Reflection functors introduced in [4] are induced by transformations of the quiver making a certain sink vertex changed into a source vertex. Let \( \Lambda \) be a finite dimensional algebra over a field \( K \). In [3], it was shown that reflection functors are of the form \( \text{Hom}_\Lambda(T,-) \) with \( T \) a certain type of tilting modules. Let \( P_1, \ldots, P_n \) be a complete set of nonisomorphic indecomposable projective modules in \( \text{mod-}\Lambda \), the category of finitely generated \( \Lambda \)-modules. Set \( I = \{1, \ldots, n\} \). Assume that there exists a simple projective module \( S \in \text{mod-}\Lambda \) which is not injective. Take \( t \in I \) with \( P_t \cong S \) and set

\[
T = T_1 \oplus \tau^{-1}S \quad \text{with} \quad T_1 = \bigoplus_{i \in I \setminus \{t\}} P_i,
\]

where \( \tau \) denotes the Auslander-Reiten translation. Then \( T \) is a tilting module, called an APR-tilting module, and \( \text{Hom}_\Lambda(T,-) \) is a reflection functor.

In [5], APR-tilting modules were generalized as follows. Assume that there exists a simple module \( S \in \text{mod-}\Lambda \) with \( \text{Ext}_\Lambda^1(S,S) = 0 \) and \( \text{Hom}_\Lambda(D\Lambda,S) = 0 \), where \( D = \text{Hom}_K(-,K) \). Let \( P_t \) be the projective cover of \( S \) and let \( T \) be the same as above. Then \( T \) is a tilting module, called a BB-tilting module. We are interested in a minimal projective presentation of \( T \), which is a two-term tilting complex. Take a minimal injective presentation \( 0 \rightarrow S \rightarrow E^0 \xrightarrow{f} E^1 \) and define a complex \( E^\bullet \) as the mapping cone of \( f : E^0 \rightarrow E^1 \). Then \( \text{Hom}_\Lambda^\bullet(D\Lambda,E^\bullet) \) is a minimal projective presentation of \( \tau^{-1}S \) and hence

\[
T^\bullet = T_1 \oplus \text{Hom}_\Lambda^\bullet(D\Lambda,E^\bullet)
\]

is a minimal projective presentation of \( T \). In this note, we demonstrate that this type of tilting complexes play an important role in the theory of derived equivalences for selfinjective algebras.

Let \( K \) be a commutative artinian local ring and \( \Lambda \) an Artin \( K \)-algebra, i.e., \( \Lambda \) is a ring endowed with a ring homomorphism \( K \rightarrow \Lambda \) whose image is contained in the center of \( \Lambda \) and \( \Lambda \) is finitely generated as a \( K \)-module. We always assume that \( \Lambda \) is connected, basic and not simple. We denote by \( \text{mod-}\Lambda \) the category of finitely generated right \( \Lambda \)-modules and by \( \mathcal{P}_\Lambda \) the full subcategory of \( \text{mod-}\Lambda \) consisting of projective modules. For a module \( M \in \text{mod-}\Lambda \), we denote by \( P(M) \) (resp., \( E(M) \)) the projective cover (resp., injective

The detailed version of this note has been submitted for publication elsewhere.
envelope) of $M$. We denote by $\mathcal{K}(\text{mod-}\Lambda)$ the homotopy category of cochain complexes over $\text{mod-}\Lambda$ and by $\mathcal{K}^b(\mathcal{P}_\Lambda)$ the full triangulated subcategory of $\mathcal{K}(\text{mod-}\Lambda)$ consisting of bounded complexes over $\mathcal{P}_\Lambda$. We consider modules as complexes concentrated in degree zero.

Throughout the rest of this note, we assume that $\Lambda$ is selfinjective. Let $S_2 \mod-\Lambda$ be a simple module with $\text{Ext}_\Lambda^1(S, S) = 0$ and $E(S) \cong P(S)$. Note that $E(S) \cong P(S)$ if and only if $\text{Hom}_\Lambda(D\Lambda, S) \cong S$, where $D$ denotes the Matlis dual over $K$. Take a minimal injective presentation $0 \to S \to E^0 \xrightarrow{f} E^1$ and define a complex $E^* \in \mathcal{K}^b(\mathcal{P}_\Lambda)$ as the mapping cone of $f : E^0 \to E^1$. Note that $E^1$ is the 0th term of $E^*$ and $E^0$ is the $(-1)$th term of $E^*$. Let $P_1, \ldots, P_n$ be a complete set of nonisomorphic indecomposable modules in $\mathcal{P}_\Lambda$ and set $I = \{1, \ldots, n\}$. We assume that $n > 1$. Take $t \in I$ with $P_t \cong P(S)$ and set

$$T^* = T_1 \oplus E^* \quad \text{with} \quad T_1 = \bigoplus_{i \in \Lambda \setminus \{t\}} P_i.$$

The following holds.

**Theorem 1.** The complex $T^*$ is a tilting complex for $\Lambda$ and $\text{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^*)$ is a selfinjective Artin $K$-algebra whose Nakayama permutation coincides with that of $\Lambda$.

**Definition 2.** The derived equivalence induced by the tilting complex $T^*$ is said to be the reflection for $\Lambda$ at $t$. Sometimes, we also say that $\text{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^*)$ is the reflection of $\Lambda$ at $t$.

We will apply Theorem 1 to Brauer tree algebras and determine the transformations of Brauer tree algebras induced by reflections. We assume that $K$ is an algebraically closed field. Recall that a Brauer tree $(B, v, m)$ consists of a finite tree $B$, called the underlying tree, together with a distinguished vertex $v$, called the exceptional vertex and a positive integer $m$, called the multiplicity. In case $m = 1$, $(B, v, m)$ is identified with the underlying tree $B$ and is called a Brauer tree without exceptional vertex. The pair of the number of edges of $B$ and the multiplicity $m$ is said to be the numerical invariants of $(B, v, m)$. Each Brauer tree determines a symmetric $K$-algebra $\Lambda$ up to Morita equivalence (see [2] for details), called a Brauer tree algebra, which is given as the path algebra defined by some quiver with relations $(\Lambda_0, \Lambda_1, \rho)$, where $\Lambda_0$ is the set of vertices, $\Lambda_1$ is the set of arrows between vertices and $\rho$ is the set of relations (see [6] for details). We have the following.

**Remark 3.** Let $\Lambda$ be a Brauer tree algebra.

1. Every ring $\Gamma$ derived equivalent to $\Lambda$ is a Brauer tree algebra having the same numerical invariants as $\Lambda$ ([7, Theorem 4.2]).

2. For any simple module $S \in \text{mod-}\Lambda$ we have $E(S) \cong P(S)$.

Throughout the rest of this note, we deal only with Brauer trees without exceptional vertex. Let $\Lambda$ be a Brauer tree algebra, $(\Lambda_0, \Lambda_1, \rho)$ the quiver with relations of $\Lambda$ and
$t \in \Lambda_0$. We consider the following cycles in $(\Lambda_0, \Lambda_1, \rho)$:

with $p, q, r, s \geq 0$, where $a_{p,1} = a_p$ and $b_{r,1} = b_r$ in case $p, r \geq 1$. We denote by $S_t$ the simple module corresponding to $t$ and by $P_t$ the projective cover of $S_t$.

**Lemma 4.** The following hold.

1. We have a minimal injective presentation

   $$0 \to S_t \to P_1 \overset{f}{\to} P_{a_p} \oplus P_{b_r} \text{ with } f = \begin{pmatrix} f_{t,a} \\ f_{t,b} \end{pmatrix}.$$ 

2. For any $t \in \Lambda_0$, we have $\text{Ext}_\Lambda^1(S_t, S_t) = 0$.

Take a minimal injective presentation $0 \to S_t \to E_t^0 \overset{f}{\to} E_t^1$ and define a complex $E_t^\bullet$ as the mapping cone of $f : E_t^0 \to E_t^1$. Set

$$T_t^\bullet = T_1 \oplus E_t^\bullet \text{ with } T_1 = \bigoplus_{i \in \Lambda_0 \setminus \{t\}} P_i.$$ 

Then $T_t^\bullet$ is a tilting complex and $\text{End}_{X(\mod-\Lambda)}(T_t^\bullet)$ is the reflection of $\Lambda$ at $t$. Set $\Gamma = \text{End}_{X(\mod-\Lambda)}(T_t^\bullet)$ and let $(\Gamma_0, \Gamma_1, \sigma)$ be the quiver with relations of $\Gamma$. Note that $\Gamma_0 = (\Lambda_0 \setminus \{t\}) \cup \{t'\}$, where $t'$ is the vertex corresponding to $E_t^\bullet$. Since $\Gamma$ is a Brauer tree algebra, the relations $\sigma$ is determined automatically by $\Gamma_0$ and $\Gamma_1$. To determine $\Gamma_1$, we need the next lemma.

**Lemma 5.** The following hold.

1. There exist $\zeta_{a_p} \in \text{Hom}_{X(\mod-\Lambda)}(P_{a_p}, E_t^\bullet)$ with $\zeta_{a_p} \in \text{rad}(\Gamma) \setminus \text{rad}^2(\Gamma)$ and $\zeta_{b_r} \in \text{Hom}_{X(\mod-\Lambda)}(P_{b_r}, E_t^\bullet)$ with $\zeta_{b_r} \in \text{rad}(\Gamma) \setminus \text{rad}^2(\Gamma)$.

2. There exist $\eta_{a_p, q} \in \text{Hom}_{X(\mod-\Lambda)}(E_t^\bullet, P_{a_p, q})$ with $\eta_{a_p, q} \in \text{rad}(\Gamma) \setminus \text{rad}^2(\Gamma)$ and $\eta_{b_r, s} \in \text{Hom}_{X(\mod-\Lambda)}(E_t^\bullet, P_{b_r, s})$ with $\eta_{b_r, s} \in \text{rad}(\Gamma) \setminus \text{rad}^2(\Gamma)$.

3. There exist $\theta_{a_p} \in \text{Hom}_{X(\mod-\Lambda)}(P_{a_1, a_p})$ with $\theta_{a_p} \in \text{rad}(\Gamma) \setminus \text{rad}^2(\Gamma)$ and $\theta_{b_r} \in \text{Hom}_{X(\mod-\Lambda)}(P_{b_1, b_r})$ with $\theta_{b_r} \in \text{rad}(\Gamma) \setminus \text{rad}^2(\Gamma)$. 


According to Lemma 5, we have the following new arrows in $\Gamma_1$. We denote by $\longrightarrow$ the arrows defined by $\zeta_*$, by $\sim$ the arrows defined by $\eta_*$ and by $\dashrightarrow$ the arrows defined by $\theta_*$. In the next theorem, the left hand side diagram denotes cycles in $(\Lambda_0, \Lambda_1, \rho)$ and the right hand side diagram denotes cycles in $(\Gamma_0, \Gamma_1, \sigma)$.

**Theorem 6.** The reflection for $\Lambda$ at $t$ gives rise to the following transformation:

Let $\Lambda$ be determined by a Brauer tree $B$ whose edges are identified with the vertices of $(\Lambda_0, \Lambda_1, \rho)$. We will describe a way to transform $B$ into a Brauer tree $B'$ determining $\Gamma$. Consider the tree

with $p, q, r, s \geq 0$, where $a_p = a_{p, 1}, b_r = b_{r, 1}$ in case $p, r \geq 1$. Turn the edge $t$ anti-clockwise around the vertex $x$ and select the edge $a_p$ which $t$ first meets. Then select the vertex $z$ of the edge $a_p$ different from $x$. Similarly, turn the edge $t$ anti-clockwise around the vertex $y$ and select the edge $b_r$ which $t$ first meets. Then select the vertex $w$ of the edge $b_r$ different from $y$. Add a new edge $t'$ connecting the vertices $z$ and $w$, and remove the
edge $t$. As a consequence, we get the following Brauer tree $B'$:

![Brauer tree diagram]

**Corollary 7.** The Brauer tree $B'$ determines $\Gamma$.

**Corollary 8** (cf. [1, Theorem 3.7]). There exists a sequence of Brauer tree algebras $\Lambda = \Delta_0, \Delta_1, \ldots, \Delta_l$ such that $\Delta_{i+1}$ is the reflection of $\Delta_i$ at a suitable vertex for $0 \leq i < l$ and $\Delta_l$ is a Brauer line algebra, i.e., the path algebra defined by the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n$$

with relations

$$\alpha_{i+1} \alpha_i = \beta_i \beta_{i+1} = 0, \quad \alpha_i \beta_i = \beta_{i+1} \alpha_{i+1}$$

for $1 \leq i < n - 1$, where $n$ is the number of vertices of $(\Delta_0, \Lambda_1, \rho)$.

**References**


**Institute of Mathematics**
**University of Tsukuba**
**Ibaraki 305-8571 JAPAN**

**E-mail address:** abeh@math.tsukuba.ac.jp