HIGH ORDER CENTERS AND LEFT DIFFERENTIAL OPERATORS

HIROAKI KOMATSU

Abstract. We give a new frame for derivations and high order left differential operators of algebras. This frame is based on high order centers of bimodules. We show the relation between separable algebras and high order centers.

Key Words: Center, Derivation, Bimodule, Separable algebra.

2010 *Mathematics Subject Classification*: Primary 16U70; Secondary 16W25.

1. Introduction

Hattori [1] and Sweedler [10] generalized the notion of high order differential operators of commutative algebras to the notion of high order *left* differential operators of noncommutative algebras. The author generalized them from algebras to noncommutative ring extensions and studied them in [2], [3], [4], and [5]. Recently in [6] and [7], he gave a new view point under which derivations and left differential operators are treated in the same frame. This new frame is based on high order centers of bimodules. He also studied the relation between separable algebras and high order centers. The purpose of this note is to introduce the results of [6] and [7].

Throughout this note, all rings have identity 1, all ring homomorphisms preserve 1, all modules are unitary, and k represents a commutative ring and $\mathbb{N}$ the set of nonnegative integers. For a k-algebra A, we denote by $\mathfrak{M}_k(A)$ the category of bimodules over a k-algebra A. An object of $\mathfrak{M}_k(A)$ is an A-bimodule M such that $\alpha u = u\alpha$ for all $\alpha \in k$ and $u \in M$.

2. High order centers (simple version)

In this section we introduce the notion of high order centers for algebras.

Definition 1. Let A be a k-algebra and let $M \in \mathfrak{M}_k(A)$. For $u \in M$ and $a \in A$, we set $[u, a] = ua - au$. For $U \subseteq M$, we set $[U, A] = \{[u, a] \mid u \in U,\ a \in A\}$. Furthermore, we set $[U, A]_0 = U$ and $[U, A]_{q+1} = [[U, A]_q,\ A]$ $(q \in \mathbb{N})$. If U is a singleton $\{u\}$, we use the notations $[u, A]$ and $[u, A]_q$ instead of $[U, A]$ and $[U, A]_q$, respectively. For $q \in \mathbb{N}$, we set

$$\mathcal{C}^q_A(M) = \{u \in M \mid [u, A]_q = 0\},$$

which is called the *qth order center of* M.

If $\varphi : M \to N$ is a morphism of $\mathfrak{M}_k(A)$, then it is easy to see that $\varphi\big(\mathcal{C}^q_A(M)\big) \subseteq \mathcal{C}^q_A(N)$. Hence $\mathcal{C}^q_A(-)$ gives a functor from $\mathfrak{M}_k(A)$ to the category of k-modules. We shall show that the functor $\mathcal{C}^q_A$ is representable.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. For a k-algebra A and $q \in \mathbb{N}$, we set

$$\mathcal{J}_A^q = (A \otimes_k A)/A[1 \otimes 1,\, A]_q A \quad \text{and} \quad j_A^q = 1 \otimes 1 + A[1 \otimes 1,\, A]_q A \in \mathcal{J}_A^q.$$

Theorem 3. *Let A be a k-algebra and $q \in \mathbb{N}$. Then we have a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{M}_k(A)}(\mathcal{J}_A^q,\, M) \ni \varphi \mapsto \varphi(j_A^q) \in \mathcal{C}_A^q(M)$$

for $M \in \mathfrak{M}_k(A)$.

For any k-algebra A, we have a sequence of surjective A-bimodule homomorphisms

$$A = \mathcal{J}_A^1 \twoheadleftarrow \mathcal{J}_A^2 \twoheadleftarrow \mathcal{J}_A^3 \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{J}_A^q \twoheadleftarrow \cdots \twoheadleftarrow A \otimes_k A.$$

Therefore $\mathcal{J}_A^q$ is closely related to the separability of A. An k-algebra A is said to be *separable* if $A[1 \otimes 1,\, A]A$ is a direct summand of $A \otimes_k A$ as A-bimodule. According to [9], A is said to be *purely inseparable* if $A[1 \otimes 1,\, A]A$ is a small submodule of $A \otimes_k A$ as A-bimodule. The next theorem is known.

Theorem 4. *Let A be a k-algebra. Then the following hold.*

(1) ([10, Theorem 1.21] and [4, Theorem 2.4]) *If A is separable, then $\mathcal{J}_A^q = A$ for all $q > 0$.*

(2) ([10, Theorem 2.1]) *If $\mathcal{J}_A^q = A \otimes_k A$ for some $q \in \mathbb{N}$, then A is purely inseparable.*

Combining Theorems 3 and 4, we get the next

Corollary 5. *Let A be a k-algebra. Then the following hold.*

(1) *If A is separable, then $\mathcal{C}_A^q(M) = \mathcal{C}_A^1(M)$ for all $M \in \mathfrak{M}_k(A)$ and for all $q > 1$.*

(2) *If there exists $q \in \mathbb{N}$ such that $\mathcal{C}_A^q(M) = M$ for all $M \in \mathfrak{M}_k(A)$, then A is purely inseparable.*

Remark 6. In case that A is a finite dimensional algebra over a field k, Sweedler [10, Theorem 2.1] showed that A is purely inseparable if and only if $\mathcal{J}_A^q = A \otimes_k A$ for some $q \in \mathbb{N}$.

3. High order left differential operators (old version)

In this section, we introduce the results of Sweedler [10].

Definition 7 ([10])**.** Let A be a k-algebra, and let M and N be left A-modules. As usual, $\mathrm{Hom}_k(M, N)$ belongs to $\mathfrak{M}_k(A)$ by the multiplications

$$(afb)(u) = af(bu) \quad (f \in \mathrm{Hom}_k(M, N),\ a,\ b \in A,\ u \in M).$$

We set

$$\mathcal{D}_A^q(M, N) = \mathcal{C}_A^{q+1}(\mathrm{Hom}_k(M, N)) \quad \text{and}$$
$$\mathrm{LDer}_k^q(A, M) = \{d \in \mathcal{D}_A^q(A, M) \mid d(1) = 0\}.$$

An element of $\mathcal{D}_A^q(M, N)$ is called a *qth order left differential operator*, and an element of $\mathrm{LDer}_k^q(A, M)$ is called a *qth order left derivation.*

Remark 8. For a left A-module M and $d \in \mathrm{Hom}_k(A, M)$, $d \in \mathrm{LDer}_k^1(A, M)$ if and only if $d(xy) = xd(y) + yd(x)$ for all $x,\ y \in A$. In commutative ring theory, d is regarded as a derivation, i.e., $d(xy) = xd(y) + d(x)y$ for all $x,\ y \in A$.

Example 9. Set $A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a,\ b,\ c,\ d \in k \right\}$. Then the mapping

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mapsto \begin{pmatrix} 0 & b & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

belongs to $\mathrm{LDer}_k^1(A, A)$. We can see that $\mathcal{J}_A^5 = A \otimes_k A$. Hence we have $\mathcal{D}_A^4(M, N) = \mathrm{Hom}_k(M, N)$ for all left A-modules M and N, and A is a purely inseparable algebra.

Definition 10. In $\mathcal{J}_A^{q+1}$, we set $\mathcal{K}_A^q = A[j_A^{q+1},\ A]A$, which is called the *qth Kähler module*, and define $d_A^q \in \mathrm{LDer}_k^q(A, \mathcal{K}_A^q)$ by $d_A^q(x) = [j_A^{q+1},\ x]$.

The next theorem corresponds to Theorems 1.17, 1.18 and 5.12 of [10].

Theorem 11. *Let A be a k-algebra and $q \in \mathbb{N} \setminus \{0\}$. Then we have the following natural isomorphisms for left A-modules M and N.*

$$\mathrm{Hom}_A(\mathcal{J}_A^{q+1} \otimes_A M,\ N) \ni \varphi \mapsto \varphi(j_A^{q+1} \otimes -) \in \mathcal{D}_A^q(M, N)$$
$$\mathrm{Hom}_A(\mathcal{K}_A^q,\ M) \ni \varphi \mapsto \varphi d_A^q \in \mathrm{LDer}_k^q(A, M)$$

Sweedler used $\mathcal{C}_A^q$ only to define differential operators, and did not investigate the functor $\mathcal{C}_A^q$. And so he did not know that $\mathcal{J}_A^q$ also represents $\mathcal{C}_A^q$.

4. High order centers (general version)

We shall generalize the notion of high order centers defined in §2.

Definition 12. We denote by $\mathbf{Alg}$ the category of k-algebras and by $\mathbf{Alg}^n$ the product of n copies of $\mathbf{Alg}$. For any $\mathbf{A} = (A_1, \ldots, A_n) \in \mathbf{Alg}^n$, we set $\widehat{\mathbf{A}} = A_1 \otimes_k \cdots \otimes_k A_n$. For any morphism $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbf{A} \to \mathbf{B}$ in $\mathbf{Alg}^n$, we set $\widehat{\alpha} = \alpha_1 \otimes \cdots \otimes \alpha_n : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$.

Let A be a k-algebra, and let $M,\ N \in \mathfrak{M}_k(A)$. Then $\mathrm{Hom}_k(M, N)$ has two A-bimodule structures.

$$\begin{cases} (afb)(u) = af(bu) \\ (a * f * b)(u) = f(ua)b \end{cases} \qquad (f \in \mathrm{Hom}_k(M, N),\ a,\ b \in A,\ u \in M)$$

We set $[f, a] = fa - af$ and $[f, a]^* = f * a - a * f$. As was mentioned in the previous section, $d \in \mathrm{Hom}_k(A, M)$ is a left derivation if and only if $[[d, A], A] = 0$ and $d(1) = 0$. Furthermore we can see that $d \in \mathrm{Hom}_k(A, M)$ is a derivation if and only if $[[d, A], A]^* = 0$ and $d(1) = 0$. This situation leads the next

Definition 13. Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$ and $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{N}^n$. Suppose that $\mathbf{B} = (B_1, \ldots, B_n)$. For $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$ and $u \in M$, we set

$$[u, \mathbf{B}]_{\mathbf{q}} = [\cdots [[u,\ B_1]_{q_1},\ B_2]_{q_2},\ \cdots,\ B_n]_{q_n}.$$

We note that $[[U,\ B_i],\ B_j] = [[U,\ B_j],\ B_i]$ for any $U \subseteq M$. We set

$$\mathcal{C}_\alpha^{\mathbf{q}}(M) = \left\{ u \in M \;\middle|\; [u, \mathbf{B}]_{\mathbf{q}} = [u, \widehat{\mathbf{A}}] = 0 \right\},$$

which is called the *center of M of type* $\mathbf{q}$.

If $\varphi : M \to N$ is a morphism of $\mathfrak{M}_k(\widehat{\mathbf{B}})$, then it is easy to see that $\varphi\big(\mathcal{C}^{\mathbf{q}}_{\alpha}(M)\big) \subseteq \mathcal{C}^{\mathbf{q}}_{\alpha}(N)$. Hence $\mathcal{C}^{\mathbf{q}}_{\alpha}(-)$ gives a functor from $\mathfrak{M}_k(\widehat{\mathbf{B}})$ to the category of k-modules. We shall show that the functor $\mathcal{C}^{\mathbf{q}}_{\alpha}$ is representable.

Definition 14. For a morphism $\alpha : \mathbf{A} \to \mathbf{B}$ of $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$, we set

$$\mathcal{J}^{\mathbf{q}}_{\alpha} = (\widehat{\mathbf{B}} \otimes_{\widehat{\mathbf{A}}} \widehat{\mathbf{B}}) \,/\, \widehat{\mathbf{B}}\,[1 \otimes 1,\, \mathbf{B}]_{\mathbf{q}}\widehat{\mathbf{B}} \quad \text{and} \quad j^{\mathbf{q}}_{\alpha} = 1 \otimes 1 + \widehat{\mathbf{B}}\,[1 \otimes 1,\, \mathbf{B}]_{\mathbf{q}}\widehat{\mathbf{B}} \in \mathcal{J}^{\mathbf{q}}_{\alpha}.$$

Theorem 15. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. Then we have a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{M}_k(\widehat{\mathbf{B}})}(\mathcal{J}^{\mathbf{q}}_{\alpha},\, M) \ni \varphi \mapsto \varphi(j^{\mathbf{q}}_{\alpha}) \in \mathcal{C}^{\mathbf{q}}_{\alpha}(M)$$

for $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$.

5. High order left differential operators (general version)

Using new high order centers, we can define new high order differential operators.

Definition 16. Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. For $M, N \in \mathfrak{M}_k(\widehat{\mathbf{B}})$, we set

$$\mathcal{D}^{\mathbf{q}}_{\alpha}(M, N) = \mathcal{C}^{\mathbf{q}}_{\alpha}(\mathrm{Hom}_k(M, N)) \quad \text{and}$$
$$\mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M) = \big\{ d \in \mathcal{D}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M) \;\big|\; d(1) = 0 \big\}.$$

An element of $\mathcal{D}^{\mathbf{q}}_{\alpha}(M, N)$ is called a *left differential operators of type* $\mathbf{q}$, and an element of $\mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{B}, M)$ is called a *left derivation of type* $\mathbf{q}$.

Example 17. Let $\mathbf{A} = (k,\, k)$, $\mathbf{B} = (R,\, R^{\circ})$, $\alpha = (\rho,\, \rho)$, $\mathbf{q} = (1,\, 1)$, where R° is the opposite algebra of R and $\rho : k \to R$ is the structure morphism of k-algebra R. Then, for any $M \in \mathfrak{M}_k(R)$, the set $\{d \in \mathcal{D}^{\mathbf{q}}_{\alpha}(R, M) \mid d(1) = 0\}$ coincides with the set of derivations of R to M.

Definition 18. In $\mathcal{J}^{\mathbf{q}}_{\alpha}$, we set $\mathcal{K}^{\mathbf{q}}_{\alpha} = \widehat{\mathbf{B}}[j^{\mathbf{q}}_{\alpha},\, \widehat{\mathbf{B}}]\widehat{\mathbf{B}}$, and define $d^{\mathbf{q}}_{\alpha} \in \mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}},\, \mathcal{K}^{\mathbf{q}}_{\alpha})$ by $d^{\mathbf{q}}_{\alpha}(x) = [j^{\mathbf{q}}_{\alpha},\, x]$.

Lemma 19. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. Then the following hold.*

(1) $\mathcal{D}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M) = \mathrm{Hom}_{\widehat{\mathbf{B}}}(\widehat{\mathbf{B}}, M) \oplus \mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M)$.

(2) $\mathcal{J}^{\mathbf{q}}_{\alpha} = \widehat{\mathbf{B}} j^{\mathbf{q}}_{\alpha} \oplus \mathcal{K}^{\mathbf{q}}_{\alpha} = j^{\mathbf{q}}_{\alpha}\widehat{\mathbf{B}} \oplus \mathcal{K}^{\mathbf{q}}_{\alpha}$ *and* $\big\{x \in \widehat{\mathbf{B}} \;\big|\; x j^{\mathbf{q}}_{\alpha} = 0\big\} = \big\{x \in \widehat{\mathbf{B}} \;\big|\; j^{\mathbf{q}}_{\alpha} x = 0\big\} = 0$.

Theorem 20. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$. Then we have following natural isomorphisms for left $\widehat{\mathbf{B}}$-modules M and N.*

$$\mathrm{Hom}_{\widehat{\mathbf{B}}}(\mathcal{J}^{\mathbf{q}}_{\alpha} \otimes_{\widehat{\mathbf{B}}} M,\, N) \ni \varphi \mapsto \varphi(j^{\mathbf{q}}_{\alpha} \otimes -) \in \mathcal{D}^{\mathbf{q}}_{\alpha}(M, N)$$
$$\mathrm{Hom}_{\widehat{\mathbf{B}}}(\mathcal{K}^{\mathbf{q}}_{\alpha},\, M) \ni \varphi \mapsto \varphi d^{\mathbf{q}}_{\alpha} \in \mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}},\, M)$$

6. Fundamental properties of $\mathcal{J}^{\mathbf{q}}_{\alpha}$ and $\mathcal{K}^{\mathbf{q}}_{\alpha}$

Theorem 21. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n \setminus \{(0,\ldots,0)\}$. Then $\mathcal{J}^{\mathbf{q}}_{\beta} \simeq \mathcal{J}^{\mathbf{q}}_{\beta\alpha}/\widehat{\mathbf{C}}[j^{\mathbf{q}}_{\beta\alpha}, \widehat{\mathbf{B}}]\widehat{\mathbf{C}}$ and $\mathcal{K}^{\mathbf{q}}_{\beta} \simeq \mathcal{K}^{\mathbf{q}}_{\beta\alpha}/\widehat{\mathbf{C}}d^{\mathbf{q}}_{\beta\alpha}\widehat{\beta}(\widehat{\mathbf{B}})\widehat{\mathbf{C}}$ as $\widehat{\mathbf{C}}$-bimodules.*

Corollary 22. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n \setminus \{(0,\ldots,0)\}$. Then there exist exact sequences of $\widehat{\mathbf{C}}$-bimodules*

$$\widehat{\mathbf{C}} \otimes_{\widehat{\mathbf{B}}} \mathcal{K}^{\mathbf{q}}_{\alpha} \otimes_{\widehat{\mathbf{B}}} \widehat{\mathbf{C}} \to \mathcal{J}^{\mathbf{q}}_{\beta\alpha} \to \mathcal{J}^{\mathbf{q}}_{\beta} \to 0 \quad \text{and} \quad \widehat{\mathbf{C}} \otimes_{\widehat{\mathbf{B}}} \mathcal{K}^{\mathbf{q}}_{\alpha} \otimes_{\widehat{\mathbf{B}}} \widehat{\mathbf{C}} \to \mathcal{K}^{\mathbf{q}}_{\beta\alpha} \to \mathcal{K}^{\mathbf{q}}_{\beta} \to 0.$$

Theorem 23. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ such that $\widehat{\beta} : \widehat{\mathbf{B}} \to \widehat{\mathbf{C}}$ is a surjective mapping, and let $\mathbf{q} \in \mathbb{N}^n \setminus \{(0,\ldots,0)\}$. Set $I = \operatorname{Ker}\widehat{\beta}$. Then the following hold.*

1. $\mathcal{J}^{\mathbf{q}}_{\beta\alpha} \simeq \mathcal{J}^{\mathbf{q}}_{\alpha}/(I\mathcal{J}^{\mathbf{q}}_{\alpha} + \mathcal{J}^{\mathbf{q}}_{\alpha}I) \simeq \widehat{\mathbf{C}} \otimes_{\widehat{\mathbf{B}}} \mathcal{J}^{\mathbf{q}}_{\alpha} \otimes_{\widehat{\mathbf{B}}} \widehat{\mathbf{C}}$
2. $\mathcal{K}^{\mathbf{q}}_{\beta\alpha} \simeq \mathcal{K}^{\mathbf{q}}_{\alpha}/\widehat{\mathbf{B}}\delta^{\mathbf{q}}_{\alpha}(I)\widehat{\mathbf{B}}$
3. *There exists an exact sequence of $\widehat{\mathbf{C}}$-bimodules: $I/I^2 \to \widehat{\mathbf{C}}\otimes_{\widehat{\mathbf{B}}}\mathcal{K}^{\mathbf{q}}_{\alpha}\otimes_{\widehat{\mathbf{B}}}\widehat{\mathbf{C}} \to \mathcal{K}^{\mathbf{q}}_{\beta\alpha} \to 0$.*

Theorem 24. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ such that $\widehat{\beta} : \widehat{\mathbf{B}} \to \widehat{\mathbf{C}}$ is a surjective mapping. Suppose that $\mathbf{B} = (B_1,\ldots,B_n)$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, and $\beta = (\beta_1,\ldots,\beta_n)$. Set $B'_i = \operatorname{Im}\alpha_i + \operatorname{Ker}\beta_i$ and denote by $\iota_i : B'_i \to B_i$ the inclusion mapping ($i = 1, \ldots, n$). Set $\iota = (\iota_1,\ldots,\iota_n) : (B'_1,\ldots,B'_n) \to \mathbf{B}$. Then $\mathcal{K}^{\mathbf{q}}_{\beta\alpha} \simeq \mathcal{K}^{\mathbf{q}}_{\iota}$.*

7. Separability

We shall generalize the separability of algebras to the separability of morphisms in $\mathbf{Alg}^n$.

Definition 25. Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$. For $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$, we set

$$\mathcal{C}_{\alpha}(M) = \sum_{i=1}^{n} \bigl\{u \in M \bigm| [u, B_i] = [u, \widehat{\mathbf{A}}] = 0\bigr\}.$$

α is called $\mathbf{q}$-*quasi-separable* if $j^{\mathbf{q}}_{\alpha} \in \mathcal{C}_{\alpha}(\mathcal{J}^{\mathbf{q}}_{\alpha})$. α is called *left* $\mathbf{q}$-*differentially separable* if

$$\mathcal{D}^{\mathbf{q}}_{\alpha}(M,N) \subseteq \sum_{i=1}^{n} \operatorname{Hom}_{B_i}(M,N) \cap \operatorname{Hom}_{\widehat{\mathbf{A}}}(M,N) \quad \bigl(= \mathcal{C}_{\alpha}(\operatorname{Hom}_k(M,N))\bigr)$$

for all left $\widehat{\mathbf{B}}$-modules M and N.

Lemma 26. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. Then the following hold.*

1. *α is $\mathbf{q}$-quasi-separable if and only if $\mathcal{C}^{\mathbf{q}}_{\alpha}(M) \subseteq \mathcal{C}_{\alpha}(M)$ for all $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$.*
2. *If α is $\mathbf{q}$-quasi-separable, then α is left $\mathbf{q}$-differentially-separable.*

Theorem 27. *Let $\mathbf{A} = (k, k)$, $\mathbf{B} = (R, R^{\circ})$, and $\alpha = (\rho, \rho)$, where R° is the opposite algebra of R and $\rho : k \to R$ is the structure morphism of k-algebra R. Then the following are equivalent:*

1. *R is a separable algebra.*
2. *α is $(1,1)$-quasi-separable.*

(3) *α is $\mathbf{q}$-quasi-separable for all $\mathbf{q} \in \mathbb{N}^2 \setminus \{(0,0)\}$.*
(4) *α is left $(1,1)$-differentially-separable.*
(5) *α is left $\mathbf{q}$-differentially-separable for all $\mathbf{q} \in \mathbb{N}^2 \setminus \{(0,0)\}$.*

Let $\rho : R \to S$ be a ring homomorphism. According to [8], ρ is said to be *separable* if $S[1 \otimes 1, S]S$ is a direct summand of $S \otimes_R S$ as S-bimodule. Usually, S is called a *separable extension* of R.

Theorem 28. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$. Suppose that all α_i are separable. Then the following hold:*
(1) *α is $(1, \ldots, 1)$-quasi-separable.*
(2) *If $[1 \otimes 1, [B_i, A_i]] = 0$ in $B_i \otimes_{A_i} B_i$ $(i = 1, \ldots, n)$, then α is $\mathbf{q}$-quasi-separable for all $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$.*

References

[1] A. Hattori, *On high order derivations from the view-point of two-sided modules*, Sci. Papers College Gen. Ed. Univ. Tokyo **20** (1970), 1–11.
[2] M. Hongan and H. Komatsu, *On the module of differentials of a noncommutative algebras and symmetric biderivations of a semiprime algebra*, Comm. Algebra **28** (2000), 669–692.
[3] H. Komatsu, *The module of differentials of a noncommutative ring extension*, International Symposium on Ring Theory, Birkhäuser, 2001, pp. 171–177.
[4] H. Komatsu, *Quasi-separable extensions of noncommutative rings*, Comm. Algebra **29** (2001), 1011–1019.
[5] H. Komatsu, *High order Kähler modules of noncommutative ring extensions*, Comm. Algebra **29** (2001), 5499–5524.
[6] H. Komatsu, *Differential operators of bimodules*, preprint.
[7] H. Komatsu, *High order centers and differential operators of modules*, preprint.
[8] Y. Miyashita, *Finite outer Galois theory of non-commutative rings*, J. Fac. Sci. Hokkaido Univ. Ser. I **19** (1966), 114–134.
[9] M. E. Sweedler, *Purely inseparable algebras*, J. Algebra **35** (1975), 342–355.
[10] M. E. Sweedler, *Right derivations and right differential operators*, Pacific J. Math. **86** (1980), 327–360.

Faculty of Computer Science and System Engineering
Okayama Prefectural University
Soja, Okayaka 719-1197 JAPAN

E-mail address: `komatsu@cse.oka-pu.ac.jp`