AUSLANDER-GORENSTEIN RESOLUTION

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Abstract. We introduce the notion of Auslander-Gorenstein resolution and show that a noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring.

1. Introduction

1.1. **Notation.** Let A be a ring. We denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. In particular, we denote by $\mathrm{Hom}_A(-,-)$ (resp., $\mathrm{Hom}_{A^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}). Sometimes, we use the notation M_A (resp., ${}_AM$) to stress that the module M considered is a right (resp., left) A-module. We denote by $\mathrm{Hom}^\bullet(-,-)$ the associated single complex of the double hom complex. As usual, we consider modules as complexes concentrated in degree zero. For an object X of an additive category $\mathcal{A}$ we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of copies of X. For a commutative ring R, we denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$ and for each $M \in$ Mod-R we denote by $\mathrm{Supp}_R(M)$ the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $M_\mathfrak{p} \neq 0$.

1.2. **Introduction.** In this note, a noetherian ring A is a ring which is left and right noetherian, and a noetherian R-algebra A is a ring endowed with a ring homomorphism $R \to A$, with R a commutative noetherian ring, whose image is contained in the center of A and A is finitely generated as an R-module. Note that a noetherian algebra is a noetherian ring.

Let R be a commutative Gorenstein local ring and A a noetherian R-algebra with $\mathrm{Ext}^i_R(A,R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A,R)$. Then proj dim ${}_A\Omega < \infty$ and proj dim $\Omega_A < \infty$ if and only if Ω_A is a tilting module in the sense of [12] (see Remark 4). In Section 2, we will show that inj dim ${}_AA \leq \dim R + 1$ if and only if inj dim $A_A \leq \dim R + 1$ (Theorem 5). In case inj dim $A_A = \dim R$, such an algebra A is called a Gorenstein algebra and extensively studied in [10]. In particular, a Gorenstein algebra is an Auslander-Gorenstein ring (see Definition 9). On the other hand, even if A is an Auslander-Gorenstein ring, it may happen that inj dim $A_A \neq \dim R$. For instance, if $A = \mathrm{T}_m(R)$, the ring of $m \times m$ upper triangular matrices over R, for $m \geq 2$, then A is an Auslander-Gorenstein ring with inj dim $A_A = \dim R + 1$ (see Example 16). Also, consider the case where R is a complete Gorenstein local ring of dimension one

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and Λ is a noetherian R-algebra with $\mathrm{Ext}^i_R(\Lambda, R) = 0$ for $i \neq 0$. Denote by $\mathcal{L}_\Lambda$ the full subcategory of mod-Λ consisting of modules X with $\mathrm{Ext}^i_R(X, R) = 0$ for $i \neq 0$ and assume that $\mathcal{L}_\Lambda = \mathrm{add}(M)$ with $M \in$ mod-Λ non-projective. Then we know from [3] that $A = \mathrm{End}_\Lambda(M)$ is an Auslander-Gorenstein ring of global dimension two (see Example 15). These examples can be formulated as follows. If Ω admits a projective resolution $0 \to P^{-1} \to P^0 \to \Omega \to 0$ in mod-A^{op} with $P^0 \in \mathrm{add}(\Omega)$, then A is an Auslander-Gorenstein ring with inj dim $A_A \leq \dim R + 1$ (see Example 14), the converse of which holds true if R is complete (see Proposition 7).

Consider the case where Ω_A is a tilting module of arbitrary finite projective dimension. Take a projective resolution $P^\bullet \to \Omega$ in mod-A^{op}. Then, setting $Q^\bullet = \mathrm{Hom}^\bullet_R(P^\bullet, R)$, we have a right resolution $A \to Q^\bullet$ in mod-A such that every $Q^i \in$ mod-R is a reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_R(Q^i, R), R) = 0$ for $j \neq 0$, $\oplus_{i \geq 0} \mathrm{Hom}_R(Q^i, R) \in$ mod-A^{op} is a projective generator and proj dim $Q^i < \infty$ in mod-A for all $i \geq 0$ (Remark 6). We will show that A is an Auslander-Gorenstein ring if proj dim $Q^i \leq i$ in mod-A for all $i \geq 0$ and that the converse holds true if R is complete and $P^\bullet \to \Omega$ is a minimal projective resolution (Proposition 7). In Section 3, formulating these facts, we will introduce the notion of Auslander-Gorenstein resolution. Let R, A be noetherian rings. In this note, a right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in Mod-A is said to be an Auslander-Gorenstein resolution of A over R if the following conditions are satisfied: (1) every Q^i is an R-A-bimodule; (2) every $Q^i \in$ Mod-R^{op} is a finitely generated reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R), R) = 0$ for $j \neq 0$; (3) $\oplus_{i \geq 0} \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R) \in$ Mod-A^{op} is faithfully flat; and (4) flat dim $Q^i \leq i$ in Mod-A for all $i \geq 0$. We will show that A is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over R and if R is an Auslander-Gorenstein ring (Theorem 13). In Section 4, we will provide several examples of Auslander-Gorenstein resolution.

We refer to [6], [7], [2], [9] and so on for information on Auslander-Gorenstein rings. Also, we refer to [8] for standard homological algebra, to [11] for standard commutative ring theory.

2. Auslander-Gorenstein algebras

Throughout this section, R is a commutative noetherian ring with a minimal injective resolution $R \to I^\bullet$ and A is a noetherian R-algebra such that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$.

In this section, assuming R being a complete Gorenstein local ring, we will provide a necessary and sufficient condition for A to be an Auslander-Gorenstein ring (see Definition 9 below). We refer to [5] for commutative Gorenstein rings.

Definition 1 ([4])**.** A family of idempotents $\{e_\lambda\}_{\lambda \in \Lambda}$ is said to be orthogonal if $e_\lambda e_\mu = 0$ unless $\lambda = \mu$. An idempotent $e \in A$ is said to be primitive if eA_A is indecomposable and to be local if $eAe \cong \mathrm{End}_A(eA)$ is local. A ring A is said to be semiperfect if $1 = e_1 + \cdots + e_n$ in A with the e_i orthogonal local idempotents.

Remark 2. Assume that R is a complete local ring. Then every noetherian R-algebra A is semiperfect.

Definition 3 ([12])**.** A module $T \in \text{Mod-}A$ is said to be a tilting module if for some integer $m \geq 0$ the following conditions are satisfied:

(1) T admits a projective resolution $0 \to P^{-m} \to \cdots \to P^{-1} \to P^0 \to T \to 0$ in Mod-A with $P^{-i} \in \mathcal{P}_A$ for all $i \geq 0$.
(2) $\mathrm{Ext}^i_A(T, T) = 0$ for $i \neq 0$.
(3) A admits a right resolution $0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$ in Mod-A with $T^i \in \mathrm{add}(T)$ for all $i \geq 0$.

Remark 4. The following hold:

(1) A has Gorenstein dimension zero as an R-module, i.e., $A \xrightarrow{\sim} \mathrm{Hom}_R(\Omega, R)$ and $\mathrm{Ext}^i_R(\Omega, R) = 0$ for $i \neq 0$.
(2) $A \xrightarrow{\sim} \mathrm{End}_A(\Omega)$ and $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}$ canonically.
(3) $\mathrm{Ext}^i_A(\Omega, \Omega) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega, \Omega) = 0$ for $i \neq 0$.
(4) The following are equivalent:
   (i) proj dim $\Omega_A < \infty$ and proj dim ${}_A\Omega < \infty$;
   (ii) Ω_A is a tilting module with proj dim ${}_A\Omega =$ proj dim Ω_A;
   (iii) inj dim ${}_AA =$ inj dim $A_A < \infty$.

Theorem 5. *Assume that R is a Gorenstein local ring. Then the following are equivalent:*

(1) inj dim $A_A \leq \dim R + 1$.
(2) inj dim ${}_AA \leq \dim R + 1$.

Throughout the rest of this section, we assume that R is a Gorenstein local ring and that proj dim ${}_A\Omega =$ proj dim $\Omega_A = m < \infty$. Take a projective resolution $P^\bullet \to \Omega$ in mod-A^{op} and set $Q^\bullet = \mathrm{Hom}_R(P^\bullet, R)$. Then we have a right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in mod-A with $Q^i = \mathrm{Hom}_R(P^{-i}, R) \in \mathrm{add}(\Omega)$ for all $i \geq 0$. Recall that a module $M \in \text{Mod-}A$ is said to be reflexive if the canonical homomorphism

$$M \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism.

Remark 6. The following hold:

(1) Every $Q^i \in \text{mod-}R$ is a reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_R(Q^i, R), R) = 0$ for $j \neq 0$.
(2) $\oplus_{i \geq 0} \mathrm{Hom}_R(Q^i, R) \in \text{mod-}A^{\mathrm{op}}$ is a projective generator.
(3) proj dim $Q^i < \infty$ in mod-A for all $i \geq 0$.

In the following, we assume further that R is complete and that $P^\bullet \to \Omega$ is a minimal projective resolution. Let $A \to E^\bullet$ be a minimal injective resolution in Mod-A.

In the next proposition, the implication (1) $\Rightarrow$ (2) holds true without the completeness of R.

Proposition 7. *The following are equivalent:*

(1) proj dim $Q^i \leq i$ *in* mod-A *for all* $i \geq 0$.
(2) flat dim $E^n \leq n$ *in* Mod-A *for all* $n \geq 0$.

3. Auslander-Gorenstein resolution

In this section, formulating Remark 6 and Proposition 7, we will introduce the notion of Auslander-Gorenstein resolution and show that a noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring.

We start by recalling the Auslander condition. In the following, Λ stands for an arbitrary noetherian ring.

Proposition 8 (Auslander)**.** *For any $n \geq 0$ the following are equivalent:*

(1) *In a minimal injective resolution $\Lambda \to I^\bullet$ in* Mod-Λ*,* flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
(2) *In a minimal injective resolution $\Lambda \to J^\bullet$ in* Mod-Λ^{op}*,* flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
(3) *For any $1 \leq i \leq n+1$, any $M \in$* mod-Λ *and any submodule X of* $\mathrm{Ext}^i_\Lambda(M, \Lambda) \in$ mod-Λ^{op} *we have* $\mathrm{Ext}^j_{\Lambda^{\mathrm{op}}}(X, \Lambda) = 0$ *for all* $0 \leq j < i$.
(4) *For any $1 \leq i \leq n+1$, any $X \in$* mod-Λ^{op} *and any submodule M of* $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(X, \Lambda) \in$ mod-Λ *we have* $\mathrm{Ext}^j_\Lambda(M, \Lambda) = 0$ *for all* $0 \leq j < i$.

Definition 9 ([6])**.** We say that Λ satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 8 for all $n \geq 0$, and that Λ is an Auslander-Gorenstein ring if inj dim ${}_\Lambda\Lambda =$ inj dim $\Lambda_\Lambda < \infty$ and if it satisfies the Auslander condition.

Definition 10. We denote by $\mathcal{G}_\Lambda$ the full subcategory of mod-Λ consisting of reflexive modules $M \in$ mod-Λ with $\mathrm{Ext}^i_{\Lambda^{\mathrm{op}}}(\mathrm{Hom}_\Lambda(M, \Lambda), \Lambda) = 0$ for $i \neq 0$.

Throughout the rest of this section, R and A are noetherian rings. We do not require the existence of a ring homomorphism $R \to A$. Also, even if we have a ring homomorphism $R \to A$ with R commutative, the image of which may fail to be contained in the center of A (cf. [1]).

Definition 11. A right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in Mod-A is said to be a Gorenstein resolution of A over R if the following conditions are satisfied:

(1) Every Q^i is an R-A-bimodule.
(2) $Q^i \in \mathcal{G}_{R^{\mathrm{op}}}$ in Mod-R^{op} for all $i \geq 0$.
(3) $\oplus_{i \geq 0} \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R) \in$ Mod-A^{op} is faithfully flat.
(4) flat dim $Q^i < \infty$ in Mod-A for all $i \geq 0$.

Definition 12. A Gorenstein resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ of A over R is said to be an Auslander-Gorenstein resolution if the following stronger condition is satisfied:

(4)′ flat dim $Q^i \leq i$ in Mod-A for all $i \geq 0$.

Theorem 13. *Assume that A admits a Gorenstein resolution*

$$0 \to A \to Q^0 \to \cdots \to Q^m \to 0$$

over R and that inj dim ${}_R R =$ inj dim $R_R = d < \infty$. *Then the following hold:*

(1) *For an injective resolution $R \to I^\bullet$ in* Mod-R *we have an injective resolution $A \to E^\bullet$ in* Mod-A *such that*

$$E^n = \bigoplus_{i+j=n} I^j \otimes_R Q^i$$

for all $n \geq 0$. In particular, $\text{inj dim}\ {}_AA = \text{inj dim}\ A_A \leq m+d$ *and*

$$\text{flat dim}\ E^n \leq \sup\{\text{flat dim}\ I^j + \text{flat dim}\ Q^i \mid i+j=n\}$$

for all $n \geq 0$.

(2) *If R is an Auslander-Gorenstein ring, and if $A \to Q^\bullet$ is an Auslander-Gorenstein resolution, then A is an Auslander-Gorenstein ring.*

In case $m = 0$, a Gorenstein resolution of A over R is just an R-A-bimodule Q such that $Q \cong A$ in Mod-A, $Q \in \mathcal{G}_{R^{\text{op}}}$ in Mod-R^{op} and $\text{Hom}_{R^{\text{op}}}(Q, R) \in \text{Mod-}A^{\text{op}}$ is faithfully flat. In particular, if A is a Frobenius extension of R in the sense of [1], then both A itself and $\text{Hom}_R(A, R)$ are Gorenstein resolutions of A over R, where $A \cong \text{Hom}_R(A, R)$ in Mod-A but $A \not\cong \text{Hom}_R(A, R)$ as R-A bimodules in general.

4. Examples

In this section, we will provide several examples of Auslander-Gorenstein resolution.

Example 14. Let R be a commutative noetherian ring and A a noetherian R-algebra such that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \text{Supp}_R(A)$ and $\text{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \text{Hom}_R(A, R)$ and assume that Ω admits a projective resolution $0 \to P^{-1} \to P^0 \to \Omega \to 0$ in mod-A^{op} with $P^0 \in \text{add}(\Omega)$. Then applying $\text{Hom}_R(-, R)$ we have a right resolution $0 \to A \to Q^0 \to Q^1 \to 0$ in mod-A with $Q^0 \in \text{add}(\Omega)$, where $Q^i = \text{Hom}_R(P^{-i}, R)$ for $0 \leq i \leq 1$, which must be an Auslander-Gorenstein resolution of A over R.

Example 15 (cf. [3])**.** Let R be a complete Gorenstein local ring of dimension one and Λ a noetherian R-algebra with $\text{Ext}^i_R(\Lambda, R) = 0$ for $i \neq 0$. Denote by $\mathcal{L}_\Lambda$ the full subcategory of mod-Λ consisting of modules X with $\text{Ext}^i_R(X, R) = 0$ for $i \neq 0$. It should be noted that $\mathcal{L}_\Lambda$ is closed under submodules. Assume that $\mathcal{L}_\Lambda = \text{add}(M)$ with $M \in \text{mod-}\Lambda$ non-projective and set $A = \text{End}_\Lambda(M)$.

Set $F = \text{Hom}_\Lambda(M, -) : \mathcal{L}_\Lambda \xrightarrow{\sim} \mathcal{P}_A$ and $D = \text{Hom}_R(-, R)$. Take a minimal projective presentation $P^{-1} \to P^0 \to DM \to 0$ in mod-Λ^{op}. Applying $F \circ D$, we have an exact sequence in mod-A

$$0 \to A \to F(DP^{-1}) \xrightarrow{f} F(DP^0).$$

Setting $Q^0 = F(DP^{-1})$ and $Q^1 = \text{Im}\ f$, we have an Auslander-Gorenstein resolution of A over R.

Example 16. Let $m \geq 2$ be an integer and $A = \text{T}_m(R)$, the ring of $m \times m$ upper triangular matrices over a noetherian ring R. Namely, A is a free right R-module with a basis $\mathfrak{B} = \{e_{ij} \mid 1 \leq i \leq j \leq m\}$ and the multiplication in A is defined subject to the following axioms: (A1) $e_{ij}e_{kl} = 0$ unless $j = k$ and $e_{ij}e_{jk} = e_{ik}$ for all $i \leq j \leq k$; and (A2) $xv = vx$ for all $x \in R$ and $v \in \mathfrak{B}$. Set $e_i = e_{ii}$ for all i. Then A is a noetherian ring with $1 = e_1 + \cdots + e_m$, where the e_i are orthogonal idempotents. We consider R as a subring of A via the injective ring homomorphism $\varphi : R \to A, x \mapsto x1$. Denote by

$\mathfrak{B}^* = \{e_{ij}^* \mid 1 \leq i \leq j \leq m\}$ the dual basis of $\mathfrak{B}$ for the left R-module $\mathrm{Hom}_R(A, R)$, i.e., we have $a = \Sigma_{v \in \mathfrak{B}} v v^*(a)$ for all $a \in A$. It is not difficult to check the following:

(1) $e_1 A \xrightarrow{\sim} \mathrm{Hom}_R(Ae_m, R), a \mapsto e_{1m}^* a$ as R-A-bimodules.

(2) For each $2 \leq i \leq m$, setting $f : e_1 A \to \mathrm{Hom}_{R^{\mathrm{op}}}(Ae_{i-1}, R), a \mapsto e_{1,i-1}^* a$ and $g : e_i A \to e_1 A, a \mapsto e_{1i} a$, we have an exact sequence of R-A-bimodules

$$0 \to e_i A \xrightarrow{g} e_1 A \xrightarrow{f} \mathrm{Hom}_R(Ae_{i-1}, R) \to 0.$$

(3) $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(Ae_i, R), R) \cong Ae_i$ as A-R-bimodules for all $1 \leq i \leq m$.

Consequently, we have an exact sequence of R-A-bimodules

$$0 \to A \to \overset{m}{\oplus} e_1 A \to \bigoplus_{i=2}^{m} \mathrm{Hom}_R(Ae_{i-1}, R) \to 0,$$

which is an Auslander-Gorenstein resolution of A over R.

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