leq j - i$ *for all* $j < d-1$*.*

Hence, Cohen-Macaulay (resp. Gorenstein, Buchsbaum) property and Serre's condition* (S_i) *of* A *are topological properties of* X*, while it may depend on* $\mathrm{char}(\Bbbk)$*.*

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