# REFLECTION FOR SELFINJECTIVE ALGEBRAS

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Abstract. We introduce the notion of reflections for selfinjective algebras and determine the transformations of Brauer trees associated with reflections. In particular, we provide a way to transform every Brauer tree into a Brauer line.

Reflection functors introduced in [4] are induced by transformations of the quiver making a certain sink vertex changed into a source vertex. Let $\Lambda$ be a finite dimensional algebra over a field $K$. In [3], it was shown that reflection functors are of the form $\mathrm{Hom}_\Lambda(T, -)$ with $T$ a certain type of tilting modules. Let $P_1, \cdots, P_n$ be a complete set of nonisomorphic indecomposable projective modules in mod-$\Lambda$, the category of finitely generated right $\Lambda$-modules. Set $I = \{1, \cdots, n\}$. Assume that there exists a simple projective module $S \in$ mod-$\Lambda$ which is not injective. Take $t \in I$ with $P_t \cong S$ and set

$$T = T_1 \oplus \tau^{-1}S \quad \text{with} \quad T_1 = \bigoplus_{i \in I \setminus \{t\}} P_i,$$

where $\tau$ denotes the Auslander-Reiten translation. Then $T$ is a tilting module, called an APR-tilting module, and $\mathrm{Hom}_\Lambda(T, -)$ is a reflection functor.

In [5], APR-tilting modules were generalized as follows. Assume that there exists a simple module $S \in$ mod-$\Lambda$ with $\mathrm{Ext}^1_\Lambda(S, S) = 0$ and $\mathrm{Hom}_\Lambda(D\Lambda, S) = 0$, where $D = \mathrm{Hom}_K(-, K)$. Let $P_t$ be the projective cover of $S$ and let $T$ be the same as above. Then $T$ is a tilting module, called a BB-tilting module. We are interested in a minimal projective presentation of $T$, which is a two-term tilting complex. Take a minimal injective presentation $0 \to S \to E^0 \xrightarrow{f} E^1$ and define a complex $E^\bullet$ as the mapping cone of $f : E^0 \to E^1$. Then $\mathrm{Hom}^\bullet_\Lambda(D\Lambda, E^\bullet)$ is a minimal projective presentation of $\tau^{-1}S$ and hence

$$T^\bullet = T_1 \oplus \mathrm{Hom}^\bullet_\Lambda(D\Lambda, E^\bullet)$$

is a minimal projective presentation of $T$. In this note, we demonstrate that this type of tilting complexes play an important role in the theory of derived equivalences for selfinjective algebras.

Let $K$ be a commutative artinian local ring and $\Lambda$ an Artin $K$-algebra, i.e., $\Lambda$ is a ring endowed with a ring homomorphism $K \to \Lambda$ whose image is contained in the center of $\Lambda$ and $\Lambda$ is finitely generated as a $K$-module. We always assume that $\Lambda$ is connected, basic and not simple. We denote by mod-$\Lambda$ the category of finitely generated right $\Lambda$-modules and by $\mathcal{P}_\Lambda$ the full subcategory of mod-$\Lambda$ consisting of projective modules. For a module $M \in$ mod-$\Lambda$, we denote by $P(M)$ (resp., $E(M)$) the projective cover (resp., injective

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envelope) of $M$. We denote by $\mathcal{K}(\text{mod-}\Lambda)$ the homotopy category of cochain complexes over mod-$\Lambda$ and by $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_\Lambda)$ the full triangulated subcategory of $\mathcal{K}(\text{mod-}\Lambda)$ consisting of bounded complexes over $\mathcal{P}_\Lambda$. We consider modules as complexes concentrated in degree zero.

Throughout the rest of this note, we assume that $\Lambda$ is selfinjective. Let $S \in \text{mod-}\Lambda$ be a simple module with $\mathrm{Ext}^1_\Lambda(S, S) = 0$ and $E(S) \cong P(S)$. Note that $E(S) \cong P(S)$ if and only if $\mathrm{Hom}_\Lambda(D\Lambda, S) \cong S$, where $D$ denotes the Matlis dual over $K$. Take a minimal injective presentation $0 \to S \to E^0 \xrightarrow{f} E^1$ and define a complex $E^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_\Lambda)$ as the mapping cone of $f : E^0 \to E^1$. Note that $E^1$ is the 0th term of $E^\bullet$ and $E^0$ is the $(-1)$th term of $E^\bullet$. Let $P_1, \cdots, P_n$ be a complete set of nonisomorphic indecomposable modules in $\mathcal{P}_\Lambda$ and set $I = \{1, \cdots, n\}$. We assume that $n > 1$. Take $t \in I$ with $P_t \cong P(S)$ and set

$$T^\bullet = T_1 \oplus E^\bullet \quad \text{with} \quad T_1 = \bigoplus_{i \in I \setminus \{t\}} P_i.$$

The following holds.

**Theorem 1.** *The complex $T^\bullet$ is a tilting complex for $\Lambda$ and $\mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^\bullet)$ is a selfinjective Artin $K$-algebra whose Nakayama permutation coincides with that of $\Lambda$.*

**Definition 2.** The derived equivalence induced by the tilting complex $T^\bullet$ is said to be *the reflection for $\Lambda$ at $t$*. Sometimes, we also say that $\mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^\bullet)$ is the reflection of $\Lambda$ at $t$.

We will apply Theorem 1 to Brauer tree algebras and determine the transformations of Brauer tree algebras induced by reflections. We assume that $K$ is an algebraically closed field. Recall that a Brauer tree $(B, v, m)$ consists of a finite tree $B$, called the underlying tree, together with a distinguished vertex $v$, called the exceptional vertex and a positive integer $m$, called the multiplicity. In case $m = 1$, $(B, v, m)$ is identified with the underlying tree $B$ and is called a Brauer tree without exceptional vertex. The pair of the number of edges of $B$ and the multiplicity $m$ is said to be the numerical invariants of $(B, v, m)$. Each Brauer tree determines a symmetric $K$-algebra $\Lambda$ up to Morita equivalence (see [2] for details), called a Brauer tree algebra, which is given as the path algebra defined by some quiver with relations $(\Lambda_0, \Lambda_1, \rho)$, where $\Lambda_0$ is the set of vertices, $\Lambda_1$ is the set of arrows between vertices and $\rho$ is the set of relations (see [6] for details). We have the following.

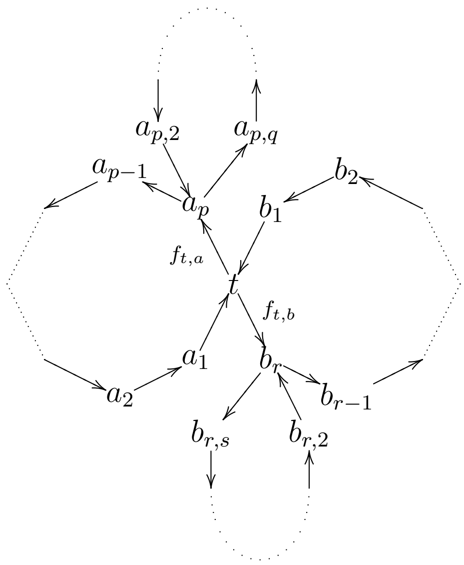
*Remark* 3. Let $\Lambda$ be a Brauer tree algebra.

(1) Every ring $\Gamma$ derived equivalent to $\Lambda$ is a Brauer tree algebra having the same numerical invariants as $\Lambda$ ([7, Theorem 4.2]).

(2) For any simple module $S \in \text{mod-}\Lambda$ we have $E(S) \cong P(S)$.

Throughout the rest of this note, we deal only with Brauer trees without exceptional vertex. Let $\Lambda$ be a Brauer tree algebra, $(\Lambda_0, \Lambda_1, \rho)$ the quiver with relations of $\Lambda$ and

$t \in \Lambda_0$. We consider the following cycles in $(\Lambda_0, \Lambda_1, \rho)$:




with $p, q, r, s \geq 0$, where $a_{p,1} = a_p$ and $b_{r,1} = b_r$ in case $p, r \geq 1$. We denote by $S_t$ the simple module corresponding to $t$ and by $P_t$ the projective cover of $S_t$.

**Lemma 4.** *The following hold.*

(1) *We have a minimal injective presentation*

$$0 \to S_t \to P_t \xrightarrow{f} P_{a_p} \oplus P_{b_r} \ \text{with}\ f = \begin{pmatrix} f_{t,a} \\ f_{t,b} \end{pmatrix}.$$

(2) *For any $t \in \Lambda_0$, we have $\mathrm{Ext}^1_\Lambda(S_t, S_t) = 0$.*

Take a minimal injective presentation $0 \to S_t \to E_t^0 \xrightarrow{f} E_t^1$ and define a complex $E_t^\bullet$ as the mapping cone of $f : E_t^0 \to E_t^1$. Set

$$T_t^\bullet = T_1 \oplus E_t^\bullet \quad \text{with} \quad T_1 = \bigoplus_{i \in \Lambda_0 \setminus \{t\}} P_i.$$

Then $T_t^\bullet$ is a tilting complex and $\mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T_t^\bullet)$ is the reflection of $\Lambda$ at $t$. Set $\Gamma = \mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T_t^\bullet)$ and let $(\Gamma_0, \Gamma_1, \sigma)$ be the quiver with relations of $\Gamma$. Note that $\Gamma_0 = (\Lambda_0 \setminus \{t\}) \cup \{t'\}$, where $t'$ is the vertex corresponding to $E_t^\bullet$. Since $\Gamma$ is a Brauer tree algebra, the relations $\sigma$ is determined automatically by $\Gamma_0$ and $\Gamma_1$. To determine $\Gamma_1$, we need the next lemma.

**Lemma 5.** *The following hold.*

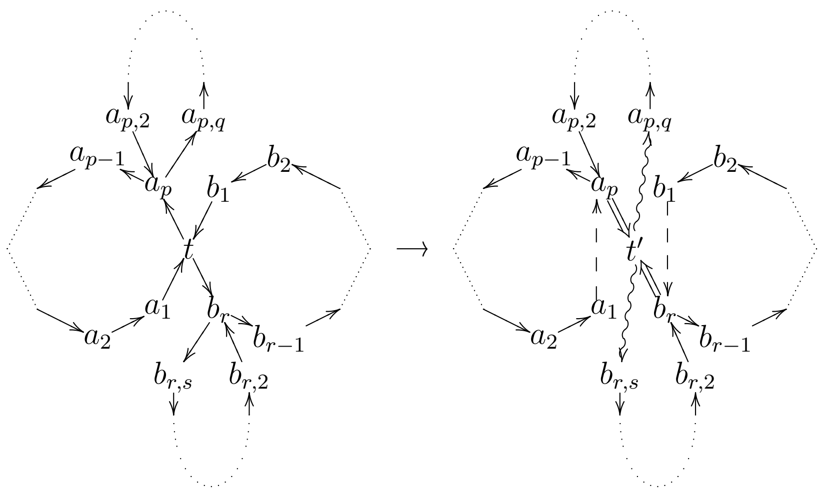
(1) *There exist $\zeta_{a_p} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{a_p}, E_t^\bullet)$ with $\zeta_{a_p} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$and $\zeta_{b_r} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{b_r}, E_t^\bullet)$ with $\zeta_{b_r} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$.*

(2) *There exist $\eta_{a_{p,q}} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(E_t^\bullet, P_{a_{p,q}})$ with $\eta_{a_{p,q}} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$and $\eta_{b_{r,s}} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(E_t^\bullet, P_{b_{r,s}})$ with $\eta_{b_{r,s}} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$.*

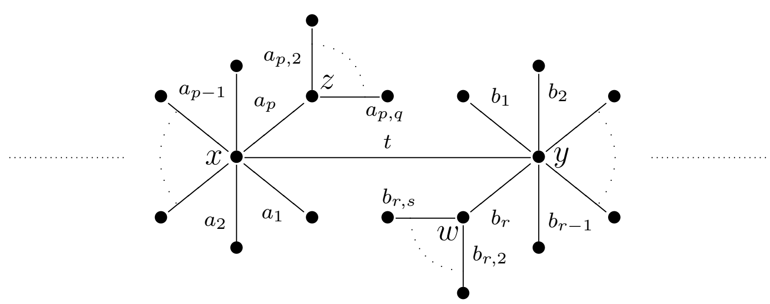
(3) *There exist $\theta_{a_p} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{a_1}, P_{a_p})$ with $\theta_{a_p} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$ and $\theta_{b_r} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{b_1}, P_{b_r})$ with $\theta_{b_r} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$.*

According to Lemma 5, we have the following new arrows in $\Gamma_1$. We denote by $\Longrightarrow$ the arrows defined by $\zeta_*$, by $\rightsquigarrow$ the arrows defined by $\eta_*$ and by $\dashrightarrow$ the arrows defined by $\theta_*$. In the next theorem, the left hand side diagram denotes cycles in $(\Lambda_0, \Lambda_1, \rho)$ and the right hand side diagram denotes cycles in $(\Gamma_0, \Gamma_1, \sigma)$.

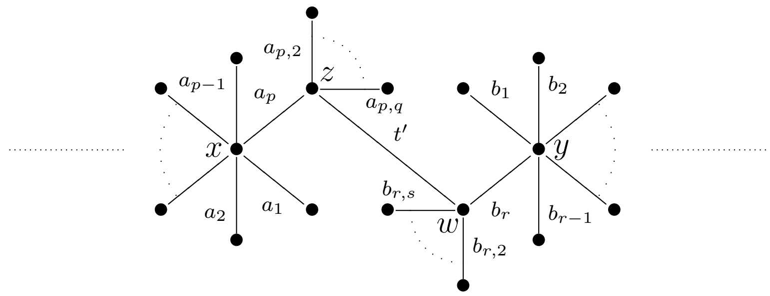
**Theorem 6.** *The reflection for $\Lambda$ at $t$ gives rise to the following transformation:*




Let $\Lambda$ be determined by a Brauer tree $B$ whose edges are identified with the vertices of $(\Lambda_0, \Lambda_1, \rho)$. We will describe a way to transform $B$ into a Brauer tree $B'$ determining $\Gamma$. Consider the tree




with $p, q, r, s \geq 0$, where $a_p = a_{p,1}, b_r = b_{r,1}$ in case $p, r \geq 1$. Turn the edge $t$ anti-clockwise around the vertex $x$ and select the edge $a_p$ which $t$ first meets. Then select the vertex $z$ of the edge $a_p$ different from $x$. Similarly, turn the edge $t$ anti-clockwise around the vertex $y$ and select the edge $b_r$ which $t$ first meets. Then select the vertex $w$ of the edge $b_r$ different from $y$. Add a new edge $t'$ connecting the vertices $z$ and $w$, and remove the edge $t$. As a consequence, we get the following Brauer tree $B'$:




**Corollary 7.** *The Brauer tree $B'$ determines $\Gamma$.*

**Corollary 8** (cf. [1, Theorem 3.7])**.** *There exists a sequence of Brauer tree algebras $\Lambda = \Delta_0, \Delta_1, \cdots, \Delta_l$ such that $\Delta_{i+1}$ is the reflection of $\Delta_i$ at a suitable vertex for $0 \leq i < l$ and $\Delta_l$ is a Brauer line algebra, i.e., the path algebra defined by the quiver*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

*with relations*

$$\alpha_{i+1}\alpha_i = \beta_i\beta_{i+1} = 0, \quad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1}$$

*for $1 \leq i < n-1$, where $n$ is the number of vertices of $(\Lambda_0, \Lambda_1, \rho)$.*

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