WEAKLY SECTIONAL PATHS AND THE SHAPES OF AUSLANDER-REITEN QUIVERS

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Abstract. We introduce weakly sectional paths in the Auslander-Reiten quiver of an artin algebra, which are generalizations of sectional paths as well as pre-sectional paths of Liu [10]. We show that there are n-irreducible maps lying on a weakly sectional path of length n such that their composite does not fall into the $(n+1)$-th power of the radical of the module category. As a corollary we see that there is no weakly sectional oriented cycle in the Auslander-Reiten quiver.

1. Introduction

Throughout this report let K be a commutative Artinian ring and A an artin algebra over K ([1]). Denote by $\operatorname{mod} A$ the category of all finitely generated right A-modules and by Γ_A the Auslander-Reiten quiver of A. We also denote by τ the Auslander-Reiten translation $D\mathrm{Tr}$ in $\operatorname{mod} A$ and by $\mathfrak{R}$ the Jacobson radical of $\operatorname{mod} A$.

Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a path in Γ_A. Then an integer i with $1 \le i \le n-1$ is a hook of Ω, if $X_{i+1} \simeq \tau X_{i-1}$ holds. Moreover Ω is called a sectional path if Ω has no hook.

Recall from [1] that a map f in $\operatorname{mod} A$ is called an irreducible map, if it satisfies the following conditions:

(1) f is neither a section nor a retraction.
(2) If $f = hg$ for some maps g and h in $\operatorname{mod} A$, then either g is a section or h is a retraction.

Let $f : X \to Y$ be a map in $\operatorname{mod} A$ with X and Y indecomposable. Then it is well-known, as a connection between irreducible maps and $\mathfrak{R}$, that f is an irreducible map if and only if f belongs to $\mathfrak{R}(X,Y)\backslash\mathfrak{R}^2(X,Y)$. Based on this fact we study here the composite of irreducible maps lying on a certain path in Γ_A, which is called a weakly sectional path.

The following question is essential in the investigation of the composite of irreducible maps.

Question. For each $i = 1, \ldots, n$, let $f_i : X_i \to X_{i-1}$ be an irreducible map in $\operatorname{mod} A$ with X_i and X_{i-1} indecomposable. When do

$$f_1 f_2 \cdots f_n \neq 0 \qquad \text{and} \qquad f_1 f_2 \cdots f_n \in \mathfrak{R}^{n+1}(X_n, X_0)$$

hold?

This question has been studied for two irreducible maps in Γ_A (i.e. for the case $n = 2$)

The detailed version of this paper will be submitted for publication elsewhere.

[4], for irreducible maps lying on an almost sectional path in Γ_A [5], and for irreducible maps lying on a path in a standard component in Γ_A [7].

On the other hand, K. Igusa and G. Todorov [9] proved the following fact, which is a pioneering result in the study of the composite of irreducible maps:

Theorem 1 ([9])**.** *Let $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a sectional path in Γ_A. Then for all irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ we have $f_1 \cdots f_n \notin \mathfrak{R}^{n+1}(X_n, X_0)$.*

Also, in [10], S. Liu introduced the following path which is a generalization of a sectional path and proved an analogue for Theorem 1.

Definition 2 ([10])**.** Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a path in Γ_A. Then Ω is called a pre-sectional path, if, for each hook i of Ω, $\tau X_{i-1} \oplus X_{i+1}$ $(\simeq X_{i+1} \oplus X_{i+1})$ is a summand of the domain of the sink map for X_i.

Theorem 3 ([10])**.** *Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a pre-sectional path in Γ_A. Then there are irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ such that $f_1 \cdots f_n \notin \mathfrak{R}^{n+1}(X_n, X_0)$.*

The aim of this report is to introduce new paths called weakly sectional paths in Γ_A (Definition 4). These paths are clearly generalizations of sectional paths as well as pre-sectional paths. We generalize Theorem 3 and show, as a corollary, that there is no weakly sectional oriented cycle in Γ_A (Threorem 7 and Corollary 8).

2. Weakly sectional paths

In this section we define weakly sectional paths in Γ_A and give some examples of them.

First recall that the pair (d_{XY}, d'_{XY}) of integers is the valuation of an arrow $X \to Y$ in Γ_A if X appears d_{XY}-times in the domain of the sink map for Y, and Y appears d'_{XY}-times in the codomain of the source map for X. Let I be one of the sets $\{0, 1, \ldots, n\}$ $(n \geq 1)$, $\mathbb{N} \cup \{0\}$, or $\{0, -1, -2, \ldots\}$. Moreover, if Ω is a path $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in Γ_A where the set of the indices i of X_i is I, then we set

$$J_\Omega := \{j \in I \mid j \text{ is a hook in } \Omega \text{ with } d_{X_{j+1}X_j} = 1\}.$$

Definition 4 ([8])**.** Let $\Omega = \cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ be a path in Γ_A, where the set of the indices i of X_i is I. Then Ω is said to be a *weakly sectional path* in Γ_A, if there is a set of (non-zero) indecomposable modules $\{M_i\}_{i \in J_\Omega}$, called a *support* of Ω, such that

(1) $X_j \oplus M_j \oplus \tau X_{j-2}$ is a summand of the domain of the sink map for X_{j-1} for all $j \in J_\Omega$ such that $j - 2 \notin J_\Omega$. (Here, if $I = \{0, 1, \ldots, n\}$ or $I = \mathbb{N} \cup \{0\}$ and if $1 \in J_\Omega$, then define τX_{-1} to be an indecomposable module in $\operatorname{mod} A$.)

(2) $X_j \oplus M_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ is a summand of the domain of the sink map for X_{j-1} for all $j \in J_\Omega$ such that $j - 2 \in J_\Omega$.

(3) $X_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ is a summand of the domain of the sink map for X_{j-1} for all $j \in I \backslash J_\Omega$ such that $j - 2 \in J_\Omega$.

It is easy to see that any subpath of a weakly sectional path is also a weakly sectional path.

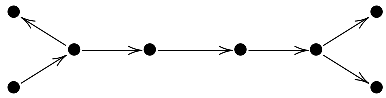
Remark 5. Let Ω be a path $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in Γ_A, where the set of indices i of X_i is I.

(1) Ω is a pre-sectional path if and only if $J_\Omega = \emptyset$.
(2) Suppose that Ω is a weakly sectional path with a support $\{M_i\}_{i \in J_\Omega}$. If $j \in J_\Omega$, then $j + 1 \notin J_\Omega$.

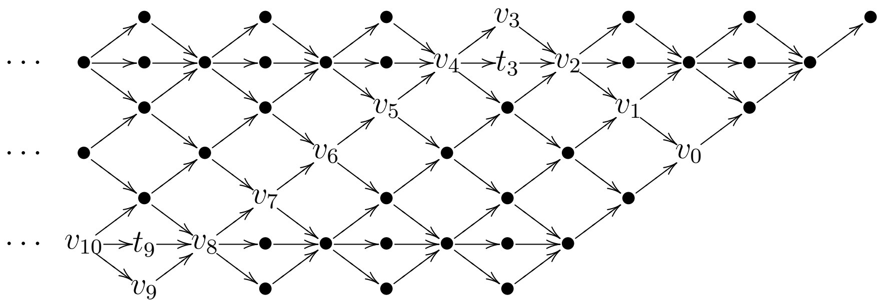
We now provide typical examples of weakly sectional paths.

Example 6. Suppose that K is an algebraically closed field.

(a) Let Δ be the quiver

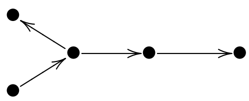


of the Euclidean type $\widetilde{\mathbb{D}}_7$. Then the pre-injective component $\mathcal{Q}$ of the path algebra $K\Delta$ is of the form:

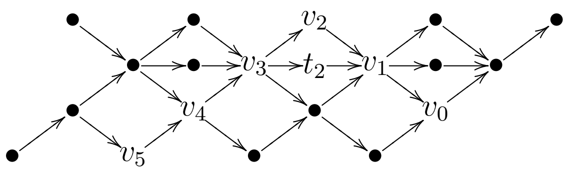


The infinite path $\cdots \to v_{n+1} \to v_n \to v_{n-1} \to \cdots \to v_1 \to v_0$ in $\mathcal{Q}$ is not a pre-sectional path but is a weakly sectional path with a support $\{t_j \mid j = 6i + 3 \text{ for } i \geq 0\}$. (Note that, in this case, for each vertex v there is a weakly sectional path ending with v.)

(b) Let Δ be the quiver



of the Dynkin type $\mathbb{D}_5$. Then it is well-known that $K\Delta$ is of representation-finite, and the Auslander-Reiten quiver of $K\Delta$ is of the form:



The path $v_5 \to v_4 \to \cdots \to v_1 \to v_0$ is not a pre-sectional path but is a weakly sectional path with a support $\{t_2\}$.

3. Main result

Now, using same technique in the proof of Theorem 3 given in [10], we have the following result. (see [8] for the detail of proof).

Theorem 7 ([8]). *Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a weakly sectional path in Γ_A. Then there are irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ such that $f_1 \cdots f_n \in \mathfrak{R}^{n+1}(X_n, X_0)$.*

Using Harada-Sai lemma (see for example [1]) we immediately have the following.

Corollary 8. *There is no weakly sectional oriented cycle in Γ_A.*

Remark 9. It is shown by Bautista and Smalø [3] (see also [2]) that there is no sectional oriented cycle in Γ_A, and by Liu [10] that there is no pre-sectional oriented cycle in Γ_A.

In particular, if A is a finite-dimensional algebra over an algebraically closed field K, we have the following.

Corollary 10. *Let A be finite-dimensional algebra over an algebraically closed field K, and let $\mathcal{C}$ be a component in Γ_A such that the valuation of every arrow in $\mathcal{C}$ is trivial. Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a weakly sectional path in $\mathcal{C}$. Then for any irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ in $\mathcal{C}$ we have $f_1 \cdots f_n \notin \mathfrak{R}^{n+1}(X_n, X_0)$.*

Example 11. Consider again the paths of Example 6. Then since the path $\cdots \to v_{n+1} \to v_n \to v_{n-1} \to \cdots \to v_1 \to v_0$ of Example 6 (a) is weakly sectional, it follows by Corollary 10 that for all irreducible maps $f_i : v_i \to v_{i-1}$ $(i = 1, 2, \ldots)$ we have $f_m \cdots f_\ell \notin \mathfrak{R}^{m+1}(v_m, v_0)$ for any integers $\ell > m > 0$. Similarly since the path $v_5 \to v_4 \to \cdots \to v_1 \to v_0$ of Example 6 (b) is a weakly sectional path, it follows that, for all irreducible maps $f_i : v_i \to v_{i-1}$ $(i = 1, \ldots, 5)$, the composite $f_1 \cdots f_5$ is not in $\mathfrak{R}^6(v_5, v_0)$.

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