High Order Centers and Left Differential Operators

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Sweedler* generalized the notion of **high order derivations** of commutative algebras to the notion of **high order right derivations** of noncommutative algebras. However they have strange shapes.

In this talk, we introduce the notion of **high order centers** and try to unify high order left derivations and derivations.

Notation We use the following notations.

- $\mathbb{N} = \{0,\ 1,\ 2,\ 3,\ \ldots\ldots\}$
- $\boldsymbol{k}$: a commutative ring
- $\boldsymbol{k}$-**Alg** : the category of $\boldsymbol{k}$-algebras
- $\boldsymbol{A}$-**Mod** : the category of left $\boldsymbol{A}$-modules
- $\mathfrak{M}_{\boldsymbol{k}}(\boldsymbol{A})$: the category of bimodules over a $\boldsymbol{k}$-algebra $\boldsymbol{A}$

$$\boldsymbol{M} \in \mathfrak{M}_{\boldsymbol{k}}(\boldsymbol{A}) \iff \begin{cases} \boldsymbol{M} \text{ is an } \boldsymbol{A}\text{-bimodule} \\ \boldsymbol{\alpha u} = \boldsymbol{u\alpha} \quad (\forall \boldsymbol{\alpha} \in \boldsymbol{k},\ \forall \boldsymbol{u} \in \boldsymbol{M}) \end{cases}$$

Notation Let $\boldsymbol{M} \in \mathfrak{M}_{\boldsymbol{k}}(\boldsymbol{A})$.

- For $\boldsymbol{u} \in \boldsymbol{M}$ and $\boldsymbol{a} \in \boldsymbol{A}$, we set $[\boldsymbol{u}, \boldsymbol{a}] = \boldsymbol{ua} - \boldsymbol{au}$.
- For $\boldsymbol{U} \subseteq \boldsymbol{M}$, we set $[\boldsymbol{U}, \boldsymbol{A}] = \{[\boldsymbol{u}, \boldsymbol{a}] \mid \boldsymbol{u} \in \boldsymbol{U},\ \boldsymbol{a} \in \boldsymbol{A}\}$.
  Set $[\boldsymbol{U}, \boldsymbol{A}]_0 = \boldsymbol{U}$ and set $[\boldsymbol{U}, \boldsymbol{A}]_{\boldsymbol{q}+1} = [[\boldsymbol{U}, \boldsymbol{A}]_{\boldsymbol{q}},\ \boldsymbol{A}] \quad (\boldsymbol{q} \in \mathbb{N})$.
- If $\boldsymbol{U} = \{\boldsymbol{u}\}$, we set $[\boldsymbol{u}, \boldsymbol{A}] = [\boldsymbol{U}, \boldsymbol{A}]$ and $[\boldsymbol{u}, \boldsymbol{A}]_{\boldsymbol{q}} = [\boldsymbol{U}, \boldsymbol{A}]_{\boldsymbol{q}}$.

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* M. E. Sweedler: Right derivations and right differential operators, Pacific J. Math. **86** (1980), 327–360.

§ 1. High Order Centers (Simple Version)

Definition For $M \in \mathfrak{M}_k(A)$ and $q \in \mathbb{N}$,
we define the **qth order center of M** by

$$\mathcal{C}_A^q(M) = \{u \in M \mid [u, A]_q = 0\},$$

and set $\mathcal{C}_A(M) = \bigcup_{q=0}^{\infty} \mathcal{C}_A^q(M)$, which is a k-submodule of M.

- If $\alpha : A \to B$ is a k-algebra homomorphism,
  then $B \in \mathfrak{M}_k(A)$ via α and $\mathcal{C}_A(B)$ is a subalgebra of B.

Definition Set $\mathcal{J}_A^q = (A \otimes_k A)/A[1 \otimes 1,\ A]_q A$
and $j_A^q = 1 \otimes 1 + A[1 \otimes 1,\ A]_q A$ in $\mathcal{J}_A^q$.

Theorem 1 We have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{M}_k(A)}(\mathcal{J}_A^q,\ M) \ni \varphi \mapsto \varphi(j_A^q) \in \mathcal{C}_A^q(M) \quad (M \in \mathfrak{M}_k(A)).$$

Thoerem 2 (Relation to Separability)

(1) A is a separable algebra*

$$\Longrightarrow \quad \mathcal{J}_A^q = A \quad (\forall q > 0) \qquad \text{[Komatsu, 2001]}$$

$$\Longleftrightarrow \quad \mathcal{C}_A^q(M) = \mathcal{C}_A^1(M) \quad (\forall M \in \mathfrak{M}_k(A),\ \forall q > 1)$$

(2) A is a purely inseparable algebra**

$$\Longleftarrow \quad \mathcal{J}_A^q = A \otimes_k A \quad (\exists q) \qquad \text{[Sweedler]}$$

$$\Longleftrightarrow \quad \mathcal{C}_A^q(M) = M \quad (\exists q,\ \forall M \in \mathfrak{M}_k(A)).$$

- In general, $A = \mathcal{J}_A^1 \twoheadleftarrow \mathcal{J}_A^2 \twoheadleftarrow \mathcal{J}_A^3 \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{J}_A^q \twoheadleftarrow \cdots \twoheadleftarrow A \otimes_k A$.

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* A is separable $\Longleftrightarrow$ $A[1 \otimes 1, A]A$ is a direct summand of ${}_A A \otimes_k A_A$

** A is purely inseparable $\Longleftrightarrow$ $A[1 \otimes 1, A]A$ is small in ${}_A A \otimes_k A_A$

§ 2. High Oder Left Derivations (Simple Version)

Definition [Sweedler] Let $\boldsymbol{M}$, $\boldsymbol{N} \in \boldsymbol{A}$-**Mod**.*

(1) Set $\mathcal{D}_{\boldsymbol{A}}^{\boldsymbol{q}}(\boldsymbol{M}, \boldsymbol{N}) = \mathcal{C}_{\boldsymbol{A}}^{\boldsymbol{q}+1}(\mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{M}, \boldsymbol{N}))$,

whose element is called a $\boldsymbol{q}$**th order left differential operator**.

(2) Set $\mathrm{LDer}_{\boldsymbol{k}}^{\boldsymbol{q}}(\boldsymbol{A}, \boldsymbol{M}) = \left\{\boldsymbol{d} \in \mathcal{D}_{\boldsymbol{A}}^{\boldsymbol{q}}(\boldsymbol{A}, \boldsymbol{M}) \;\middle|\; \boldsymbol{d}(1) = 0\right\}$,

whose element is called a $\boldsymbol{q}$**th order left derivation**.

(3) Set $\mathcal{D}_{\boldsymbol{A}}(\boldsymbol{M}) = \mathcal{C}_{\boldsymbol{A}}(\mathrm{End}_{\boldsymbol{k}}(\boldsymbol{M})) \ \left(= \bigcup_{\boldsymbol{q}=0}^{\infty} \mathcal{D}_{\boldsymbol{A}}^{\boldsymbol{q}}(\boldsymbol{M}, \boldsymbol{M})\right)$,

which is called the **left differential operator algebra**.

Remark For $\boldsymbol{M} \in \boldsymbol{A}$-**Mod** and $\boldsymbol{d} \in \mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{A}, \boldsymbol{M})$, we have

$$\boldsymbol{d} \in \mathrm{LDer}_{\boldsymbol{k}}^{1}(\boldsymbol{A}, \boldsymbol{M}) \iff \boldsymbol{d}(\boldsymbol{xy}) = \boldsymbol{x}\boldsymbol{d}(\boldsymbol{y}) + \boldsymbol{y}\boldsymbol{d}(\boldsymbol{x}) \quad (\forall \boldsymbol{x}, \boldsymbol{y}).$$

In commutative ring theory, $\boldsymbol{d}$ is regarded as a derivation, i.e.,

$$\boldsymbol{d}(\boldsymbol{xy}) = \boldsymbol{x}\boldsymbol{d}(\boldsymbol{y}) + \boldsymbol{d}(\boldsymbol{x})\boldsymbol{y} \quad (\forall \boldsymbol{x}, \boldsymbol{y}).$$

Example

$$\boldsymbol{A} = \left\{ \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 0 & \boldsymbol{a} & \boldsymbol{d} \\ 0 & 0 & \boldsymbol{a} \end{pmatrix} \;\middle|\; \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \boldsymbol{K} \right\}$$

$\mathcal{J}_{\boldsymbol{A}}^{5} = \boldsymbol{A} \otimes_{\boldsymbol{k}} \boldsymbol{A}$ (Hence $\boldsymbol{A}$ is a purely inseparable algebra.)

$\mathcal{D}_{\boldsymbol{A}}^{4}(\boldsymbol{M}, \boldsymbol{N}) = \mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{M}, \boldsymbol{N}) \quad (\forall \boldsymbol{M}, \boldsymbol{N} \in \boldsymbol{A}$-**Mod**$)$.

$$\left(\begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 0 & \boldsymbol{a} & \boldsymbol{d} \\ 0 & 0 & \boldsymbol{a} \end{pmatrix} \mapsto \begin{pmatrix} 0 & \boldsymbol{b} & \boldsymbol{d} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \in \mathrm{LDer}_{\boldsymbol{k}}^{1}(\boldsymbol{A}, \boldsymbol{A}).$$

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* $\mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{M}, \boldsymbol{N}) \in \mathfrak{M}_{\boldsymbol{k}}(\boldsymbol{A})$

$(\boldsymbol{afb})(\boldsymbol{u}) = \boldsymbol{af}(\boldsymbol{bu}) \quad (\boldsymbol{f} \in \mathrm{Hom}_{\boldsymbol{A}}(\boldsymbol{M}, \boldsymbol{N}),\ \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{A},\ \boldsymbol{u} \in \boldsymbol{M})$

Definition In $\mathcal{J}_A^{q+1}$, we set $\Omega_A^q = A[j_A^{q+1}, A]A$,
which is called the **qth Kähler module**,
and define $d_A^q \in \mathrm{LDer}_k^q(A, \Omega_A^q)$ by $d_A^q(x) = [j_A^{q+1}, x]$.

Theorem 3 [Sweedler] We have following two natural isomorphisms.

- $\mathrm{Hom}_A(\mathcal{J}_A^{q+1} \otimes_A M, N) \ni \varphi \mapsto \varphi(j_A^{q+1} \otimes -) \in \mathcal{D}_A^q(M, N)$
- $\mathrm{Hom}_A(\Omega_A^q, M) \ni \varphi \mapsto \varphi d_A^q \in \mathrm{LDer}_k^q(A, M)$ $(M, N \in A\text{-}\mathbf{Mod})$.

Sweedler used $\mathcal{C}_A^q(-)$ only to define differential operators, and did not investigate the functor $\mathcal{C}_A^q$. And so he did not know that $\mathcal{J}_A^q$ also represents $\mathcal{C}_A^q$.

§ 3. Derivations

Definition For $M \in \mathfrak{M}_k(A)$ and $d \in \mathrm{Hom}_k(A, M)$,
d is called a **derivation** if $d(xy) = xd(y) + d(x)y$ $(\forall\, x, y)$

Important Fact Let $M, N \in \mathfrak{M}_k(A)$.

- $\mathrm{Hom}_k(M, N)$ has two A-bimodule structures.

$$\begin{cases} (afb)(u) = af(bu) \\ (a * f * b)(u) = f(ua)b \end{cases} \quad (f \in \mathrm{Hom}_A(M, N),\ a,\ b \in A,\ u \in M)$$

We set $[f, a] = fa - af$ and $[f, a]^* = f * a - a * f$.

- For $d \in \mathrm{Hom}_K(A, M)$, the following hold:

d is a derivation $\iff$ $[[d, A], A]^* = 0$ and $d(1) = 0$

d is a left derivation $\iff$ $[[d, A], A] = 0$ and $d(1) = 0$

Regarding this fact, we can unify derivations and left derivations.

§ 4. High Order Centers (General Version)

Notation

- Fix $\boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_n) \in (\boldsymbol{k}\text{-}\mathbf{Alg})^{\boldsymbol{n}}$
  $\boldsymbol{B} = (\boldsymbol{B}_1, \ldots, \boldsymbol{B}_n) \in (\boldsymbol{k}\text{-}\mathbf{Alg})^{\boldsymbol{n}}$
  $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n) : \boldsymbol{A} \to \boldsymbol{B}$ a morphism in $(\boldsymbol{k}\text{-}\mathbf{Alg})^{\boldsymbol{n}}$
  $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) \in \mathbb{N}^{\boldsymbol{n}} \setminus \{(0, \ldots, 0)\}$
- Set $\widehat{\boldsymbol{A}} = \boldsymbol{A}_1 \otimes_{\boldsymbol{k}} \cdots \otimes_{\boldsymbol{k}} \boldsymbol{A}_n$
  $\widehat{\boldsymbol{B}} = \boldsymbol{B}_1 \otimes_{\boldsymbol{k}} \cdots \otimes_{\boldsymbol{k}} \boldsymbol{B}_n$
  $\widehat{\boldsymbol{\alpha}} = \boldsymbol{\alpha}_1 \otimes \cdots \otimes \boldsymbol{\alpha}_n : \widehat{\boldsymbol{A}} \to \widehat{\boldsymbol{B}}$
- For $\boldsymbol{M} \in \mathfrak{M}_{\boldsymbol{k}}(\widehat{\boldsymbol{B}})$ and $\boldsymbol{u} \in \boldsymbol{M}$,

  we set $[\boldsymbol{u}, \boldsymbol{B}]_{\boldsymbol{q}} = [\cdots[[\boldsymbol{u}, \boldsymbol{B}_1]_{\boldsymbol{q}_1}, \boldsymbol{B}_2]_{\boldsymbol{q}_2}, \cdots, \boldsymbol{B}_n]_{\boldsymbol{q}_n}$.

  We note that $[[\boldsymbol{U}, \boldsymbol{B}_i], \boldsymbol{B}_j] = [[\boldsymbol{U}, \boldsymbol{B}_j], \boldsymbol{B}_i]$ for any $\boldsymbol{U} \subseteq \boldsymbol{M}$.

Definition For $\boldsymbol{M} \in \mathfrak{M}_{\boldsymbol{k}}(\widehat{\boldsymbol{B}})$,
we define the **center of $\boldsymbol{M}$ of type $\boldsymbol{q}$** by

$$\mathcal{C}_{\boldsymbol{\alpha}}^{\boldsymbol{q}}(\boldsymbol{M}) = \{\boldsymbol{u} \in \boldsymbol{M} \mid [\boldsymbol{u}, \boldsymbol{B}]_{\boldsymbol{q}} = [\boldsymbol{u}, \widehat{\boldsymbol{A}}] = 0\}.$$

Definition Set $\mathcal{J}_{\boldsymbol{\alpha}}^{\boldsymbol{q}} = (\widehat{\boldsymbol{B}} \otimes_{\widehat{\boldsymbol{A}}} \widehat{\boldsymbol{B}}) \,/\, \widehat{\boldsymbol{B}}\,[1 \otimes 1, \boldsymbol{B}]_{\boldsymbol{q}} \widehat{\boldsymbol{B}}$
and $j_{\boldsymbol{\alpha}}^{\boldsymbol{q}} = 1 \otimes 1 + \widehat{\boldsymbol{B}}\,[1 \otimes 1, \boldsymbol{B}]_{\boldsymbol{q}} \widehat{\boldsymbol{B}}$ in $\mathcal{J}_{\boldsymbol{\alpha}}^{\boldsymbol{q}}$.

Theorem 4 We have a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{M}_{\boldsymbol{k}}(\widehat{\boldsymbol{B}})}(\mathcal{J}_{\boldsymbol{\alpha}}^{\boldsymbol{q}}, \boldsymbol{M}) \ni \varphi \mapsto \varphi(j_{\boldsymbol{\alpha}}^{\boldsymbol{q}}) \in \mathcal{C}_{\boldsymbol{\alpha}}^{\boldsymbol{q}}(\boldsymbol{M}) \quad (\boldsymbol{M} \in \mathfrak{M}_{\boldsymbol{k}}(\widehat{\boldsymbol{B}})).$$

§ 5. Left Derivations (General Version)

Definition Let $\boldsymbol{\alpha} : \boldsymbol{A} \to \boldsymbol{B}$ be a morphism in $(\boldsymbol{k}\text{-}\mathbf{Alg})^{\boldsymbol{n}}$, and let $\boldsymbol{M}$, $\boldsymbol{N} \in \widehat{\boldsymbol{B}}\text{-}\mathbf{Mod}$.

(1) Set $\mathcal{D}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\boldsymbol{M}, \boldsymbol{N}) = \mathcal{C}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\mathrm{Hom}_{\boldsymbol{k}}(\boldsymbol{M}, \boldsymbol{N}))$ whose element is called a **left differential operators of type $\boldsymbol{q}$**.

(2) Set $\mathrm{LDer}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\widehat{\boldsymbol{B}}, \boldsymbol{M}) = \{\boldsymbol{d} \in \mathcal{D}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\widehat{\boldsymbol{B}}, \boldsymbol{M}) \mid \boldsymbol{d}(1) = 0\}$ whose element is called a **left derivation of type $\boldsymbol{q}$**.

Motivation Let $\boldsymbol{A} = (\boldsymbol{k}, \boldsymbol{k})$, $\boldsymbol{B} = (\boldsymbol{R}, \boldsymbol{R}^{\circ})$, $\boldsymbol{\alpha} = (\boldsymbol{\rho}, \boldsymbol{\rho})$, $\boldsymbol{q} = (1, 1)$, where $\boldsymbol{R}^{\circ}$ is the opposite algebra of $\boldsymbol{R}$ and $\boldsymbol{\rho} : \boldsymbol{k} \to \boldsymbol{R}$ is the structure morphism of $\boldsymbol{k}$-algebra $\boldsymbol{R}$. Then we have $\widehat{\boldsymbol{B}}\text{-}\mathbf{Mod} = \mathfrak{M}_{\boldsymbol{k}}(\boldsymbol{R})$ and $\{\boldsymbol{d} \in \mathcal{D}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\boldsymbol{R}, \boldsymbol{M}) \mid \boldsymbol{d}(1) = 0\}$ coincides with the set of derivations of $\boldsymbol{R}$ to $\boldsymbol{M}$ for all $\boldsymbol{M} \in \mathfrak{M}_{\boldsymbol{k}}(\boldsymbol{R})$.

Definition In $\mathcal{J}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}$, we set $\Omega^{\boldsymbol{q}}_{\boldsymbol{\alpha}} = \widehat{\boldsymbol{B}}\,[\boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}, \widehat{\boldsymbol{B}}]\widehat{\boldsymbol{B}}$, and define $\boldsymbol{d}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} \in \mathrm{LDer}_{\boldsymbol{\alpha}}(\widehat{\boldsymbol{B}}, \boldsymbol{M})$ by $\boldsymbol{d}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\boldsymbol{x}) = [\boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}, \boldsymbol{x}]$.

Lemma

(1) $\mathcal{D}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\widehat{\boldsymbol{B}}, \boldsymbol{M}) = \mathrm{Hom}_{\widehat{\boldsymbol{B}}}(\widehat{\boldsymbol{B}}, \boldsymbol{M}) \oplus \mathrm{LDer}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\widehat{\boldsymbol{B}}, \boldsymbol{M})$.

(2) $\mathcal{J}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} = \widehat{\boldsymbol{B}}\boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} \oplus \Omega^{\boldsymbol{q}}_{\boldsymbol{\alpha}} = \boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}\widehat{\boldsymbol{B}} \oplus \Omega^{\boldsymbol{q}}_{\boldsymbol{\alpha}}$ and $\{\boldsymbol{x} \in \widehat{\boldsymbol{B}} \mid \boldsymbol{x}\boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} = 0\} = \{\boldsymbol{x} \in \widehat{\boldsymbol{B}} \mid \boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}\boldsymbol{x} = 0\} = 0$.

Theorem 5 We have following two natural isomorphisms.

- $\mathrm{Hom}_{\widehat{\boldsymbol{B}}}(\mathcal{J}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} \otimes_{\widehat{\boldsymbol{B}}} \boldsymbol{M}, \boldsymbol{N}) \ni \boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}(\boldsymbol{j}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} \otimes -) \in \mathcal{D}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\boldsymbol{M}, \boldsymbol{N})$
- $\mathrm{Hom}_{\widehat{\boldsymbol{B}}}(\Omega^{\boldsymbol{q}}_{\boldsymbol{\alpha}}, \boldsymbol{M}) \ni \boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}\boldsymbol{d}^{\boldsymbol{q}}_{\boldsymbol{\alpha}} \in \mathrm{LDer}^{\boldsymbol{q}}_{\boldsymbol{\alpha}}(\widehat{\boldsymbol{B}}, \boldsymbol{M})$ $(\boldsymbol{M}, \boldsymbol{N} \in \widehat{\boldsymbol{B}}\text{-}\mathbf{Mod})$.

§ 6. Fundamental Properties of $\mathcal{J}^q_\alpha$ and Ω^q_α

<u>**Theorem 6**</u> Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be morphisms in $(k\text{-}\mathbf{Alg})^n$. Then

$$\mathcal{J}^q_\beta \simeq \mathcal{J}^q_{\beta\alpha}/\widehat{C}[j^q_{\beta\alpha}, \widehat{B}]\widehat{C} \quad \text{and} \quad \Omega^q_\beta \simeq \Omega^q_{\beta\alpha}/\widehat{C}d^q_{\beta\alpha}\hat{\beta}(\widehat{B})\widehat{C}.$$

<u>**Corollary**</u> Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be morphisms in $(k\text{-}\mathbf{Alg})^n$.

Then there exist exact sequences of $\widehat{C}$-bimodules

$$\widehat{C} \otimes_{\widehat{B}} \Omega^q_\alpha \otimes_{\widehat{B}} \widehat{C} \to \mathcal{J}^q_{\beta\alpha} \to \mathcal{J}^q_\beta \to 0 \quad \text{and}$$

$$\widehat{C} \otimes_{\widehat{B}} \Omega^q_\alpha \otimes_{\widehat{B}} \widehat{C} \to \Omega^q_{\beta\alpha} \to \Omega^q_\beta \to 0.$$

<u>**Theorem 7**</u> Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be morphisms in $(k\text{-}\mathbf{Alg})^n$ such that $\widehat{\beta} : \widehat{B} \to \widehat{C}$ is a surjective mapping. Set $I = \operatorname{Ker}\widehat{\beta}$. Then the following hold:

(1) $\mathcal{J}^q_{\beta\alpha} \simeq \mathcal{J}^q_\alpha/(I\mathcal{J}^q_\alpha + \mathcal{J}^q_\alpha I) \simeq \widehat{C} \otimes_{\widehat{B}} \mathcal{J}^q_\alpha \otimes_{\widehat{B}} \widehat{C}$

(2) $\Omega^q_{\beta\alpha} \simeq \Omega^q_\alpha/\widehat{B}\delta^q_\alpha(I)\widehat{B}$

(3) There exists an exact sequence of $\widehat{C}$-bimodules

$$I/I^2 \to \widehat{C} \otimes_{\widehat{B}} \Omega^q_\alpha \otimes_{\widehat{B}} \widehat{C} \to \Omega^q_{\beta\alpha} \to 0.$$

<u>**Theorem 8**</u> Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be morphisms in $(k\text{-}\mathbf{Alg})^n$ such that $\widehat{\beta} : \widehat{B} \to \widehat{C}$ is a surjective mapping. Set $B'_i = \operatorname{Im}\alpha_i + \operatorname{Ker}\beta_i$ and denote by $\iota_i : B'_i \to B_i$ the inclusion mapping ($i = 1, \ldots, n$). Set $\iota = (\iota_1, \ldots, \iota_n) : (B'_1, \ldots, B'_n) \to B$. Then $\Omega^q_{\beta\alpha} \simeq \Omega^q_\iota$.

§ 7. Separablility

According to **Theorem 2**, we propose the next definition.

Definition Let $\alpha : A \to B$ be a morphism of $(k\text{-}\mathbf{Alg})^n$.

(1) For $M \in \mathfrak{M}_k(\widehat{B})$, we set

$$\mathcal{CC}_\alpha(M) = \sum_{i=1}^{n} \left\{ u \in M \;\middle|\; [u, B_i] = [u, \widehat{A}] = 0 \right\}.$$

(2) α is called **q-quasi-separable** if $j_\alpha^q \in \mathcal{CC}_\alpha(\mathcal{J}_\alpha^q)$.

(3) α is called **left q-differentially separable** if

$$\mathcal{D}_\alpha^q(M, N) \subseteq \sum_{i=1}^{n} \mathrm{Hom}_{B_i}(M, N) \cap \mathrm{Hom}_{\widehat{A}}(M, N)$$
$$\big(= \mathcal{CC}_\alpha(\mathrm{Hom}_k(M, N)) \big) \quad (\forall M,\ N \in \widehat{B}\text{-}\mathbf{Mod})$$

Lemma

(1) α is q-quasi-separable $\iff$ $\mathcal{C}_\alpha^q(M) \subseteq \mathcal{CC}_\alpha(M) \quad (\forall M)$

(2) α is q-quasi-separable $\implies$ α is left q-differentially-separable

Theorem 9 Let $A = (k, k)$, $B = (R, R^\circ)$, and $\alpha = (\rho, \rho)$, where R° is the opposite algebra of R and $\rho : k \to R$ is the structure morphism of k-algebra. Then the following are equivalent:

(1) R is a separable algebra.

(2) α is $(1, 1)$-quasi-separable.

(3) α is q-quasi-separable for all $q \neq (0, 0)$.

(4) α is left $(1, 1)$-differentially-separable.

(5) α is left q-differentially-separable for all $q \neq (0, 0)$.

Definition Let $\rho : R \to S$ be a ring homomorphism.

(1) ρ is said to be **separable** if $S[1 \otimes 1, S]S$ is a direct summand of ${}_S S \otimes_R S_S$. Usually, S is called a **separable extension** of R.

(2) Set $K(\rho) = \{x \in S \mid [1 \otimes 1,\, x] = 0 \text{ in } S \otimes_R S\}$.

Lemma $K(\rho)$ is the largest subring of S contained in the kernels of all R-derivations of S to S-bimodules.

Theorem 10 Let $\alpha = (\alpha_1, \ldots, \alpha_n) : A \to B$ be a morphism in $(k\text{-}\mathbf{Alg})^n$. Suppose that all α_i are separable. Then the following hold:

(1) α is $(1, \ldots, 1)$-quasi-separable.

(2) If $[B_i, A_i] \subseteq K(\alpha_i)$ $(i = 1,\, \ldots,\, n)$, then α is q-quasi-separable for all $q \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$.

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