MODULES LEFT ORTHOGONAL TO MODULES OF FINITE PROJECTIVE DIMENSION

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Abstract. In this proceeding, we characterize several properties of commutative noetherian local rings in terms of the left perpendicular category of the category of finitely generated modules of finite projective dimension. As an application we prove that a local ring is regular if (and only if) there exists a strong test module for projectivity having finite projective dimension.

Key Words: perpendicular category, projective dimension, semidualizing module, totally reflexive module, strong test module for projectivity.

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1. Introduction

Throughout this proceeding, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. All modules considered in this proceeding are assumed to be finitely generated.

An R-module C is said to be *semidualizing* if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^i(C, C) = 0$ for all $i > 0$. A semidualizing module admits a duality property, which has been defined by Foxby [5] and Golod [6]. A free module of rank one and a dualizing module are semidualizing modules. Various homological dimensions with respect to a fixed semidualizing R-module C are invented and investigated (cf. [2, 6, 9]). Among them, the *C-projective dimension* of a nonzero R-module M, denoted by $C\text{-}\mathrm{pd}_R\, M$, is defined as the infimum of integers n such that there exists an exact sequence of the form

$$0 \to C^{b_n} \to C^{b_{n-1}} \to \cdots \to C^{b_1} \to C^{b_0} \to M \to 0,$$

where each b_i is a positive integer.

An R-module M is called *totally C-reflexive*, where C is a semidualizing R-module, if the natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism and $\mathrm{Ext}_R^i(M, C) = \mathrm{Ext}_R^i(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$. The *complete intersection dimension* of M, which has been introduced in [4], is defined as the infimum of $\mathrm{pd}_S(M \otimes_R R') - \mathrm{pd}_S R'$ where $R \to R' \leftarrow S$ runs over all quasi-deformations. Here, a diagram $R \xrightarrow{f} R' \xleftarrow{g} S$ of homomorphisms of local rings is said to be a quasi-deformation if f is faithfully flat and g is a surjection whose kernel is generated by an S-regular sequence.

We denote by $\mathrm{mod}\, R$ the category of (finitely generated) R-modules. Let $\mathcal{G}_C(R)$, $\mathcal{I}(R)$, and $\mathrm{add}\, C$ denote the full subcategories of $\mathrm{mod}\, R$ consisting of all totally C-reflexive

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R-modules, consisting of all R-modules of complete intersection dimension zero, and consisting of all direct summands of finite direct sums of copies of C, respectively. Let $\mathcal{X}_C(R)$ be the *left perpendicular category* of the category of R-modules of finite C-projective dimension, that is, the subcategory of $\operatorname{mod} R$ consisting of all R-modules X satisfying $\operatorname{Ext}_R^1(X, M) = 0$ for each R-module M of finite C-projective dimension. We write $\mathcal{G}(R) = \mathcal{G}_R(R)$ and $\mathcal{X}(R) = \mathcal{X}_R(R)$. There are inclusion relations of subcategories of $\operatorname{mod} R$:

$$\mathcal{X}(R) \supset \mathcal{G}(R) \supset \mathcal{I}(R) \supset \operatorname{add} R,$$
$$\mathcal{X}_C(R) \supset \mathcal{G}_C(R) \supset \operatorname{add} C,\ \operatorname{add} R.$$

The main purpose of this proceeding is to find out what property is characterized by the equalities of $\mathcal{X}(R)$ (respectively, $\mathcal{X}_C(R)$) and each of $\mathcal{G}(R)$, $\mathcal{I}(R)$, $\operatorname{add} R$ (respectively, each of $\mathcal{G}_C(R)$, $\operatorname{add} C$, $\operatorname{add} R$). The main result of this proceeding is the following theorem.

Theorem 1. *Let R be a commutative noetherian local ring.*

(1) The following are equivalent for a semidualizing R-module C.

(a) C is dualizing.

(b) $\mathcal{X}_C(R) = \mathcal{G}_C(R)$ holds.

If this is the case, then R is Cohen-Macaulay.

(2) The following are equivalent.

(a) R is Gorenstein.

(b) $\mathcal{X}(R) = \mathcal{G}(R)$ holds.

(3) The following are equivalent.

(a) R is a complete intersection.

(b) $\mathcal{X}(R) = \mathcal{I}(R)$ holds.

(4) The following are equivalent.

(a) R is regular.

(b) $\mathcal{X}(R) = \operatorname{add} R$ holds.

(c) $\mathcal{X}_C(R) = \operatorname{add} C$ holds for some semidualizing R-module C.

(d) $\mathcal{X}_C(R) = \operatorname{add} R$ holds for some semidualizing R-module C.

On the other hand, the notion of a strong test module for projectivity has been introduced and studied by Ramras [8]. An R-module M is called a *strong test module for projectivity* if every R-module N with $\operatorname{Ext}_R^1(N, M) = 0$ is projective, or equivalently, free. The residue field k is a typical example of a strong test module for projectivity. Ramras shows that the maximal ideal $\mathfrak{m}$ is a strong test module for projectivity. He also proves that every strong test module for projectivity has depth at most one. Using the rigidity theorem for Tor modules, Jothilingam [7] proves that when R is a regular local ring, every R-module of depth at most one is a strong test module for projectivity. Our Theorem 1 yields that the converse of this Jothilingam's result also holds true.

Corollary 2. *The following seven conditions are equivalent.*

(1) R is regular.

(2) Every R-module of depth at most one is a strong test module for projectivity.

(3) Every R-module of depth zero is a strong test module for projectivity.

(4) Every R-module of depth zero and of finite projective dimension is a strong test module for projectivity.

(5) There exists a strong test R-module for projectivity of depth zero and of finite projective dimension.

(6) There exists a strong test R-module for projectivity of finite projective dimension.

(7) There exist a semidualizing R-module C and a strong test R-module for projectivity of finite C-projective dimension.

Now let us give a proof of the corollary.

Proof. (1) $\Rightarrow$ (2): This implication follows from [7, Theorem 1].

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6): These implications are obvious.

(4) $\Rightarrow$ (5): Take a maximal R-regular sequence $x_1, x_2, \ldots, x_t$. Then $R/(x_1, x_2, \ldots, x_t)$ is an R-module of depth zero and of finite projective dimension.

(6) $\Rightarrow$ (7): Letting $C = R$ shows this implication.

(7) $\Rightarrow$ (1): Let M be a strong test R-module for projectivity with $C\text{-pd}_R M < \infty$. Let N be a module in $\mathcal{X}_C(R)$. Then we have $\operatorname{Ext}^1_R(N, M) = 0$. Since M is a strong test module for projectivity, N is a free R-module. Thus $\mathcal{X}_C(R)$ is contained in $\operatorname{add} R$. Therefore $\mathcal{X}_C(R) = \operatorname{add} R$, and R is regular by Theorem 1(4). $\square$

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