WEAKLY SECTIONAL PATHS AND THE SHAPES OF AUSLANDER-REITEN QUIVERS

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ABSTRACT. We introduce weakly sectional paths in the Auslander-Reiten quiver of an artin algebra, which are generalizations of sectional paths as well as pre-sectional paths of Liu [10]. We show that there are \( n \)-irreducible maps lying on a weakly sectional path of length \( n \) such that their composite does not fall into the \((n+1)\)-th power of the radical of the module category. As a corollary we see that there is no weakly sectional oriented cycle in the Auslander-Reiten quiver.

1. INTRODUCTION

Throughout this report let \( K \) be a commutative Artinian ring and \( A \) an artin algebra over \( K \) ([1]). Denote by \( \text{mod} A \) the category of all finitely generated right \( A \)-modules and by \( \Gamma_A \) the Auslander-Reiten quiver of \( A \). We also denote by \( \tau \) the Auslander-Reiten translation \( D \text{Tr} \) in \( \text{mod} A \) and by \( \mathcal{R} \) the Jacobson radical of \( \text{mod} A \).

Let \( \Omega = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \) be a path in \( \Gamma_A \). Then an integer \( i \) with \( 1 \leq i \leq n - 1 \) is a hook of \( \Omega \), if \( X_{i+1} \cong \tau X_{i-1} \) holds. Moreover \( \Omega \) is called a sectional path if \( \Omega \) has no hook.

Recall from [1] that a map \( f \) in \( \text{mod} A \) is called an irreducible map, if it satisfies the following conditions:

(1) \( f \) is neither a section nor a retraction.
(2) If \( f = hg \) for some maps \( g \) and \( h \) in \( \text{mod} A \), then either \( g \) is a section or \( h \) is a retraction.

Let \( f : X \rightarrow Y \) be a map in \( \text{mod} A \) with \( X \) and \( Y \) indecomposable. Then it is well-known, as a connection between irreducible maps and \( \mathcal{R} \), that \( f \) is an irreducible map if and only if \( f \) belongs to \( \mathcal{R}(X, Y) \backslash \mathcal{R}^2(X, Y) \). Based on this fact we study here the composite of irreducible maps lying on a certain path in \( \Gamma_A \), which is called a weakly sectional path.

The following question is essential in the investigation of the composite of irreducible maps.

**Question.** For each \( i = 1, \ldots, n \), let \( f_i : X_i \rightarrow X_{i-1} \) be an irreducible map in \( \text{mod} A \) with \( X_i \) and \( X_{i-1} \) indecomposable. When do

\[
 f_1 f_2 \cdots f_n \neq 0 \quad \text{and} \quad f_1 f_2 \cdots f_n \in \mathcal{R}^{n+1}(X_n, X_0)
\]

hold?

This question has been studied for two irreducible maps in \( \Gamma_A \) (i.e. for the case \( n = 2 \))

The detailed version of this paper will be submitted for publication elsewhere.
[4], for irreducible maps lying on an almost sectional path in $\Gamma_A$ [5], and for irreducible maps lying on a path in a standard component in $\Gamma_A$ [7].

On the other hand, K. Igusa and G. Todorov [9] proved the following fact, which is a pioneering result in the study of the composite of irreducible maps:

**Theorem 1** ([9]). Let $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a sectional path in $\Gamma_A$. Then for all irreducible maps $f_i: X_i \to X_{i-1}$ ($i = 1, \ldots, n$) we have $f_1 \cdots f_n \notin R^{n+1}(X_n, X_0)$.

Also, in [10], S. Liu introduced the following path which is a generalization of a sectional path and proved an analogue for Theorem 1.

**Definition 2** ([10]). Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a path in $\Gamma_A$. Then $\Omega$ is called a pre-sectional path, if, for each hook $i$ of $\Omega$, $\tau X_{i-1} \oplus X_{i+1} (\simeq X_{i+1} \oplus X_{i+1})$ is a summand of the domain of the sink map for $X_i$.

**Theorem 3** ([10]). Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a pre-sectional path in $\Gamma_A$. Then there are irreducible maps $f_i: X_i \to X_{i-1}$ ($i = 1, \ldots, n$) such that $f_1 \cdots f_n \notin R^{n+1}(X_n, X_0)$.

The aim of this report is to introduce new paths called weakly sectional paths in $\Gamma_A$ (Definition 4). These paths are clearly generalizations of sectional paths as well as pre-sectional paths. We generalize Theorem 3 and show, as a corollary, that there is no weakly sectional oriented cycle in $\Gamma_A$ (Theorem 7 and Corollary 8).

2. **Weakly sectional paths**

In this section we define weakly sectional paths in $\Gamma_A$ and give some examples of them. First recall that the pair $(d_{XY}, d'_{XY})$ of integers is the valuation of an arrow $X \to Y$ in $\Gamma_A$ if $X$ appears $d_{XY}$-times in the domain of the sink map for $Y$, and $Y$ appears $d'_{XY}$-times in the codomain of the source map for $X$. Let $I$ be one of the sets $\{0, 1, \ldots, n\}$ ($n \geq 1$), $\mathbb{N} \cup \{0\}$, or $\{0, -1, -2, \ldots\}$. Moreover, if $\Omega$ is a path $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in $\Gamma_A$ where the set of the indices $i$ of $X_i$ is $I$, then we set

$$J_I := \{ j \in I \mid j \text{ is a hook in } \Omega \text{ with } d_{X_{j+1}X_j} = 1 \}.$$

**Definition 4** ([8]). Let $\Omega = \cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ be a path in $\Gamma_A$, where the set of the indices $i$ of $X_i$ is $I$. Then $\Omega$ is said to be a weakly sectional path in $\Gamma_A$, if there is a set of (non-zero) indecomposable modules $\{M_i\}_{i \in J_I}$, called a support of $\Omega$, such that

1. $X_j \oplus M_j \oplus \tau X_{j-2}$ is a summand of the domain of the sink map for $X_{j-1}$ for all $j \in J_I$ such that $j - 2 \notin J_I$. (Here, if $I = \{0, 1, \ldots, n\}$ or $I = \mathbb{N} \cup \{0\}$ and if $1 \in J_I$, then define $\tau X_{-1}$ to be an indecomposable module in mod $A$.)
2. $X_j \oplus M_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ is a summand of the domain of the sink map for $X_{j-1}$ for all $j \in J_I$ such that $j - 2 \in J_I$.
3. $X_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ is a summand of the domain of the sink map for $X_{j-1}$ for all $j \in \Gamma \setminus J_I$ such that $j - 2 \in J_I$.

It is easy to see that any subpath of a weakly sectional path is also a weakly sectional path.

**Remark 5.** Let $\Omega$ be a path $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in $\Gamma_A$, where the set of indices $i$ of $X_i$ is $I$. 
(1) $\Omega$ is a pre-sectional path if and only if $J_\Omega = \emptyset$.
(2) Suppose that $\Omega$ is a weakly sectional path with a support $\{M_i\}_{i \in J_\Omega}$. If $j \in J_\Omega$, then $j + 1 \not\in J_\Omega$.

We now provide typical examples of weakly sectional paths.

**Example 6.** Suppose that $K$ is an algebraically closed field.

(a) Let $\Delta$ be the quiver

of the Euclidean type $\tilde{D}_7$. Then the pre-injective component $Q$ of the path algebra $K\Delta$ is of the form:

The infinite path $\cdots \rightarrow v_{n+1} \rightarrow v_n \rightarrow v_{n-1} \rightarrow \cdots \rightarrow v_1 \rightarrow v_0$ in $Q$ is not a pre-sectional path but is a weakly sectional path with a support $\{\tau_j \mid j = 6i + 3 \text{ for } i \geq 0\}$. (Note that, in this case, for each vertex $v$ there is a weakly sectional path ending with $v$.)

(b) Let $\Delta$ be the quiver

of the Dynkin type $D_5$. Then it is well-known that $K\Delta$ is of representation-finite, and the Auslander-Reiten quiver of $K\Delta$ is of the form:

The path $v_5 \rightarrow v_4 \rightarrow \cdots \rightarrow v_1 \rightarrow v_0$ is not a pre-sectional path but is a weakly sectional path with a support $\{\tau_2\}$.

3. Main result

Now, using same technique in the proof of Theorem 3 given in [10], we have the following result. (see [8] for the detail of proof).

**Theorem 7 ([8]).** Let $\Omega = X_n \rightarrow X_{n-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0$ be a weakly sectional path in $\Gamma_A$. Then there are irreducible maps $f_i : X_i \rightarrow X_{i-1}$ ($i = 1, \ldots, n$) such that $f_1 \cdots f_n \in \mathcal{R}^{n+1}(X_n, X_0)$. 

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Using Harada-Sai lemma (see for example [1]) we immediately have the following.

**Corollary 8.** There is no weakly sectional oriented cycle in $\Gamma_A$.

**Remark 9.** It is shown by Bautista and Smalø [3] (see also [2]) that there is no sectional oriented cycle in $\Gamma_A$, and by Liu [10] that there is no pre-sectional oriented cycle in $\Gamma_A$.

In particular, if $A$ is a finite-dimensional algebra over an algebraically closed field $K$, we have the following.

**Corollary 10.** Let $A$ be finite-dimensional algebra over an algebraically closed field $K$, and let $C$ be a component in $\Gamma_A$ such that the valuation of every arrow in $C$ is trivial. Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a weakly sectional path in $C$. Then for any irreducible maps $f_i : X_i \to X_{i-1}$ ($i = 1, \ldots, n$) in $C$ we have $f_1 \cdots f_n \notin R^{n+1}(X_n, X_0)$.

**Example 11.** Consider again the paths of Example 6. Then since the path $\cdots \to v_{n+1} \to v_n \to v_{n-1} \to \cdots \to v_1 \to v_0$ of Example 6 (a) is weakly sectional, it follows by Corollary 10 that for all irreducible maps $f_i : v_i \to v_{i-1}$ ($i = 1, 2, \ldots$) we have $f_m \cdots f_\ell \notin R^{m+1}(v_m, v_0)$ for any integers $\ell > m > 0$. Similarly since the path $v_5 \to v_4 \to \cdots \to v_1 \to v_0$ of Example 6 (b) is a weakly sectional path, it follows that, for all irreducible maps $f_i : v_i \to v_{i-1}$ ($i = 1, \ldots, 5$), the composite $f_1 \cdots f_5$ is not in $R^6(v_5, v_0)$.

**REFERENCES**


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