HOCHSCHILD COHOMOLOGY AND GORENSTEIN NAKAYAMA ALGEBRAS

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Abstract. Let A be a Nakayama algebra over an algebraically closed field k, $\mathrm{HH}(A)$ the Hochschild cohomology ring. We will study the condition when $\mathrm{HH}(A)$ is a finitely generated algebra and $\mathrm{Ext}^*_A(A/J, A/J)$ is a finitely generated $\mathrm{HH}(A)$-module, where J is the Jacobson radical of A. In [4], it is shown that if an algebra satisfies the both finiteness conditions, then the algebra is Gorenstein. We will investigate the Hochschild cohomology of Gorenstein Nakayama algebras and show that Gorenstein Nakayama algebras satisfy the both finiteness conditions above.

1. Introduction

Let A be a finite dimensional algebra over an algebraically closed field k and H a noetherian commutative graded subalgebra of the Hochschild cohomology algebra $HH(A)$ with $H^0 = HH^0(A)$. In [9], Snashall and Solberg defined the support variety of a finitely generated A-module M over H as the set of maximal ideals of H containing the annihilator $\mathrm{Ann}_H \mathrm{Ext}^*_A(M, M)$, where the H-action on $\mathrm{Ext}^*_A(M, M)$ is given by the graded algebra homomorphism

$$H \xrightarrow{incl.} \mathrm{HH}^*(A) \xrightarrow{-\otimes M} \mathrm{Ext}^*_A(M, M).$$

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer showed that some geometric properties of the support variety and some representation theoretic properties are related if A satisfies the following finiteness condition:

$$\mathrm{Ext}^*_A(A/J, A/J) \text{ is a finitely generated } H\text{-module},$$

where J is the Jacobson radical of A. This finiteness condition holds for group algebras of finite groups and, in [4], various results for finite groups are generalized to those for the class of selfinjective algebras satisfying the finiteness condition. It is known that the condition holds for any block of a finite dimensional cocomutative Hopf algebra [6], for any complete intersection in commutative setting [7], and so on [5].

In this paper, we consider this finiteness condition in the case of Nakayama algebras. In [9], Hochschild cohomology rings of Nakayama algebras with a single relation are investigated and some of them do not satisfy the finiteness condition. On the other hand, in [4], it is shown that any algebra A is Gorenstein if A satisfies the finiteness condition. We are, therefore, interested in to determine when Gorenstein Nakayama algebras satisfy the finiteness condition. One of our main results, Theorem 9 answers to this question.

The detailed version of this paper will be submitted for publication elsewhere.

2. The finiteness condition (Fg)

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer introduce some finiteness conditions (Fg1) and (Fg2) for an algebra A and a graded subalgebra H of $\mathrm{HH}(A)$. These conditions are the followings:

(Fg1) H is a commutative noetherian algebra with $H^0 = HH^0(A)$.
(Fg2) $\mathrm{Ext}_A^*(A/J, A/J)$ is a finitely generated H-module.

In [4], some geometric properties of the support variety and some representation theoretic properties are related if A satisfies the finiteness condition above. Moreover various results for finite groups are generalized to those for selfinjective algebras satisfying the finiteness conditions.

On the other hand, in [10], Solberg showed the following.

Proposition 1. *Let A be a finite dimensional algebra. Then there exists a graded subalgebra H of* $\mathrm{HH}(A)$ *such that A and H satisfy (Fg1) and (Fg2) if and only if* $\mathrm{HH}(A)$ *is a finitely generated algebra and* $\mathrm{Ext}_A^*(A/J, A/J)$ *is a finitely generated* $\mathrm{HH}(A)$*-module.*

Definition 2. We denote by (Fg) the latter condition in the proposition above.

3. Stratifying ideals

In this section, we will give some results on algebras with stratifying ideals. The stratifying ideal is defined as follows.

Definition 3. Let A be an algebra and $e = e^2$ an idempotent. The two-sided ideal AeA generated by e is called a *stratifying ideal* if the following conditions are satisfied:
(a) The multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism.
(b) $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0$ for all $n > 0$.

The following lemma will be used to check if an ideal is stratifying [8].

Lemma 4. *Let e be an idempotent element in A. If AeA is projective as a right or left A-module, then AeA is stratifying.*

In [8], it is shown that there exist several long exact sequences relating Hochschild cohomology of algebras with a stratifying ideal. The followings are the sequences, which we will use to prove Proposition 6.

Theorem 5. *Let A be an algebra with a stratifying ideal AeA and B the factor algebra A/AeA. Then there are long exact sequences as follows:*
(1) $\to \mathrm{Ext}^n_{A^e}(A, AeA) \to \mathrm{HH}^n(A) \to \mathrm{HH}^n(B) \to \mathrm{Ext}^{n+1}_{A^e}(A, AeA) \to$;
(2) $\to \mathrm{Ext}^n_{A^e}(B, A) \to \mathrm{HH}^n(A) \to \mathrm{HH}^n(eAe) \to \mathrm{Ext}^{n+1}_{A^e}(B, A) \to$; *and*
(3) $\to \mathrm{Ext}^n_{A^e}(B, AeA) \to \mathrm{HH}^n(A) \to \mathrm{HH}^n(B) \oplus \mathrm{HH}^n(eAe) \to \mathrm{Ext}^{n+1}_{A^e}(B, AeA) \to$.
Moreover these sequences induce graded algebra homomorphisms between Hochschild cohomology algebras, especially, the second sequence is induced from the functor $eA \otimes_A - \otimes_A Ae$.

The following proposition is one of our main results and we will apply this for the class of Nakayama algebras in the next section.

Proposition 6. *Let A be an algebra with a stratifying ideal AeA. Suppose* $\mathrm{pd}_{A^e} A/AeA < \infty$. *Then we have*

(1) $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(eAe)$ *as graded algebras, where* $n = \mathrm{pd}_{A^e} A/AeA + 1$,
(2) *A satisfies (Fg) if and only if so does eAe,*
(3) *A is Gorenstein if and only if so is eAe.*

Proof. By the second long exact sequence in theorem 5, the first assertion (1) holds.

For the proof of (2), applying the functor $\mathrm{Hom}_{A^e}(-, \mathrm{Hom}_k(A/J, A/J))$ to the short exact sequense $0 \to AeA \to A \to A/AeA \to 0$ we obtain the isomorphism

$$\mathrm{Ext}^n_{A^e}(A, \mathrm{Hom}_k(A/J, A/J)) \cong \mathrm{Ext}^n_{A^e}(AeA, \mathrm{Hom}_k(A/J, A/J))$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$. This gives the following isomorphism

$$\mathrm{Ext}^n_A(A/J, A/J) \cong \mathrm{Ext}^n_{eAe}(eA/eJ, eA/eJ)$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$, which is induced from the exact functor $eA \otimes_A -$. Then we have the following commutative diagram of graded algebra homomorphism,

$$\begin{tikzcd} \mathrm{HH}(A) \arrow[r, "-\otimes_A A/J"] \arrow[d, "eA\otimes_A - \otimes_A Ae"] & \mathrm{Ext}^*_A(A/J, A/J) \arrow[d, "eA\otimes_A -"] \\ \mathrm{HH}(eAe) \arrow[r, "-\otimes_{eAe} eA/eJ"'] & \mathrm{Ext}^*_{eAe}(eA/eJ, eA/eJ), \end{tikzcd}$$

both columns are isomorphic on all but finite degrees. Hence (2) holds.

For the proof of (3), applying the functor $\mathrm{Hom}_{A^e}(-, \mathrm{Hom}_k(X, A))$ to the short exact sequense $0 \to AeA \to A \to A/AeA \to 0$ we obtain the isomorphism

$$\mathrm{Ext}^n_{A^e}(A, \mathrm{Hom}_k(X, A)) \cong \mathrm{Ext}^n_{A^e}(AeA, \mathrm{Hom}_k(X, A))$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$. This gives the following isomorphism

$$\mathrm{Ext}^n_A(X, A) \cong \mathrm{Ext}^n_{eAe}(eX, eA)$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA+1$. Therefore we have that $\mathrm{id}_A A < \infty$ if and only if $\mathrm{id}_{eAe} eA < \infty$. Hence if $\mathrm{id}_A A < \infty$ then $\mathrm{id}_{eAe} eAe < \infty$. On the other hand, since

$$\mathrm{Ext}^n_A(AeA, X) \cong \mathrm{Ext}^n_{eAe}(eA, eX)$$

for any i, we have that $\mathrm{pd}_A AeA{=}\mathrm{pd}_{eAe} eA$. By the assumption $\mathrm{pd}_{A^e} A/AeA < \infty$, it follows that $\mathrm{pd}_A AeA < \infty$. Hence if $\mathrm{id}_{eAe} eAe < \infty$ then $\mathrm{id}_{eAe} eA < \infty$, so that $\mathrm{id}_A A < \infty$.

Similarly we can show that $\mathrm{id}\, A_A < \infty$ if and only if $\mathrm{id}\, eAe_{eAe} < \infty$. Hence (3) holds. □

4. Nakayama algebras

Throughout this section, we assume that the algebras are basic for simplicity. Because Hochschild cohomology is a Morita-invariance, Theorem 9 holds for any algebra. In this section, we will prove our main theorem, which states that Gorenstein Nakayama algebras satisfy the finiteness condition (Fg). An algebra A is called *Nakayama* if the indecomposable projective right and left modules are uniserial. It is known that if the indecomposable projective modules over a Nakayama algebra have the same length, then the algebra is selfinjective (see [1, Proposition 3.8.]). Especially, any local Nakayama algebra is selfinjective. Using this fact, it is easy to show the following.

Lemma 7. *Let A be a Nakayama algebra. If A is not self injective, then there exists a primitive idempotent f in A such that Jf is a non-zero projective A-module.*

Lemma 8. *Let A be a Gorenstein Nakayama algebra. If A is not selfinjective, then there exists an idempotent $e \neq 1$ such that*

(1) *AeA is projective as left A-module;*
(2) $\mathrm{pd}_{A^e} A/AeA < \infty$*; and*
(3) *eAe is a Gorenstein Nakayama algebra.*

Proof. Assume that A is not selfinjective. By Lemma 7, there exists a primitive idempotent f in A such that Jf is a non-zero projective A-module, so that there exists a primitive idempotent $f' \neq f$ such that $Jf \cong Af'$. Put $e = 1 - f$. Since $f' \neq f$, $AfJf < AeJf$, so that $Jf = AfJf + AeJf = AeJf \leq AeAf < Af$. We obtain that $Jf = AeAf$ because Jf is a maximal submodule of Af. Since $Jf = AeAf$, $J \leq AeA$, so that $fJ \leq fAeA < fA$. Therefore we obtain that $fJ = fAeA$ because fJ is a maximal submodule of fA.

(1) Since $AeAf = Jf$, it follows that $AeA = AeAe \oplus AeAf = Ae \oplus Jf$, so that AeA is projective as left A-module.

(2) Since $(A/AeA)e = 0$, it follows that $A/AeA \cong Af/AeAf = Af/Jf$. Similarly we have that $A/AeA \cong fA/fJ$. Thus A/AeA is simple A^e-module and $A/AeA \cong Af/Jf \otimes_k fA/fJ$. Since Jf is projective, the left projective dimension of Af/Jf is finite and the right injective dimension of $fA/fJ \cong D(Af/Jf)$ is finite. Since A is Gorenstein, the right projective dimension of fA/fJ is finite. Hence $\mathrm{pd}_{A^e} A/AeA < \infty$.

(3)By Lemma 4, AeA is a stratifying ideal. By the assertion (2) above and Proposition 6, eAe is Gorenstein. It is clear that eAe is a Nakayama algebra. □

Theorem 9. *Let A be a Gorenstein Nakayama algebra. Then we have*

(1) *There exists a selfinjective Nakayama algebra B such that* $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(B)$ *as graded algebras for some n,*
(2) *A satisfies the finiteness condition (Fg).*

Proof. By Proposion 6 and Lemma 8, if A is not selfinjective, then there exists an idempotent $e \neq 1$ such that $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(eAe)$ as graded algebras for some n. Since the number of the simple modules of eAe is less than that of A and local Nakayama algebras are selfinjective, the assertion (1) holds.

By Proposion 6 and Lemma 8, if A is not selfinjective, then there exists an idempotent $e \neq 1$ such that A satisfies (Fg) if and only if so does eAe. By [2, Section 4], selfinjective Nakayama algebras satisfy (Fg). Hence assertion (2) holds. □

Corollary 10. *Let A be a Nakayama algebra. Then*

A is Gorenstein if and only if A satisfies the finiteness condition (Fg).

Proof. By [4] and [10], if an algebra satisfies the finiteness condition (Fg), then the algebra is Gorenstein. Hence, by Theorem 9, the assertion holds. □

This corollary gives us a way to check whether a given Nakayama algebra satisfies the finiteness condition (Fg) or not without computing Hochschild cohomology, because we can check whether a given Nakayama algebra is Gorenstein or not by using the Kupish series.

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